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Analytical Methods for Markov Semigroups

Luca Lorenzi

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Luca Lorenzi
University of Parma
Parma, Italy

Marcello Bertoldi
Banca Intesa
Milano, Italy

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Preface

The main objective of this book is to study Markov semigroups, starting from the partial differential equations associated (parabolic and elliptic equations involving a second-order elliptic operator which has unbounded coefficients).

The theory of Markov processes and of related transition semigroups has numerous applications in many fields of science, engineering and economics. Historically, in the mathematical literature the subject is studied using several approaches, with ideas and methods from partial differential equations, Dirichlet forms, stochastic processes, stochastic differential equations, martingale theory.

Somehow the classical semigroup theory is unfit to study this particular class of equations, as well as the classical theory of elliptic differential operators. Indeed, the fact that the coefficients of the operator are unbounded is not merely a technical difficulty, but has significant consequences for the solutions (nonuniqueness of continuous bounded solutions, semigroup not strongly continuous, failure of regularity properties).

These facts lead to looking for specific techniques and results, and here we describe some of them.

The semigroup is studied in spaces of continuous functions and in L^p-spaces of the invariant measure, which is the stationary distribution of the Markov process and it exists under suitable assumptions.

In the first part of the book we study the general properties of the semigroup in spaces of continuous functions: the existence of solutions to the elliptic and to the parabolic equation, the uniqueness properties (and counterexamples to uniqueness), the definition and the properties of the weak generator, which is a specific notion that substitutes the infinitesimal generator of strongly continuous semigroups.

We see also some properties of the associated Markov process and the connection with the uniqueness of the solutions.

Then we focus on the proof of regularity results: global and pointwise estimates of the space derivatives of the semigroup. In particular, first we prove global estimates for the space derivatives and Schauder estimates, similar to those for operators with bounded coefficients. Then, we prove a number of pointwise estimates, which relate the derivatives of the semigroup at a point with the semigroup applied to the derivatives of the function, or to the function itself, at the same point. These estimates are truly characteristic of this class of operators and have interesting consequences (for instance in terms of

Liouville theorems).

We make the same analysis for boundary value problems in unbounded domains, with Dirichlet or Neumann boundary conditions, and for problems involving degenerate operators.

In the part about the invariant measure, we study some different approaches to the problem of the existence of the invariant measure, and we study the properties of the semigroup in L^p-spaces, including the asymptotic behaviour, the Poincaré inequality and the log–Sobolev inequality.

Besides we devote a chapter to the Ornstein–Uhlenbeck semigroup, the most studied example of an operator with unbounded coefficients.

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Marcello Bertoldi

Luca Lorenzi

Symbol Descriptions

Sets

$\mathbb{N}$	set of all positive natural numbers
$\mathbb{Z}$	set of all relative integers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
$\mathbb{R}^N$	set of all real N-tuples
$\mathbb{C}^N$	set of all complex N-tuples
$A \subset\subset B$	given two subsets $A, B \subset \mathbb{R}^N$ with B open, it means that $\overline{A}$ is contained in B
$B(R)$	open disk in $\mathbb{R}^N$ with centre at 0 and radius $R > 0$
$\overline{B}(R)$	the closure of $B(R)$
$x + B(R)$	open disk in $\mathbb{R}^N$ with centre at x and radius $R > 0$
$\overline{F}$	the closure of F
∂F	the boundary of F
$E + F$	given a space X and two subsets of $E, F \subset X$ it denotes the set of $x \in X$ such that $x = e + f$ for some $e \in E$ and some $f \in F$
$E \oplus F$	it denotes the set $E + F$ when E and F are two subspaces of a vector space X and $E \cap F = \{0\}$
$L(X, Y)$	the set of all the bounded linear operators from X to Y
$L(X)$	$:= L(X, X)$
X'	the dual space of the Banach space X, i.e., the sets of bounded linear functionals from X to $\mathbb{R}$ ($\mathbb{C}$)

Functions

δ_x	the delta function, i.e., $\delta_x(x) = 1$, $\delta_x(y) = 0$ if $y \neq x$
$\operatorname{div} f$	the divergence of $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$, i.e., $\operatorname{div} f = \sum_{i=1}^N D_i f$
$\|f\|_\infty$	the sup-norm of $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}$, i.e., $\|f\|_\infty := \sup_\Omega \|f\|$ (whenever is finite)
f' (or f_x)	the first-order derivative of a function f of one variable. We use a similar notation for higher order derivatives
$f^{(k)}$	the k-th-order derivative of a function f of one variable
$D_{i_1,\ldots,i_r} f$	the derivative $\frac{\partial^r f}{\partial x_{i_1},\ldots,\partial x_{i_r}}$ of the function $f : \mathbb{R}^N \to \mathbb{R}$, where $i_1, \ldots, i_r \in \{1, \ldots, N\}$
$D^\alpha f$	the derivative $\frac{\partial^{\|\alpha\|} f}{\partial x^{\alpha_1},\ldots,\partial x^{\alpha_N}}$ of the function f, where $\alpha = (\alpha_1, \ldots, \alpha_N)$
$D^k f$	for $k \in \mathbb{N}$, it denotes the vector consisting of all the k-th order derivatives of f
$\|D^k f\|_\infty$	for $k \in \mathbb{N}$ it denotes the sup-norm of the vector-valued function $D^k f$, namely $\|D^k f\|_\infty^2 = \sum_{\|\alpha\|=k\|} \|D^\alpha f\|_\infty^2$
χ_A	the characteristic function of the set A, i.e., the function defined by $\chi_A(x) = 1$ for any $x \in \mathbb{R}^N$ and $\chi_A(x) = 0$ for any $x \notin A$
$\mathbb{1}$	the characteristic function of $\mathbb{R}^N$
$\operatorname{Jac} f$	the Jacobian matrix of the vector-valued function $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}^m$ ($m, N \in \mathbb{N}$), i.e., the matrix whose j-th line is the vector $(D_1 f_j, \ldots, D_N f_j)$,

	where f_j is the j-th component of f
$\operatorname{supp} f$	the support of the function $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}$
$g \circ f$	given two functions $f : X \to Y$ and $g : Y \to Z$, it denotes the function defined by $(g \circ f)(x) = g(f(x))$ for any $x \in X$
$\\|Q\\|_\infty$	the Euclidean norm of the matrix $Q = (q_{ij})$ belonging to $L(\mathbb{R}^N)$, i.e., $\\|Q\\|_\infty^2 = \sum_{i,j=1}^N q_{ij}^2$. If the entries of Q depend on $x \in \Omega$, we use the same notation to denote the norm $\\|Q\\|_\infty^2 = \sum_{i,j=1}^N \\|q_{ij}\\|_\infty^2$
$\\|DQ\\|_\infty$	$:= \sup_\Omega \sum_{i,j,h=1}^N \|D_h q_{ij}\|^2$ for any matrix Q whose entries are continuously differentiable in open set Ω with bounded derivatives

Operators

$D(A)$	the domain of the (linear) operator A
$\rho(A)$	resolvent set of a linear operator A
$\sigma(A)$	spectrum of a linear operator A
I	(in a Banach space X) the identity operator
$[A, B]$	the commutator between the operators A and B, i.e., $[A, B] = AB - BA$ defined on $D(AB) \cap D(BA)$

Matrix and linear algebra

$\det B$	the determinant of the matrix B
$\operatorname{diag}(\lambda_1, ..., \lambda_N)$	the diagonal matrix whose entries on the main diagonal are $\lambda_1, \ldots, \lambda_N$
e_j	the j-th vector of the canonical basis of $\mathbb{R}^N$
I_r	the identity matrix with r rows and r columns. When there is no damage of confusion we simply write I
$\lambda_{\min}(A)$	the minimum eigenvalue of the matrix A
$\lambda_{\max}(A)$	the maximum eigenvalue of the matrix A
$\operatorname{rank} A$	the maximum number of columns linearly independent of the matrix A
$\operatorname{span}(E)$	given a subset E of a vector space V, it denotes the set of finite linear combinations of elements of E
$\operatorname{Tr} B$	the trace of the $N \times N$ matrix B, i.e., $\operatorname{Tr}(B) = \sum_{i=1}^N b_{ii}$
$x \otimes x$	the $N \times N$-matrix whose entries are $(x \otimes x)_{ij} = x_i x_j$ for any $i, j = 1, \ldots, N$
$\langle x, y\rangle$	the Euclidean inner product between the vectors $x, y \in \mathbb{R}^N$
$x \cdot y$	$:= \langle x, y\rangle$

Miscellanea

a^+	the positive part of $a \in \mathbb{R}$, i.e., the maximum between a and 0
a^-	the negative part of $a \in \mathbb{R}$, i.e., the minimum between a and 0
$a \vee b$	the maximum between a and b
$a \wedge b$	the minimum between a and b
$\|\alpha\|$	the length of the multi-index α, i.e., $\|\alpha\| = \alpha_1 + \ldots + \alpha_N$
$\alpha!$	the factorial of the multi-

	index α, i.e., $\alpha! = \prod_{i=1}^{N} \alpha_i!$
$\deg p$	the degree of the polynomial $p : \mathbb{R}^N \to \mathbb{R}$
δ_{ij}	the Kronecker delta, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise
$\mathrm{dist}(x, F)$	the distance of the point x from the set x, i.e., $\mathrm{dist}(x, F) = \inf_{y \in F} \lvert x - y \rvert$
$\mathrm{dist}(F, G)$	the distance of the set F from the set G, i.e., the number $\inf_{x \in F} d(x, G)$
dx	the Lebesgue measure in $\mathbb{R}^N$ $(N \geq 1)$
$\mathrm{Re}\,\lambda$	the real part of the complex number λ
$\mathrm{Im}\,\lambda$	the imaginary part of the complex number λ

Chapter 1

Introduction

Elliptic operators with bounded coefficients have been widely studied in the literature both in $\mathbb{R}^N$ and in open subsets of $\mathbb{R}^N$, starting from the 1950's, and nowadays they are well understood.

The study of operators with unbounded coefficients is a generalization of this case and recently the interest in this class of operators has grown considerably due to their numerous applications in many fields of science, engineering and economics. The most famous example of an elliptic operator with unbounded coefficients in $\mathbb{R}^N$ $(N \geq 1)$ is the Ornstein-Uhlenbeck operator, defined on smooth functions by

$$\mathcal{A}\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{N} q_{ij}D_{ij}\varphi(x) + \sum_{i,j=1}^{N} b_{ij}x_jD_i\varphi(x), \qquad x \in \mathbb{R}^N,$$

where (q_{ij}) is a constant strictly positive definite matrix and (b_{ij}) is a constant real matrix whose eigenvalues have nonpositive real parts. Such an operator displays all the main peculiarities of the operators with unbounded coefficients. For instance, the associated semigroup $\{T(t)\}$ in $BUC(\mathbb{R}^N)$ is neither strongly continuous nor analytic, whereas the semigroups associated with operators with bounded coefficients are analytic in $C_b(\mathbb{R}^N)$. Nevertheless, it has smoothing effects similar to those enjoyed by analytic semigroups associated with uniformly elliptic operators with bounded coefficients.

Starting from the pioneering papers [10, 76, 90, 91, 92, 93] the literature on elliptic operators with unbounded coefficients has spread out considerably and now we are able to treat uniformly elliptic operators of the type

$$\mathcal{A}\varphi(x) = \sum_{i,j=1}^{N} q_{ij}(x)D_{ij}\varphi(x) + \sum_{i=1}^{N} b_i(x)D_i\varphi(x) + c(x)\varphi(x), \qquad (1.0.1)$$

under rather weak assumptions on the coefficients, both with analytic and probabilistic methods.

The aim of this book is to present most of the old and recent results on the Markov semigroups associated with elliptic operators with unbounded coefficients, using analytic methods. We mainly consider the case when the coefficients of the operator $\mathcal{A}$ are defined in $\mathbb{R}^N$, but we also give some very recent results in the case when they are defined in open and unbounded domains of $\mathbb{R}^N$. The book is divided in three parts.

Part I

In the first part of the book, we are concerned with the following topics:

Existence of the Markov semigroup associated in $C_b(\mathbb{R}^N)$ with the operator $\mathcal{A}$ in (1.0.1). In the case when the coefficients of the operator $\mathcal{A}$ are bounded, the natural way to construct analytically such a semigroup consists in defining, for any $f \in C_b(\mathbb{R}^N)$ (the space of bounded and continuous functions defined in $\mathbb{R}^N$) and any $t > 0$, $T(t)f$ as the value at t of the classical solution to the Cauchy problem

$$\begin{cases} D_t u(t,x) - \mathcal{A}u(t,x) = 0, & t > 0, \ x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.0.2}$$

To generalize this procedure to the case when the coefficients are unbounded, we are led to prove existence results for the bounded classical solution of the Cauchy problem (1.0.2), when $\mathcal{A}$ is given by (1.0.1). Here, by bounded classical solution of (1.0.2), we mean a function u which is bounded and continuous in $[0,+\infty) \times \mathbb{R}^N$ and admits first-order time derivative and first- and second-order spatial derivatives, which are continuous in $(0,+\infty) \times \mathbb{R}^N$. This has been done, first, by S. Itô in [76] and, more recently, by R. Azencott in [10], under quite minimal regularity assumptions on the coefficients of the operator $\mathcal{A}$. The arguments used to prove the existence of a classical solution (which we denote by u_f) are very simple and elegant. They are based both on an approximation argument with Cauchy-Dirichlet problems in bounded and smooth domains, and classical Schauder estimates. In general, u_f is not the unique classical solution to the problem (1.0.2). This is a typical feature of elliptic operators with unbounded coefficients. Nevertheless, u_f enjoys a nice property: when $f \geq 0$, u_f is the minimal positive solution to the problem (1.0.2). This minimality property allows us to define the semigroup $\{T(t)\}$ by setting $T(t)f = u_f(t,\cdot)$ for any $t \geq 0$.

In general, such a semigroup is neither strongly continuous nor analytic in $C_b(\mathbb{R}^N)$. In fact, $T(t)f$ converges to f as t goes to 0, uniformly on compact subsets, but, in general, not uniformly in $\mathbb{R}^N$, and this happens even if f is uniformly continuous. Some sufficient conditions on the coefficients of the operator $\mathcal{A}$ which imply that the semigroup $\{T(t)\}$ is (resp. is not) analytic in $C_b(\mathbb{R}^N)$ are known in the literature and here we describe some of them.

There is also a natural, purely probabilistic way to introduce the semigroup $\{T(t)\}$. It is known that, under suitable assumptions on the coefficients of the operator $\mathcal{A}$, there exists a Markov process X in $\mathbb{R}^N$ such that the semigroup $\{T(t)\}$ is the transition semigroup of X, i.e.,

$$(T(t)f)(x) = \mathbb{E}^x \mathbb{1}_{t<\tau} f(X_t), \qquad t > 0, \ x \in \mathbb{R}^N, \ f \in C_b(\mathbb{R}^N), \tag{1.0.3}$$

where $\tau \in (0,+\infty]$ is the life time of X (X_t is defined for $t \in [0,\tau)$), and $\mathbb{E}^x$ is the expectation under the probability measure $\mathbb{P}_x$. Note that $X_0 = x$

$\mathbb{P}_x$-almost surely. The process X associated with $\{T(t)\}$ is equivalent under the probability $\mathbb{P}_x$ to the solution ξ_t^x of the stochastic differential equation

$$d\xi_t^x = b(\xi_t^x)dt + \sigma(\xi_t^x)dW_t, \qquad \xi_0^x = x, \tag{1.0.4}$$

where W_t is a N-dimensional Brownian motion, $b : \mathbb{R}^N \to \mathbb{R}^N$ is the coefficient of $\mathcal{A}$ and $\sigma : \mathbb{R}^N \to L(\mathbb{R}^N)$ is such that $Q(x) = \frac{1}{2}\sigma(x)\sigma^*(x)$, where $Q(x)$ is the matrix with elements $q_{ij}(x)$.

Conversely, the stochastic equation (1.0.4) can be the starting point for the construction of $\{T(t)\}$: if ξ_t^x is a solution of the equation (1.0.4) and $\{T(t)\}$ is defined by (1.0.3) with $X_t = \xi_t^x$, then the function $u = T(\cdot)f$ is a solution of the parabolic problem (1.0.2). This follows using results from the stochastic calculus (the Itô formula is the main tool).

The equation (1.0.4) can be considered a random perturbation of the ordinary differential equation

$$\frac{d}{dt}\xi_t^x = b(\xi_t^x). \tag{1.0.5}$$

Of course, here it is not natural to assume that the function b is bounded; in almost all the significant cases b is unbounded. This is an important reason for studying operators with unbounded coefficients.

Note that, according to the formula (1.0.3), the properties of $\{T(t)\}$ may be deduced from the study of the process X. This probabilistic approach has been widely used in the literature (see, e.g., [30, 49, 53, 65, 75, 83, 88, 139]).

Under the same smoothness assumptions on the coefficients as above and using similar approximation arguments by Dirichlet problems in bounded domains, the existence of a solution $v_f \in D_{\max}(\mathcal{A})$ to the elliptic equation

$$\lambda v - \mathcal{A}v = f, \qquad \lambda > c_0 := \sup_{x\in\mathbb{R}^N} c(x), \tag{1.0.6}$$

can be proved for any $f \in C_b(\mathbb{R}^N)$. Here,

$$D_{\max}(\mathcal{A}) = \left\{u \in C_b(\mathbb{R}^N) \cap \bigcap_{1\le p<+\infty} W^{2,p}_{\rm loc}(\mathbb{R}^N) : \mathcal{A}u \in C_b(\mathbb{R}^N)\right\}. \tag{1.0.7}$$

As in the parabolic case, the problem (1.0.6) may admit several solutions in $D_{\max}(\mathcal{A})$. In any case, when $f \ge 0$, v_f is the minimal positive solution.

As in the case when the coefficients of the operator $\mathcal{A}$ are bounded, there is a connection between the functions u_f and v_f. In fact, for any $\lambda > c_0$, v_f is the Laplace transform of u_f, in the sense that

$$v_f(x) = \int_0^{+\infty} e^{-\lambda t}u_f(t,x)dt, \qquad x \in \mathbb{R}^N. \tag{1.0.8}$$

The main difference with the classical case is that now, in general, the integral term in the right-hand side of (1.0.8) does not converge in $C_b(\mathbb{R}^N)$.

Since, in general, the semigroup $\{T(t)\}$ is neither strongly continuous nor analytic, then the infinitesimal generator does not exist in the classical sense. This gap is filled introducing the concept of a "weak generator". In fact, we can define a family $\{R(\lambda),\ \lambda > c_0\}$ of bounded operators in $C_b(\mathbb{R}^N)$ by setting $R(\lambda)f = v_f$. Such a family is the resolvent family associated with some (closed) operator $\widehat{A}$. We call $\widehat{A}$ the "weak generator" of the semigroup. The reason for this name is due to the fact that $\widehat{A}$ is a generalization of the classical concept of the infinitesimal generator of a strongly continuous (or analytic) semigroup. Indeed, the weak generator $\widehat{A}$ can be defined, in a equivalent way, using the bounded pointwise convergence: a function f belongs to $D(\widehat{A})$ if and only if

$$\frac{|(T(t)f)(x) - f(x)|}{t} \le M, \qquad t \in (0,1),\ \ x \in \mathbb{R}^N,$$

for some $M > 0$, and $(T(t)f - f)/t$ converges pointwise to some bounded and continuous function g as t tends to 0^+. In such a case $\widehat{A}f = g$.

In general, $D(\widehat{A})$ is properly contained in $D_{\max}(\mathcal{A})$ and $\widehat{A}$ coincides with $\mathcal{A}$ on $D(\widehat{A})$. The previous set inclusion is actually a set equality if and only if the elliptic equation (1.0.6) is uniquely solvable in $D_{\max}(\mathcal{A})$.

As in the classical case of bounded coefficients, it is possible to associate a transition family $\{p(t,x;dy) : t > 0,\ \ x \in \mathbb{R}^N\}$ (or equivalently a Green's function G) to the semigroup, for which the following representations hold:

$$(T(t)f)(x) = \int_{\mathbb{R}^N} f(y)p(t,x;dy) = \int_{\mathbb{R}^N} G(t,x,y)f(y)dy, \qquad t > 0,\ \ x \in \mathbb{R}^N. \tag{1.0.9}$$

The formula (1.0.9) is a keystone for proving some interesting and very useful continuity properties of the semigroup, as well as for showing that the semigroup can be extended to the space $B_b(\mathbb{R}^N)$ of bounded Borel functions. Such a semigroup, still denoted by $\{T(t)\}$, is both irreducible and strong Feller.

All the previous results can be obtained without assuming the uniqueness of the solutions to the problems (1.0.2) and (1.0.6), with the claimed regularity properties. Of course, it is natural to investigate what conditions imply the uniqueness of the classical solution to the Cauchy-Dirichlet problem and of the solution to the elliptic equation in $D_{\max}(\mathcal{A})$. The two problems are not independent of one another. In fact, there exists a unique solution to the elliptic equation in $D_{\max}(\mathcal{A})$ if and only if there exists a unique classical solution to the problem (1.0.2), which is bounded in $[0,T] \times \mathbb{R}^N$ for any $T > 0$.

In the case when the coefficients are bounded, the uniqueness results are straightforward consequences of the classical maximum principles. If the coefficients of the operator $\mathcal{A}$ are unbounded, the classical maximum principle may fail. This is the reason why, in general, the elliptic equation and the parabolic Cauchy-Dirichlet problem admit more than one solution. Hence, to prove uniqueness results some additional assumptions on the operator $\mathcal{A}$ need to be imposed. The typical assumptions which we assume are those that allow us to prove a generalized maximum principle: the existence of a so-called Lyapunov function φ, i.e., a sufficiently smooth function φ such that

$$(i) \lim_{|x|\to+\infty} \varphi(x) = +\infty \text{ and } (ii) \sup_{x\in\mathbb{R}^N} \mathcal{A}\varphi(x) - \lambda\varphi(x) < +\infty, \qquad (1.0.10)$$

for some $\lambda > c_0$. At a first glance, the condition (1.0.10)(ii) may seem a condition related merely to the growth rate of the coefficients at infinity, but actually this is not the case. Indeed, the differential operators

$$\mathcal{A}\varphi(x) = \Delta\varphi(x) - |x|\langle x, D\varphi(x)\rangle, \qquad x \in \mathbb{R}^N \qquad (1.0.11)$$

and

$$\mathcal{A}\varphi(x) = \Delta\varphi(x) + |x|\langle x, D\varphi(x)\rangle, \qquad x \in \mathbb{R}^N, \qquad (1.0.12)$$

differ just in the sign of the drift term, but this difference is essential. Indeed, the Cauchy-Dirichlet problem associated with the first operator admits a unique classical solution, while the Cauchy problem associated with the second operator admits several classical solutions.

In the case when $c \equiv 0$, both the functions $\mathbb{1}$ (i.e., the constant function identically equal to 1) and $T(\cdot)\mathbb{1}$ solve the Cauchy problem (1.0.2). Hence, $T(t)\mathbb{1} \equiv \mathbb{1}$ for any $t > 0$ is a necessary condition for the problem (1.0.2) to admit a unique classical solution. Actually, such a condition is also sufficient. When this condition is satisfied we say that the semigroup is conservative or that the conservation of the probability holds. The reason for this nomenclature is based on the fact that, in this case, the family of measures $\{p(t, x; dy) : \ t > 0, \ x \in \mathbb{R}^N\}$ consists of probability measures.

There are also some conditions under which the problem (1.0.2) admits several classical solutions. For instance, this is the case when there exists a nonnegative bounded and smooth function φ (still called a Lyapunov function) such that $\lambda\varphi - \mathcal{A}\varphi \le 0$.

The one-dimensional case is easier and has been studied by Feller (see [57]). In such a setting there are necessary and sufficient conditions which give uniqueness of the solution to the elliptic (and, consequently, to the parabolic problem) which can be written in terms of the integrability at infinity of some functions which are strictly related to the coefficients of the operator $\mathcal{A}$.

The arguments that we have briefly described here are discussed in detail in Chapters 2, 3, 4 and 10.

Study of the main properties of the semigroup $\{T(t)\}$ in the space $C_b(\mathbb{R}^N)$. We discuss two main topics in this book. The first one is the compactness of the semigroup in $C_b(\mathbb{R}^N)$ which has been studied mainly by E.B. Davies in [44] and by G. Metafune, D. Pallara and M. Wacker in [115], in the case when $c \equiv 0$.

In the conservative case, the semigroup is compact if and only if, for any $t > 0$, the family of measures $\{p(t, x; dy) : \ x \in \mathbb{R}^N\}$ is tight. This means that for any $t > 0$ the measures $p(t,x;dy)$ are ε-concentrated in a compact set, uniformly with respect to $x \in \mathbb{R}^N$. This, for instance, is the case when there exist a nonnegative function $\varphi \in C^2(\mathbb{R}^N)$ and a convex function $g \in C^1([0,+\infty))$ such that

$$\lim_{|x|\to\infty} \varphi(x) = +\infty, \quad 1/g \text{ is integrable at } +\infty, \quad \mathcal{A}\varphi(x) \le -g(\varphi(x)), \quad x \in \mathbb{R}^N.$$

In the case when the family of measures $\{p(t,x;dy),\ x \in \mathbb{R}^N\}$ is tight for any $t > 0$, the semigroup, besides being compact, enjoys another important property: it is norm continuous in $(0,+\infty)$. This means that the function $t \mapsto T(t)$ is continuous with respect to the operator topology at any point $t > 0$. Of course, one cannot expect continuity up to $t = 0$ since the semigroup is not, in general, strongly continuous.

In the nonconservative case, the compactness of the semigroup can be read, roughly speaking, in terms of some regularity properties of the function $T(t)\mathbb{1}$. Indeed, $T(t)\mathbb{1}$ belongs to $C_0(\mathbb{R}^N)$ (the space of continuous functions defined in $\mathbb{R}^N$ which vanish at infinity), for any $t > 0$, if and only if the semigroup is compact and maps $C_0(\mathbb{R}^N)$ into itself. In such a case, as in the conservative one, the semigroup is norm continuous and $R(\lambda, \widehat{A})$ is compact as well.

Even though the semigroup $\{T(t)\}$ may fail to be strongly continuous in $C_b(\mathbb{R}^N)$, the function $t \mapsto T(t)f$ is continuous up to $t = 0$, with respect to the sup-norm, for any $f \in C_0(\mathbb{R}^N)$. Therefore, if the semigroup maps $C_0(\mathbb{R}^N)$ into itself, then its restriction to $C_0(\mathbb{R}^N)$ gives rise to a strongly continuous semigroup. So, it is important to determine suitable conditions on the coefficients of the operator $\mathcal{A}$ which imply that $C_0(\mathbb{R}^N)$ is invariant under the action of the semigroup. We do this in Sections 5.2 and 5.3.
Note that, in general, $\{T(t)\}$ does not map $C_0(\mathbb{R}^N)$ into itself. This is the case, for instance, when the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\varphi(x) = \Delta\varphi(x) - |x|^2\langle x, D\varphi(x)\rangle, \qquad x \in \mathbb{R}^N,$$

on smooth functions φ.

All the arguments that we have described here are discussed in Chapter 5.

Study of the smoothing effects of the semigroup. We pay attention to the estimates of the space derivatives of $T(t)f$ when $f \in C_b(\mathbb{R}^N)$ or it is smooth. These estimates can be split into two main families: uniform (in the space variables) and pointwise estimates. Starting from the pioneering papers on the Ornstein-Uhlenbeck semigroup it has become clear that under suitable assumptions on the growth rate at infinity of the coefficients of the operator $\mathcal{A}$, the space derivatives of $T(t)f$ should behave as in the case of the bounded coefficients. Therefore, under reasonable assumptions on the coefficients, one can expect that

$$\|D^kT(t)f\|_\infty \le Ct^{-\frac{k-h}{2}}e^{\omega t}\|f\|_{C_b^h(\mathbb{R}^N)}, \qquad t > 0, \tag{1.0.13}$$

for any $0 \le h \le k$, any $f \in C_b^h(\mathbb{R}^N)$ and some positive constants C and ω.

We focus our attention on the case when $k \le 3$. Indeed, in this case, the estimates (1.0.13) are a fundamental tool in order to prove existence results and Schauder estimates for the nonhomogeneous Cauchy problem and the nonhomogeneous elliptic equation associated with the operator $\mathcal{A}$.

The estimates (1.0.13) were first proved by A. Lunardi in [108] with analytic methods and by S. Cerrai in [29, 30] with probabilistic methods. Recently they have been improved in [18] in the case when $c \equiv 0$. Here, we describe the results of [18]. The choice of focussing attention on the case when $c \equiv 0$ is due to the fact that such an assumption is necessary for the validity of some of the pointwise estimates that we present.

The method used to prove (1.0.13) is very easy and at the same time very elegant. It consists in adapting to our situation the classical Bernstein method. Unfortunately, to make such a method work, one needs to assume stronger assumptions than those needed to construct the semigroup. In particular, some dissipativity conditions are needed. More precisely, some bounds from above on the sum

$$\sum_{i,j=1}^{N} D_i b_j(x)\xi_i\xi_j \qquad x, \xi \in \mathbb{R}^N$$

are essential to prove (1.0.13). Indeed, without any dissipativity assumption, the estimates (1.0.13) may fail to hold even in the one-dimensional case. Moreover, here we have to assume the uniqueness of the solution to the homogeneous problem (1.0.2).

As it has been claimed, the previous estimates allow us to prove optimal Schauder estimates for both the nonhomogeneous Cauchy problem and the nonhomogeneous elliptic equation associated with the operator $\mathcal{A}$. This is obtained by an interpolation argument under the same assumptions on f and g as in the case of bounded coefficients. As in this latter case, the solution to the nonhomogeneous Cauchy problem

$$\begin{cases} D_t u(t,x) - \mathcal{A}u(t,x) = g(t,x), & t > 0, \ x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases}$$

can be represented by means of the usual variation-of-constants formula

$$u(t,x) = (T(t)f)(x) + \int_0^t (T(t-s)g(t,\cdot))(x)ds, \qquad t \ge 0, \ x \in \mathbb{R}^N, \quad (1.0.14)$$

where now the integral term, in general, does not converge in $C_b(\mathbb{R}^N)$.

The optimal Schauder estimates may also be used to give a partial characterization of $D(\widehat{A})$, showing that $D(\widehat{A}) \subset C_b^{1+\alpha}(\mathbb{R}^N)$ for any $\alpha \in (0,1)$.

The latter type of estimates that we discuss in this book are pointwise estimates. There are two types of pointwise estimates. The first ones are of the following type:

$$|(D^kT(t)f)(x)|^p \le M_{k,p}\left(T(t)((|f|^2+\ldots+|D^kf|^2)^{\frac{p}{2}})\right)(x), \qquad (1.0.15)$$

for any $t>0$, any $x\in\mathbb{R}^N$, any $f\in C_b^k(\mathbb{R}^N)$ ($k=1,2,3$), any $p\in(1,+\infty)$ and some positive constant $M_{k,p}$. They can be proved under the same hypotheses used to get the uniform estimates. Under somewhat heavier assumptions on the coefficients of the operator $\mathcal{A}$, the estimates (1.0.15) can be improved, eliminating the dependence on f^2 from its right-hand side. In particular, it can be shown that

$$|DT(t)f|^p \le e^{\sigma_p t}T(t)(|Df|^p), \qquad t>0, \qquad (1.0.16)$$

for any $p>1$. Note that for (1.0.16) to hold it is necessary that $T(t)\mathbb{1}\equiv\mathbb{1}$ for any $t>0$ and, hence, $c\equiv 0$. Such more restrictive assumptions cover, for instance, some cases when the coefficients have polynomial growth rate at infinity.

The second type of estimate allows us to give an upper bound of the left-hand side of (1.0.15) for any $t>0$ and any $x\in\mathbb{R}^N$ in terms of

$$\psi(t)\left(T(t)((|f|^2+\ldots+|D^{k-1}f|^2)^{\frac{p}{2}})\right)(x),$$

ψ being a positive function which behaves like $t^{-kp/2}$ near $t=0$ and it is bounded or decreases exponentially at infinity. Then, iterating the arguments and taking the semigroup property into account, we can derive the estimates

$$|(D^kT(t)f)(x)|^p \le C_{k,p}t^{-\frac{pk}{2}}\psi_{k,p}(t)(T(t)(|f|^p))(x), \qquad t>0,\ x\in\mathbb{R}^N, \qquad (1.0.17)$$

for any $k=1,2,3$, where the function $\psi_{k,p}$ is bounded at 0 and it behaves like $t^{\frac{pk}{2}}$ or it decreases exponentially to 0 at $+\infty$. This latter estimate improves the uniform estimates (1.0.13), since now $\omega_{k,p}$ may be a negative constant. Consequently, the estimates (1.0.17) allow for a better asymptotic analysis of the semigroup at infinity.

In general, the estimates (1.0.15) cannot be extended to the case $p=1$, as a well-known counterexample by Wang shows. However, in the particular case when $\mathcal{A}=\Delta+\sum_{i=1}^N b_i(\cdot)D_i$ and the b_i's satisfy suitable growth conditions at infinity, they can be extended also to the case $p=1$. The importance of obtaining such estimates with $p=1$ will be clarified later on in this introduction. On the other hand, we stress that the estimates (1.0.17) with $p=1$ may fail also in the case when the coefficients of the operator $\mathcal{A}$ are bounded. A very easy counterexample is given by the Gauss-Weierstrass semigroup.

Liouville type theorems are an important consequence of the previous gradient estimates. Indeed, in the case when $\omega_{1,p}<0$ in (1.0.17), one can show that if $u\in D_{\max}(\mathcal{A})$ satisfies $\mathcal{A}u=0$, then u is constant. But this is not the only application of these estimates. Other important applications will be seen in the next paragraph.

All the arguments discussed here are contained in Chapters 6 and 7.

Study of both the invariant measures associated with the semigroup and the regularity properties of the semigroup in L^p-spaces related to this measure. By definition, an invariant measure is a probability measure μ such that

$$\int_{\mathbb{R}^N} T(t)f\,d\mu = \int_{\mathbb{R}^N} f\,d\mu, \qquad t > 0, \tag{1.0.18}$$

for any $f \in B_b(\mathbb{R}^N)$.

The spaces $L^p(\mathbb{R}^N, \mu)$ are the "right" L^p-spaces for $\{T(t)\}$. Indeed, if an invariant measure exists, then $\{T(t)\}$ maps $L^p(\mathbb{R}^N, \mu)$ into itself. Note that, in general, this property fails for the L^p-spaces related to other measures, even in the case of the Lebesgue measure. In such L^p-spaces, $\{T(t)\}$ is a strongly continuous semigroup for any $p \in [1, +\infty)$. Note that, if $c \equiv 0$ and an invariant measure exists, then the semigroup is conservative. So, from the probabilistic point of view, an invariant measure is a stationary distribution for the Markov process X.

When it exists, the invariant measure of $\{T(t)\}$ is unique and is absolutely continuous with respect to the Lebesgue measure. Moreover, its density is a positive and continuous function (not necessarily bounded). In general, an explicit expression of the density of μ is not available. In any case, in some situations (e.g., in the case when $\mathcal{A}$ is in divergence form), under suitable integrability and smoothness assumptions on the coefficients, one can prove some global L^p and Sobolev regularity properties of the density.

So, the main problem consists in determining suitable conditions ensuring the existence of the invariant measure. The main result in this direction is the Khas'minskii theorem. It states that if there exists a regular function φ such that

$$\varphi \geq 0, \qquad \lim_{|x|\to+\infty} \mathcal{A}\varphi(x) = -\infty,$$

then the invariant measure of $\{T(t)\}$ exists.

Let L_p be the infinitesimal generator of $\{T(t)\}$ in $L^p(\mathbb{R}^N, \mu)$. As far as we know, there are only a few situations in which a complete characterization of $D(L_p)$ is available. This is, for instance, the case of the Ornstein-Uhlenbeck semigroup. In any case, one can prove that $D_{\max}(\mathcal{A})$ is always a core of $D(L_p)$. This means that $D_{\max}(\mathcal{A})$ is dense in $D(L_p)$ which is endowed with the graph norm. Moreover, if the pointwise estimate (1.0.17) holds, one can partially characterize the domain of L_p, showing that it is continuously embedded in $W^{1,p}(\mathbb{R}^N, \mu)$ (the set of all the functions whose first-order distributional derivatives are in $L^p(\mathbb{R}^N, \mu)$).

In the case when $p = 2$, a description of the asymptotic behaviour of the semigroup is available. In fact, for any $f \in L^2(\mathbb{R}^N, \mu)$ we have

$$\lim_{t\to+\infty} \int_{\mathbb{R}^N} (T(t)f - \overline{f})^2 d\mu = 0, \tag{1.0.19}$$

where $\overline{f}$ is the projection of f on the subspace of the constant functions. This subspace is the eigenspace corresponding to the eigenvalue $\lambda = 0$ of the operator $\mathcal{A}$.

If there exists a *spectral gap*, i.e., if

$$\sigma(L_2) \setminus \{0\} \subset \{\mathrm{Re}\ \lambda \le -\delta\}, \tag{1.0.20}$$

for some $\delta > 0$, the convergence in (1.0.19) is of exponential type. This is the case, for instance, when the Poincaré inequality

$$\int_{\mathbb{R}^N} |f - \overline{f}|^2 d\mu \le C \int_{\mathbb{R}^N} |Df|^2 d\mu, \qquad f \in W^{1,2}(\mathbb{R}^N, \mu), \tag{1.0.21}$$

is satisfied. A sufficient condition for the validity of the Poincaré inequality is the compactness of the embedding $W^{1,2}(\mathbb{R}^N, \mu) \subset L^2(\mathbb{R}^N, \mu)$. In turn, this is implied by the validity of the log-Sobolev inequality:

$$\int_{\mathbb{R}^N} f^2 \log |f|\, d\mu \le \|f\|_2^2 \log \|f\|_2 + C \int_{\mathbb{R}^N} |Df|^2 d\mu, \quad f \in W^{1,2}(\mathbb{R}^N, \mu). \tag{1.0.22}$$

For instance, this happens whenever the gradient estimate (1.0.16), with $p = 1$ and $\sigma_1 = -\alpha < 0$, hold. Here, we see the importance of the pointwise gradient estimates with $p = 1$ and exponentials of negative type.

We summarize the relations up to now determined in the following scheme.

$|(DT(t)f)(x)| \le e^{-\alpha t}(T(t)|Df|)(x), \quad t > 0,\ x \in \mathbb{R}^N, \quad \alpha > 0,$

$\Downarrow$

log-Sobolev inequality (1.0.22)

$\Downarrow$

$W^{1,2}(\mathbb{R}^N, \mu) \subset L^2(\mathbb{R}^N, \mu)$ with a compact embedding

$\Downarrow$

Poincaré inequality (1.0.21)

$\Downarrow$

spectral gap (1.0.20) $\Longleftrightarrow$ exponential convergence to $\overline{f}$

$\Uparrow$

$T(t)$ is compact in $C_b(\mathbb{R}^N)$ for any $t > 0$

The log-Sobolev inequality (1.0.22) implies that $f^2 \log|f|$ is integrable whenever $f \in W^{1,2}(\mathbb{R}^N, \mu)$. This Sobolev-type result is very sharp, as an example of L. Gross shows: if μ is the Gaussian measure, there exists a function $f \in W^{1,2}(\mathbb{R}^N, \mu)$ such that $f^2(\log f)(\log\log f)$ is not integrable. In particular, one cannot expect any Sobolev embedding, such as $W^{1,2}(\mathbb{R}^N, \mu) \subset L^p(\mathbb{R}^N, \mu)$, when $p > 2$.

In the case when

$$\mathcal{A}\varphi(x) = \Delta\varphi(x) - \langle DU(x), D\varphi(x)\rangle, \qquad x \in \mathbb{R}^N, \tag{1.0.23}$$

on smooth functions φ, and U is a smooth function such that e^{-U} is integrable in $\mathbb{R}^N$, the previous scheme can be improved a bit. Up to a normalization factor, the invariant measure of the associated semigroup is given by

$$d\mu = e^{-U(x)}dx.$$

In this symmetric case, the log-Sobolev inequality (1.0.22) is equivalent to the *hypercontractivity* of $\{T(t)\}$, that is

$$\|T(t)f\|_{q(t)} \le \|f\|_2, \quad q(t) = 1 + e^{\lambda t}, \quad f \in L^2(\mathbb{R}^N, \mu), \tag{1.0.24}$$

where $\|\cdot\|_r$ is the norm in $L^r(\mathbb{R}^N, \mu)$ and $\lambda = 2/C$. The hypercontractivity was first proved by E. Nelson (see [121]) for the Ornstein Uhlenbeck semigroup, while the equivalence with the log-Sobolev inequality was proved by L. Gross (see [69]). On the other hand, D. Bakry and M. Émery (see [12]) showed the hypercontractivity of $\{T(t)\}$ when the operator $\mathcal{A}$ is given by (1.0.23) with U satisfying

$$\langle D^2U(x)\xi, \xi\rangle \ge \alpha|\xi|^2, \qquad x, \xi \in \mathbb{R}^N,$$

for some $\alpha > 0$. Their proof needs the gradient estimate (1.0.16) with $p = 1$ and $\sigma_1 < 0$.

Thus, when $\mathcal{A}$ is given by (1.0.23) we can complete the scheme above with the following part:

$$\boxed{\mathcal{A}\varphi = \Delta\varphi - \langle DU, D\varphi\rangle, \quad D^2U \ge \alpha > 0}$$

$$\Downarrow$$

$$\boxed{|(DT(t)f)(x)| \le e^{-\alpha t}(T(t)|Df|)(x), \quad t > 0,\ x \in \mathbb{R}^N, \quad \alpha > 0}$$

$$\Downarrow$$

$$\boxed{\text{log-Sobolev inequality (1.0.22)}}$$

$$\Updownarrow$$

$$\boxed{\text{hypercontractivity of } T(t)}$$

The arguments described here are discussed in detail in Chapter 8.

Finally, due to its importance, we devote Chapter 9 to describe most of the significant results on the Ornstein-Uhlenbeck semigroup both in $C_b(\mathbb{R}^N)$ and in the L^p-space related to its invariant measure. We mainly focus on the nondegenerate Ornstein-Uhlenbeck operator, but some of the results that we describe hold also in the degenerate case.

To conclude this first part of the introduction, we summarize, in the following table, the results that can be proved assuming the existence of a nonidentically vanishing Lyapunov function $\varphi \in C^2(\mathbb{R}^N)$. For simplicity we assume $c \equiv 0$, even though some results hold in the general case. We denote by λ and C positive constants. Moreover, the notation $\varphi \nearrow +\infty$ means $\lim_{|x|\to+\infty} \varphi(x) = +\infty$. Similarly the notation $\varphi \searrow +\infty$ means $\lim_{|x|\to+\infty} \varphi(x) = -\infty$.

1	$\varphi \nearrow +\infty$	$\mathcal{A}\varphi - \lambda\varphi \le C$	uniqueness
2	$0 \le \varphi \le C$	$\mathcal{A}\varphi - \lambda\varphi \ge 0$	nonuniqueness
3	$\varphi \ge 0,\ \varphi \in C_0(\mathbb{R}^N)$	$\mathcal{A}\varphi - \lambda\varphi \le -C$	$T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$
4	$\varphi \nearrow +\infty$, $g : \mathbb{R} \to \mathbb{R}$ convex, $\frac{1}{g}$ integrable at $+\infty$	$\mathcal{A}\varphi(x) \le -g(\varphi(x))$	$T(t)$ compact, $C_0(\mathbb{R}^N)$ not invariant
5	$\varphi > 0,\ \varphi \in C_0(\mathbb{R}^N)$	$\mathcal{A}\varphi - \lambda\varphi \le 0$	$C_0(\mathbb{R}^N)$ invariant
6	$\varphi \ge 0$	$\mathcal{A}\varphi \searrow -\infty$	$\exists$ invariant measure

Part II

In this part of the book, we consider the case when $\mathbb{R}^N$ is replaced with an open and unbounded domain of $\mathbb{R}^N$ and we associate homogeneous Dirichlet and Neumann boundary conditions with the operator $\mathcal{A}$. In these two situations the geometry of the open set Ω plays a crucial role in proving the existence of a semigroup associated with the operator $\mathcal{A}$. In all the cases when we guarantee the existence of the semigroup, we can also prove some uniform gradient estimates (and pointwise gradient estimates in the case of Neumann boundary conditions).

The case when homogeneous Neumann boundary conditions are associated with the operator $\mathcal{A}$ and Ω is a convex set is the easiest one to handle. The existence of the semigroup is proved by approximating the Cauchy-Neumann problem with a sequence of Cauchy-Neumann problems in suitable bounded and convex domains Ω_n. Using the same arguments as in the case when $\Omega = \mathbb{R}^N$, one can show that the solutions u_n to such approximating problems converge to a smooth function u, with normal derivative vanishing on

$\partial\Omega_n$, satisfying the parabolic equation $D_t u_n - \mathcal{A} u_n = 0$ in Ω_n. Showing the continuity of u up to $t = 0$ is much more difficult than in the case when $\Omega = \mathbb{R}^N$, since the sequence u_n does not satisfy any monotonicity property. The key point to overcome such a difficulty consists in determining uniform gradient estimates for the solution u_n (similar to those of the case $\Omega = \mathbb{R}^N$), where the constants appearing in the estimates are independent of n. Once such estimates are available, a localization argument shows that u is continuous up to $t = 0$. The uniform gradient estimates can be proved by adapting the Bernstein method and applying it to the functions u_n. Here, the convexity of Ω_n plays a crucial role in making this machinery work.

Unfortunately, the Bernstein method cannot be adapted to prove the estimates for the higher order derivatives and, to the best of our knowledge, such estimates are available only in the case when Ω is an exterior domain (see Section 13.5).

The semigroup $\{T(t)\}$ is not strongly continuous and, in general, it is not analytic in $C_b(\overline{\Omega})$. In any case, as in Chapter 2, it is possible to define its weak generator and to give a partial characterization of it.

As it has been already remarked, the other two situations considered in the second part of the book are much more involved. This is essentially due to the fact that it is not immediate to adapt the Bernstein method as in the previous situation.

Part III

Finally, in this part of the book, we describe how the approach used in the first part of the book can successfully be applied also to a few cases in which the operator $\mathcal{A}$ is degenerate (and with the coefficients defined on the whole of $\mathbb{R}^N$). More precisely, we consider degenerate elliptic operators $\mathcal{A}$ of the type

$$\mathcal{A}\varphi(x) = \sum_{i,j=1}^{r} q_{ij}(x) D_{ij}\varphi(x) + \sum_{i,j=1}^{N} b_{ij} x_j D_i \varphi(x), \qquad x \in \mathbb{R}^N, \qquad (1.0.25)$$

when $N/2 \le r < N$ and the matrix (q_{ij}) is strictly positive definite. It is possible to associate a semigroup $\{T(t)\}$ with the operator $\mathcal{A}$ and to prove uniform estimates for the space derivatives of the function $T(t)f$ up to the third-order, when $f \in C_b(\mathbb{R}^N)$. Such estimates are used to prove Schauder estimates for the distributional solutions to both the elliptic equation and the nonhomogeneous Cauchy problem associated with the operator $\mathcal{A}$.

Unfortunately, the techniques used in the nondegenerate case cannot be easily adapted to this situation. Indeed, such techniques rely essentially on interior Schauder estimates that are not available in the degenerate case.

The construction of the semigroup associated with the problem (1.0.2) and the determination of the uniform estimates are two topics to be treated simultaneously: we need *a priori* estimates on the behaviour of the space derivatives of the function $T(t)f$ in order to guarantee its existence. This forces us to assume stronger assumptions on the coefficients than those in the nondegenerate case.

The underlying idea used to solve the problem is simple: we approximate the operator $\mathcal{A}$ by a sequence of uniformly elliptic operators $\mathcal{A}_\varepsilon$ ($\varepsilon > 0$) converging to the operator $\mathcal{A}$ in a sense to be made precise later. With each of such operators we can associate a semigroup of bounded linear operators $\{T_\varepsilon(t)\}$. We prove uniform estimates for $\{T_\varepsilon(t)\}$, with constants being independent of ε, and, then, letting ε go to 0^+, we can show that $\{T_\varepsilon(t)\}$ converges to a semigroup of bounded operators. Such a semigroup is associated with the operator $\mathcal{A}$, in the sense that $T(\cdot)f$ turns out to be the (unique) smooth function satisfying $D_t u - \mathcal{A}u = 0$ in $(0, +\infty) \times \mathbb{R}^N$ and the initial condition $u(0, \cdot) = f$, which is bounded in $[0, T] \times \mathbb{R}^N$ for any $T > 0$.

A fundamental tool for such a machinery work is given by the maximum principle which must hold for both of the operators $\mathcal{A}$ and $\mathcal{A}_\varepsilon$.

The behaviour of the derivatives of $T(t)f$ is worse than in the nondegenerate case. For instance, for any $\omega > 0$, one can show that there exists a positive constant C such that

$$\|D_i T(t) f\|_\infty \le \begin{cases} Ct^{-\frac{1}{2}} e^{\omega t}, & i \le r, \\ Ct^{-\frac{3}{2}} e^{\omega t}, & i > r, \end{cases} \tag{1.0.26}$$

for any $t > 0$. This situation is not surprising at all, since this is just what happens for the degenerate Ornstein-Uhlenbeck operator defined by (1.0.25) with constant diffusion coefficients. In such a case, the estimates (1.0.26) are known to be optimal near $t = 0$.

Most of the results holding in the nondegenerate case can be recovered also in this situation. In particular, it is still possible to associate a weak generator with the semigroup and, taking advantage of the uniform estimates, one can (at least partially) characterize its domain. Further, one can deal with the elliptic equation

$$\lambda u - \mathcal{A}u = f \in C_b(\mathbb{R}^N)$$

whose continuous distributional solution is still formally given by the Laplace transform of the semigroup, where the integral, as in the nondegenerate case, is meant in pointwise sense.

Similarly, the continuous distributional solution to the nonhomogeneous Cauchy problem

$$\begin{cases} D_t u(t,x) - \mathcal{A}u(t,x) = g(t,x), & t > 0,\, x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases}$$

when f and g are sufficiently smooth, is still given by the variation-of-constants formula (1.0.14). Such a formula allows us, also in this situation, to determine Schauder estimates for the solution u to the Cauchy problem via an interpolation argument.

With the aim of making the book as self-contained as possible, we collect in four appendices all the classical results of functional analysis and of the theory of partial differential equations, as well as some properties of the distance function. Finally, in the fifth appendix, we collect all the function spaces that we use in this book.

Part I

Markov semigroups in $\mathbb{R}^N$

Chapter 2

The elliptic equation and the Cauchy problem in $C_b(\mathbb{R}^N)$: the uniformly elliptic case

2.0 Introduction

In this chapter we consider linear elliptic and parabolic problems in $C_b(\mathbb{R}^N)$ associated with the differential operator $\mathcal{A}$ defined by

$$\mathcal{A}u(x) = \sum_{i,j=1}^{N} q_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{N} b_i(x)D_iu(x) + c(x)u(x), \qquad x \in \mathbb{R}^N,$$

on smooth functions. We assume the following hypotheses on the coefficients of the operator $\mathcal{A}$.

Hypotheses 2.0.1 (i) $q_{ij} \equiv q_{ji}$ for any $i, j = 1, \ldots, N$ and

$$\sum_{i,j=1}^{N} q_{ij}(x)\xi_i\xi_j \geq \kappa(x)|\xi|^2, \quad \kappa(x) > 0, \quad \xi, x \in \mathbb{R}^N;$$

(ii) q_{ij}, b_i $(i, j = 1, \ldots, N)$ and c belong to $C^{\alpha}_{\rm loc}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$;

(iii) there exists $c_0 \in \mathbb{R}$ such that

$$c(x) \leq c_0, \quad x \in \mathbb{R}^N.$$

Besides, we introduce the realization A of $\mathcal{A}$ in $C_b(\mathbb{R}^N)$, with domain $D_{\max}(\mathcal{A})$, defined as follows:

$$D_{\max}(\mathcal{A}) = \left\{ u \in C_b(\mathbb{R}^N) \cap \bigcap_{1\leq p<+\infty} W^{2,p}_{\rm loc}(\mathbb{R}^N) : \mathcal{A}u \in C_b(\mathbb{R}^N) \right\}, \quad Au = \mathcal{A}u. \tag{2.0.1}$$

In Section 2.1 we prove that, for any $\lambda > c_0$ and any $f \in C_b(\mathbb{R}^N)$, the elliptic equation

$$\lambda u(x) - \mathcal{A}u(x) = f(x), \quad x \in \mathbb{R}^N, \tag{2.0.2}$$

admits a solution $u \in D_{\max}(\mathcal{A})$. The idea of the proof is the following. For each $n \in \mathbb{N}$ we consider the Dirichlet problem

$$\begin{cases} \lambda u_n(x) - \mathcal{A}u_n(x) = f(x), & x \in B(n), \\ u_n(x) = 0, & x \in \partial B(n), \end{cases} \tag{2.0.3}$$

in the ball $B(n) = \{x \in \mathbb{R}^N : \ |x| < n\}$. This problem has a unique solution $u_n \in C(\overline{B}(n))$ (in Section C we recall the results about elliptic and parabolic problems in bounded domains that we need throughout this chapter). Using an interior estimate (see Theorem C.1.1), we prove that we can define a function $u : \mathbb{R}^N \to \mathbb{R}$ by setting

$$u(x) := \lim_{n\to+\infty} u_n(x),$$

for any $x \in \mathbb{R}^N$. The function u belongs to $D_{\max}(\mathcal{A})$ is a solution of the equation (2.0.2) and it satisfies the estimate

$$\|u\|_\infty \le \frac{1}{\lambda - c_0}\|f\|_\infty.$$

We stress that, in general, u is not the unique solution of the equation (2.0.2) in $D_{\max}(\mathcal{A})$. It is the unique solution provided further conditions on the coefficients are satisfied. The problem of the uniqueness of the solution to the equation (2.0.2) will be treated in Chapter 4. At any rate, if $f \ge 0$ then u is the minimal positive solution of (2.0.2) in $D_{\max}(\mathcal{A})$.

Next, we prove that u is given by the formula

$$u(x) = \int_{\mathbb{R}^N} K_\lambda(x,y)f(y)dy, \quad x \in \mathbb{R}^N, \tag{2.0.4}$$

where K_λ is a positive function; K_λ is the so-called Green's function associated with the problem (2.0.2). To prove (2.0.4) we recall that the solution u_n of (2.0.3) is given by the formula

$$u_n(x) = \int_{B(n)} K_\lambda^n(x,y)f(y)dy, \quad x \in B(n),$$

where K_λ^n is the Green's function associated with (2.0.3). Using the classical maximum principle we prove that the sequence $\{K_\lambda^n\}$ is increasing (with respect to $n \in \mathbb{N}$). This leads to the formula (2.0.4) with

$$K_\lambda(x,y) := \lim_{n\to+\infty} K_\lambda^n(x,y), \qquad x, y \in \mathbb{R}^N.$$

Thus for any $\lambda > c_0$ we can define the linear operator $R(\lambda)$ in $C_b(\mathbb{R}^N)$ by setting

$$(R(\lambda)f)(x) = \int_{\mathbb{R}^N} K_\lambda(x,y)f(y)dy, \quad x \in \mathbb{R}^N.$$

$R(\lambda)$ is a bounded operator, with $\|R(\lambda)\|_{L(C_b(\mathbb{R}^N))} \le (\lambda - c_0)^{-1}$. In Section 2.3 we see that the operators $\{R(\lambda) : \lambda > c_0\}$ are the resolvent operators of a linear operator $\widehat{A}$ in $C_b(\mathbb{R}^N)$. The operator $\widehat{A}$ is called weak generator. This terminology is justified by the theory of parabolic problems and it will be much clearer later on in this chapter.

The parabolic problem that we consider is the following

$$\begin{cases} D_t u(t,x) - \mathcal{A}u(t,x) = 0, & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.0.5}$$

with $f \in C_b(\mathbb{R}^N)$. In Section 2.2 we prove the existence of a classical solution of such a problem (i.e., a function $u \in C([0,+\infty)\times\mathbb{R}^N) \cap C^{1,2}((0,+\infty)\times\mathbb{R}^N)$), which is bounded in $[0,T]\times\mathbb{R}^N$ for any $T > 0$ and satisfies $D_t u, D^2 u \in C^{\alpha/2,\alpha}_{\rm loc}((0,+\infty)\times\mathbb{R}^N)$. The idea of the proof is similar to that used in the elliptic case. More precisely, for each $n \in \mathbb{N}$ we consider the Cauchy-Dirichlet problem

$$\begin{cases} D_t u_n(t,x) - \mathcal{A}u_n(t,x) = 0, & t > 0,\ x \in B(n), \\ u_n(t,x) = 0, & t > 0,\ x \in \partial B(n), \\ u_n(0,x) = f(x), & x \in B(n), \end{cases} \tag{2.0.6}$$

in the ball $B(n)$. By classical results for parabolic Cauchy problems in bounded domains we know that the problem (2.0.6) admits a unique solution $u_n \in C([0,+\infty)\times\overline{B}(n) \setminus (\{0\}\times\partial B(n))) \cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times B(n))$. Using Schauder interior estimates (see Theorem C.1.4) and a compactness argument, we prove that we can define a function $u : [0,+\infty)\times\mathbb{R}^N \to \mathbb{R}$ by setting

$$u(t,x) := \lim_{n\to+\infty} u_n(t,x),$$

for any $t \in [0,+\infty)$ and any $x \in \mathbb{R}^N$. Such a function belongs to $C([0,+\infty)\times\mathbb{R}^N) \cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\mathbb{R}^N)$, is a solution of the problem (2.0.6) and satisfies the estimate

$$|u(t,x)| \le \exp(c_0 t)\|f\|_\infty, \qquad t > 0,\ x \in \mathbb{R}^N.$$

In general, as in the elliptic case, u is not the unique classical solution of the problem (2.0.5) which is bounded in $[0,T]\times\mathbb{R}^N$ for any $T > 0$. It turns out to be the unique solution under further assumptions on the coefficients (see Chapter 4). Even though u is not the unique solution to the problem (2.0.5), at any rate it has an intrinsic meaning: in fact, for any $f \ge 0$, u is the minimal solution to the problem (2.0.5) in the sense that, if v is another positive solution to the same Cauchy problem, then $v \ge u$. This is shown in Remark 2.2.3.

Then, we prove that u can be represented by the formula

$$u(t,x)=\int_{\mathbb{R}^N} G(t,x,y)f(y)dy, \quad t>0,\ x\in\mathbb{R}^N, \tag{2.0.7}$$

where G is a positive function, called the fundamental solution. To prove (2.0.7) we argue as in the elliptic case, recalling that the solution u_n of the problem (2.0.6) is given by the formula

$$u_n(t,x)=\int_{B(n)} G_n(t,x,y)f(y)dy, \quad t>0,\ x\in B(n),$$

where G_n is the fundamental solution of (2.0.6). Using the classical maximum principle we prove that the sequence $\{G_n\}$ is increasing with respect to $n\in\mathbb{N}$. This gives the formula (2.0.7) with

$$G(t,x,y)=\lim_{n\to+\infty} G_n(t,x,y), \quad t>0,\ x,y\in\mathbb{R}^N,$$

and it allows us to define the linear operator $T(t)$ in $C_b(\mathbb{R}^N)$, for any $t>0$, by setting

$$(T(t)f)(x)=\int_{\mathbb{R}^N} G(t,x,y)f(y)dy, \quad t>0,\ x\in\mathbb{R}^N.$$

We prove that the family $\{T(t)\}$ is a semigroup of linear operators in $C_b(\mathbb{R}^N)$. In general, $\{T(t)\}$ is a strongly continuous semigroup neither in $C_b(\mathbb{R}^N)$ nor in $BUC(\mathbb{R}^N)$ (see Proposition 9.2.6 and [148, Theorem 4.2]). Nevertheless, $T(t)f$ tends to f as t tends to 0, uniformly on compact sets. Then, we show that if f vanishes at infinity, then, actually, $T(t)f$ tends to f as t tends to 0, uniformly in $\mathbb{R}^N$. But this does not mean that the restriction of $\{T(t)\}$ to $C_0(\mathbb{R}^N)$ is a strongly continuous semigroup, because, in general, $C_0(\mathbb{R}^N)$ is not invariant for $\{T(t)\}$ (see Section 5.3).

In Section 2.3 we study the relation between the semigroup $\{T(t)\}$ and the weak generator $\widehat{A}$ introduced above. In fact, $\widehat{A}$ is the generator of $\{T(t)\}$ with respect to the bounded pointwise convergence. This means that, for any $f\in D(\widehat{A})$, the function $(t,x)\mapsto\{(T(t)f)(x)-f(x)\}/t$ is bounded in $(0,1)\times\mathbb{R}^N$ and

$$\lim_{t\to0^+}\frac{(T(t)f)(x)-f(x)}{t}=(\widehat{A}f)(x),$$

for any $x\in\mathbb{R}^N$. Further, $\widehat{A}$ is also the generator of the semigroup with respect to the mixed topology, which is the finest locally convex topology which agrees on every norm-bounded subsets of $C_b(\mathbb{R}^N)$ with the topology of the uniform convergence on compact sets. A sequence of functions converges in the mixed topology if and only if it is bounded and it converges locally uniformly. Thus, the convergence in the mixed topology is very similar to the bounded pointwise convergence.

In general, $D(\widehat{A})$ is a proper subset of $D_{\max}(\mathcal{A})$. It turns out to coincide with $D_{\max}(\mathcal{A})$ whenever the elliptic equation (2.0.2) has a unique solution in $D_{\max}(\mathcal{A})$.

It is immediate to see that the semigroup $\{T(t)\}$ can be extended to the space $B_b(\mathbb{R}^N)$ of Borel measurable and bounded functions. We prove that $\{T(t)\}$ is irreducible and it has the strong Feller property in $B_b(\mathbb{R}^N)$ (see Definition 2.2.11). These properties are useful for the theory of the invariant measures treated in Chapter 8.

In the case when $c \equiv 0$, the semigroup $\{T(t)\}$ is associated with a transition function. This leads to the existence of a Markov process associated with $\{T(t)\}$. We briefly deal with the probabilistic approach in Section 2.4. Of course, there is a huge literature on the subject. Here, we just recall the definitions and the main properties of the Markov processes associated with the semigroup. Among them, we see the Dynkin formula and the link with the theory of differential stochastic equations.

2.1 The elliptic equation and the resolvent $R(\lambda)$

In this section we consider the elliptic equation (2.0.2) with $f \in C_b(\mathbb{R}^N)$ and $\lambda > c_0$. We prove that such an equation admits *at least* a solution u belonging to the domain $D_{\max}(\mathcal{A})$ defined in (2.0.1).

Theorem 2.1.1 *For any $f \in C_b(\mathbb{R}^N)$ there exists $u \in D_{\max}(\mathcal{A})$ which solves* (2.0.2). *Moreover, the following estimate holds:*

$$\|u\|_\infty \le \frac{1}{\lambda - c_0}\|f\|_\infty. \tag{2.1.1}$$

Finally, if $f \ge 0$, then $u \ge 0$.

Proof. For any $n \in \mathbb{N}$, we denote by A_n the realization of the operator $\mathcal{A}$ with homogeneous Dirichlet boundary conditions in $C(\overline{B}(n))$. By Proposition C.3.4, the elliptic problem (2.0.3) admits a unique solution $u_n = R(\lambda, A_n)f$ in $\bigcap_{1\le p<+\infty} W^{2,p}(B(n))$. Moreover, by (C.2.3), u_n satisfies the estimate

$$\|u_n\|_\infty \le \frac{1}{\lambda - c_0}\|f\|_\infty. \tag{2.1.2}$$

Let us prove that the sequence $\{u_n\}$ converges uniformly on compact sets, and in $W^{2,p}_{\rm loc}(\mathbb{R}^N)$, to a function $u \in D_{\max}(\mathcal{A})$ which satisfies the statement. For this purpose, consider first the case when $f \ge 0$. By the maximum principle, the functions u_n and $u_{n+1} - u_n$ are nonnegative in $B(n)$ for any $n \in \mathbb{N}$, that is the sequence $\{u_n\}$ is nonnegative and increasing. Therefore, it converges pointwise to some nonnegative function $u : \mathbb{R}^N \to \mathbb{R}$. By (2.1.2) it follows that u satisfies (2.1.1). Moreover, according to Theorem C.1.1 and (2.1.2), the sequence $\{u_n\}$ is bounded in $W^{2,p}(B(R))$ for any $p \in [1, +\infty)$ and any fixed $R > 0$. Then, by the Sobolev embedding theorems (see [2, Theorem 5.4]), it is bounded in $C^1(\overline{B}(R))$ too, and the Ascoli-Arzelà Theorem implies that it converges to u in $C(\overline{B}(R))$. Besides, applying again Theorem C.1.1 to the function $u_n - u_m$, we deduce that u belongs to $W^{2,p}(B(R))$ and that u_n converges to u in $W^{2,p}(B(R))$, for any $p \in [1, +\infty)$.

Since $\mathcal{A}u_n = \lambda u_n - f$ in $B(n)$, it follows that $u \in D_{\max}(\mathcal{A})$ and $\mathcal{A}u = \lambda u - f$.

This concludes the proof in the case when $f \geq 0$. For an arbitrary $f \in C_b(\mathbb{R}^N)$, it suffices to split $f = f^+ - f^-$ and

$$u_n = R(\lambda, A_n)(f^+) - R(\lambda, \mathcal{A}_n)(f^-) := u_{n,1} + u_{n,2},$$

and to apply the previous arguments separately to the sequences $u_{n,1}$ and $u_{n,2}$. ∎

Remark 2.1.2 In general, the equation (2.0.2) admits more than one solution in $D_{\max}(\mathcal{A})$. In Chapters 3 and 5 we show some situations in which the elliptic equation $\lambda u - \mathcal{A}u = 0$ admits a nontrivial solution in $D_{\max}(\mathcal{A})$ (see Examples 3.2.5 and 5.2.5). Nevertheless, in the case when the datum f is nonnegative the solution u provided by Theorem 2.1.1 can be characterized as the minimal positive solution. Indeed, if v is another positive solution, by the maximum principle it follows that $v(x) \geq u_n(x)$ for any $x \in B(n)$ and any $n \in \mathbb{N}$. Letting n go to $+\infty$ gives $v \geq u$.

We now prove that we can associate a positive Green's function with the equation (2.0.2). This will then allow us to define the resolvent operator $R(\lambda)$ for any $\lambda > c_0$.

Theorem 2.1.3 *For any $\lambda > c_0$ there exists a linear operator $R(\lambda)$ in $C_b(\mathbb{R}^N)$ such that for any $f \in C_b(\mathbb{R}^N)$ the solution of the equation* (2.0.2), *provided by Theorem* 2.1.1, *is represented by*

$$u(x) = (R(\lambda)f)(x), \quad x \in \mathbb{R}^N. \tag{2.1.3}$$

The family of operators $\{R(\lambda) : \lambda > c_0\}$ satisfies the estimate

$$\|R(\lambda)f\|_\infty \leq \frac{1}{\lambda - c_0}\|f\|_\infty, \quad f \in C_b(\mathbb{R}^N), \tag{2.1.4}$$

and the resolvent identity

$$R(\lambda)f - R(\mu)f = (\mu - \lambda)R(\mu)R(\lambda)f, \quad c_0 < \lambda < \mu. \tag{2.1.5}$$

Moreover, $R(\lambda)$ is injective for any $\lambda > c_0$. Finally, there exists a positive function $K_\lambda : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ such that

$$(R(\lambda)f)(x) = \int_{\mathbb{R}^N} K_\lambda(x,y)f(y)dy, \quad x \in \mathbb{R}^N, \ f \in C_b(\mathbb{R}^N). \tag{2.1.6}$$

Proof. Let K_λ^n be the Green's function associated with the realization A_n of the operator $\mathcal{A}$ in $C(\overline{B}(n))$ with homogeneous Dirichlet conditions (see Proposition C.3.4). With any nonnegative function $f \in C_0(B(n))$, let

$$v_n(x) = \int_{B(n)} (K_\lambda^{n+1}(x,y) - K_\lambda^n(x,y)) f(y)dy, \qquad x \in B(n).$$

Since $f \geq 0$ and $v_n = u_{n+1} - u_n$ by Proposition C.2.3, we have $v_n(x) \geq 0$ for any $x \in B(n)$. The arbitrariness of $f \geq 0$ implies that

$$K_\lambda^{n+1}(x,y) - K_\lambda^n(x,y) \geq 0,$$

for any $x \in B(n)$ and any $y \in E_n(x)$, where $E_n(x)$ is a measurable set such that $B(n) \setminus E(x)$ is negligible. Hence, for any $x \in \mathbb{R}^N$ and any $y \in E_n(x)$, $\{K_\lambda^n(x,y)\}$ is an increasing sequence. Therefore, we can define the function $K_\lambda : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by setting

$$K_\lambda(x,y) := \lim_{n\to+\infty} K_\lambda^n(x,y), \qquad x \in \mathbb{R}^N, \ y \in \bigcap_{n\in\mathbb{N}} E_n. \tag{2.1.7}$$

This limit is not infinite. Indeed estimate (C.2.3) with $f \equiv \mathbb{1}$ yields

$$\int_{B(n)} K_\lambda^n(x,y)dy \leq \frac{1}{\lambda - c_0}, \qquad x \in B(n), \ n \in \mathbb{N},$$

and, then, by monotone convergence

$$\int_{\mathbb{R}^N} K_\lambda(x,y)dy \leq \frac{1}{\lambda - c_0}, \qquad x \in \mathbb{R}^N,$$

so that, for any $x \in \mathbb{R}^N$, $K(x,y)$ is finite for almost any $y \in \mathbb{R}^N$. Besides, since K_λ^n is strictly positive in $B(n) \times B(n)$ for any $n \in \mathbb{N}$, also K_λ is.

Now, we observe that the solution of the equation (2.0.2) given by Theorem 2.1.1 can be represented by

$$u(x) = \lim_{n\to+\infty} \left(\int_{\mathbb{R}^N} K_\lambda^n(x,y) f^+(y)dy - \int_{\mathbb{R}^N} K_\lambda^n(x,y) f^-(y)dy \right),$$

for any $x \in \mathbb{R}^N$. Since both f^+ and f^- are nonnegative, the monotone convergence theorem implies that

$$u(x) = \int_{\mathbb{R}^N} K_\lambda(x,y) f(y)dy, \quad x \in \mathbb{R}^N.$$

Thus, the operator $R(\lambda)$ in (2.1.6) is well defined and (2.1.3) holds. Clearly, $R(\lambda)$ is linear, and, by virtue of (2.1.1), it satisfies (2.1.4). Moreover, $R(\lambda)$ is injective. Indeed, if $u = R(\lambda)f = 0$, then $f = 0$ since $R(\lambda)f$ solves, by construction, the elliptic equation $\lambda u - \mathcal{A}u = f$.

To conclude the proof, it remains to prove the resolvent identity (2.1.5). To prove it, we fix $f \in C_b(\mathbb{R}^N)$, $n \in \mathbb{N}$ and we observe that

$$R(\lambda, A_n)f - R(\mu, A_n)f = (\mu - \lambda)R(\lambda, A_n)R(\mu, A_n)f, \quad \mu > \lambda > c_0,$$

(see Sections A.3 and C). By virtue of (C.3.8), this can be rewritten as

$$\int_{B(n)} K_\lambda^n(x,y)f(y)dy - \int_{B(n)} K_\mu^n(x,y)f(y)dy$$
$$= (\mu - \lambda)\int_{B(n)} dy \int_{B(n)} K_\mu^n(x,y)K_\lambda^n(y,z)f(z)dz.$$

Letting n go to $+\infty$, the dominated convergence theorem gives

$$\int_{\mathbb{R}^N} K_\lambda(x,y)f(y)dy - \int_{\mathbb{R}^N} K_\mu(x,y)f(y)dy$$
$$= (\mu - \lambda)\int_{\mathbb{R}^N} dy \int_{\mathbb{R}^N} K_\mu(x,y)K_\lambda(y,z)f(z)dz,$$

that is (2.1.5). ∎

2.2 The Cauchy problem and the semigroup

We now prove that for any $f \in C_b(\mathbb{R}^N)$ there exists a solution $u \in C([0,+\infty) \times \mathbb{R}^N) \cap C^{1+\alpha/2, 2+\alpha}((0,+\infty) \times \mathbb{R}^N)$ of the Cauchy problem (2.0.5).

Theorem 2.2.1 *For any $f \in C_b(\mathbb{R}^N)$, there exists a solution $u \in C([0,+\infty) \times \mathbb{R}^N)$ of the problem* (2.0.5). *The function u belongs to $C_{\rm loc}^{1+\alpha/2,2+\alpha}((0,+\infty) \times \mathbb{R}^N)$ and*

$$|u(t,x)| \le \exp(c_0 t)\|f\|_\infty, \qquad t > 0, \;\; x \in \mathbb{R}^N. \tag{2.2.1}$$

Proof. We split the proof into two steps. First, we show that there exists a solution $u \in C_{\rm loc}^{1+\alpha/2,2+\alpha}((0,+\infty) \times \mathbb{R}^N)$ to the differential equation in (2.0.5), and it satisfies (2.2.1). Then, in Step 2, we show that u is continuous up to $t = 0$, and $u(0,\cdot) = f$.

Step 1. For any $n \in \mathbb{N}$, let $u_n \in C([0,+\infty) \times \overline{B}(n) \setminus (\{0\} \times \partial B(n))\} \cap C^{1,2}((0,+\infty) \times B(n))$ be the solution of the Cauchy-Dirichlet problem (2.0.6) (see Proposition C.3.2), which is given by

$$u_n(t,x) = (T_n(t)f)(x), \quad t > 0, \; x \in B(n),$$

where $\{T_n(t)\}$ is the semigroup in $C(\overline{B}(n))$ associated with the Cauchy-Dirichlet problem (2.0.6). From Proposition C.3.2 (see (C.3.4)), for any $n \in \mathbb{N}$ we have

$$|u_n(t,x)| \le \exp(c_0 t)\|f\|_\infty, \qquad t>0,\ x \in B(n). \tag{2.2.2}$$

Now, fix $M \in \mathbb{N}$ and set $D(M) = (0,M)\times B(M)$ and $D'(M) = [1/M, M] \times B(M-1)$. From the interior Schauder estimate (C.1.15) we deduce that

$$\|u_n\|_{C^{1+\alpha/2,2+\alpha}(D'(M))} \le C_M \|u_n\|_{L^\infty(D(M))} \le C_M(\exp(c_0 M)\vee 1)\|f\|_\infty, \tag{2.2.3}$$

for any $n \ge M$, where $C_M > 0$ is a constant independent of $n \in \mathbb{N}$. Fix $\beta \in (0,\alpha)$. By (2.2.3) there exists a subsequence $\{u_n^{(M)}\}$ of $\{u_n\}$ converging in $C^{1+\beta/2,2+\beta}(D'(M))$ to some function $u_\infty^{(M)} \in C^{1+\alpha/2,2+\alpha}(D'(M))$. Without loss of generality we can assume that $\{u_n^{(M+1)}\}$ is a subsequence of $\{u_n^{(M)}\}$. Thus, the functions $u_\infty^{(M)}$ and $u_\infty^{(M+1)}$ coincide in the domain $D'(M)$ and, therefore, we can define the function $u \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\mathbb{R}^N)$ by setting

$$u = u_\infty^{(M)} \ \text{ in } D'(M).$$

Moreover, the diagonal subsequence defined by

$$\tilde{u}_n = u_n^{(n)}, \qquad n \in \mathbb{N},$$

converges to u in $C^{1+\beta/2,2+\beta}([T_1,T_2]\times K)$ for any compact set $K \subset \mathbb{R}^N$ and any $0 < T_1 < T_2$. Hence, letting n go to $+\infty$ in the differential equation satisfied by $\tilde{u}_n$, it follows that u satisfies the equation

$$D_t u(t,x) - \mathcal{A}u(t,x) = 0, \qquad t>0,\ x\in\mathbb{R}^N.$$

Besides, (2.2.1) follows from (2.2.2).

Step 2. To complete the proof we must show that $u \in C([0,+\infty)\times\mathbb{R}^N)$ and $u(0,x) = f(x)$. For this purpose, we take advantage of the semigroup theory. In particular, we will use the representation formula of solutions to Cauchy-Dirichlet problems in bounded domains through semigroups.

Fix $M \in \mathbb{N}$ and let ϑ be any smooth function such that

$$0 \le \vartheta \le 1, \qquad \vartheta \equiv 1 \ \text{ in } \ B(M-1), \qquad \vartheta \equiv 0 \ \text{ outside } \ B(M).$$

For any $n > M$, let $v_n = \vartheta\tilde{u}_n$. As it is easily seen, the function v_n belongs to $C([0,+\infty)\times\overline{B}(M))$ and is the solution of the Cauchy-Dirichlet problem

$$\begin{cases} D_t v_n(t,x) - \mathcal{A}v_n(t,x) = \psi_n(t,x), & t>0,\ x\in B(M),\\ v_n(t,x) = 0, & t>0,\ x\in\partial B(M),\\ v_n(0,x) = \vartheta(x)f(x), & x \in B(M),\end{cases}$$

where ψ_n is given by

$$\psi_n = -\sum_{i,j=1}^N q_{ij}(2D_i\tilde{u}_n D_j\vartheta + \tilde{u}_n D_{ij}\vartheta) - \tilde{u}_n \sum_{i=1}^N b_i D_i\vartheta.$$

For any $t > 0$ and any $x \in B(M)$ we have

$$|\psi_n(t,x)| \le K_M\Big(\exp(c_0 t)\|f\|_\infty + \sum_{i=1}^N \|D_i\tilde{u}_n(t,\cdot)\|_{L^\infty(B(M))}\Big), \qquad (2.2.4)$$

where $K_M > 0$ is such that

$$\sum_{i,j=1}^N \|q_{ij}D_{ij}\vartheta\|_{L^\infty(B(M))} + \sum_{i=1}^N \|b_i D_i\vartheta\|_{L^\infty(B(M))} \le K_M,$$
$$2\sum_{j=1}^N \|q_{ij}D_j\vartheta\|_{L^\infty(B(M))} \le K_M, \qquad i = 1,\dots,N.$$

We consider again the interior estimate of Theorem C.1.4. By (C.1.16), the function $\tilde{u}_n$ satisfies the estimate

$$|\sqrt{t}D\tilde{u}_n(t,x)| \le C\|\tilde{u}_n\|_{L^\infty(D(M+1))} \le C(\exp(c_0)\vee 1)\|f\|_\infty,$$

for any $x \in B(M)$, any $t < 1 = \mathrm{dist}\,(B(M), \partial B(M+1))$ and some positive constant C, independent of n. This yields

$$\|D_i\tilde{u}_n(t,\cdot)\|_{L^\infty(B(M))} \le t^{-1/2}C'\|f\|_\infty, \qquad t \le 1,$$

for any $i = 1,\dots,N$, where $C' = C(\exp(c_0)\vee 1)$. Then, by (2.2.4) it follows that

$$|\psi_n(t,x)| \le K'_M(1 + t^{-1/2})\|f\|_\infty, \qquad t \in (0,1],\ x \in B(M), \qquad (2.2.5)$$

for any $n > M$, where $K'_M > 0$ is a constant independent of n. Therefore, $\psi_n \in L^1(0,T,C_0(B(M)))$ and we can represent v_n by means of the variation-of-constants formula

$$v_n(t) = T_M(t)(\vartheta f) + \int_0^t T_M(t-s)\psi_n(s)ds, \quad t > 0,$$

where, as usual, $\{T_M(t)\}$ is the semigroup in $C(\overline{B}(M))$ associated with the operator $\mathcal{A}$ with homogeneous Dirichlet conditions on $\partial B(M)$. Since $v_n \equiv \tilde{u}_n$ and $\vartheta \equiv \mathbb{1}$ in $B(M-1)$, by (2.2.2) and (2.2.5) it follows

$$|\tilde{u}_n(t,x) - f(x)| \le \|T_M(t)(\vartheta f) - \vartheta f\|_\infty + K'_M\|f\|_\infty \int_0^t e^{c_0(t-s)}(1 + s^{-\frac{1}{2}})ds,$$

for any $t > 0$ and any $x \in B(M-1)$. Letting n go to $+\infty$ (and taking (C.3.4) into account) we get

$$|u(t,x) - f(x)| \leq \|T_M(t)(\vartheta f) - \vartheta f\|_\infty + K'_M \|f\|_\infty \int_0^t e^{c_0(t-s)}(1+s^{-\frac{1}{2}})ds,$$

which shows that u is continuous at $t = 0$ and $x \in B(M-1)$. Since $M \in \mathbb{N}$ is arbitrary, we have $u \in C([0,+\infty) \times \mathbb{R}^N)$ and $u(0,\cdot) \equiv f$. ■

Remark 2.2.2 We note that the maximum principle implies that, if $f \geq 0$, the sequence $\{u_n\}$ is positive and increasing. Therefore, u is positive and the whole sequence $\{u_n\}$ converges to u. In the general case where $f \in C_b(\mathbb{R}^N)$, we write $f = f^+ - f^-$, and conclude again that the whole sequence $\{u_n\}$ converges to u in $C^{1+\beta/2,2+\beta}([T_1,T_2] \times K)$ for any $\beta \in (0,\alpha)$, any compact set $K \subset \mathbb{R}^N$ and any $0 < T_1 < T_2$, applying the previous argument to the functions u_{f^+} and u_{f^-} which are the solutions to the problem (2.0.5) provided by Theorem 2.2.1, corresponding, respectively, to the data f^+ and f^-.

Remark 2.2.3 If $f \geq 0$, then the solution u given by Theorem 2.2.1 is the minimal positive solution of the problem (2.0.5). Indeed, if v is another positive solution, the maximum principle yields $v(t,x) \geq u_n(t,x)$ for any $t > 0$, any $x \in B(n)$ and any $n \in \mathbb{N}$, and, eventually, $v \geq u$.

We stress that, in general, the problem (2.0.5) is not uniquely solvable in $C_b([0,+\infty) \times \mathbb{R}^N) \cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty) \times \mathbb{R}^N)$. Indeed, in Example 3.2.5 (see also Proposition 4.1.1) it is proved that the Cauchy problem (2.0.5) associated with the one-dimensional operator $\mathcal{A}u = u'' + x^3 u'$ admits a nontrivial solution u satisfying $u(0,\cdot) = 0$.

We now recall the definition of transition function as it is given in [49]. Here $\mathcal{B}(E)$ denotes the σ-algebra of Borel sets of a topological space E.

Definition 2.2.4 *A family of Borel measures* $\{p(t,x;\cdot) : \ t \geq 0, \ x \in E\}$ *is a transition function if the function* $p(t,\cdot;B) : E \to \mathbb{R}$ *is Borel measurable for any* $t \geq 0$ *and any* $B \in \mathcal{B}(E)$, *and*

(*i*) $p(t,x;E) \leq 1$ *for any* $t \geq 0$ *and any* $x \in E$;

(*ii*) $p(0,x;E \setminus \{x\}) = 0$ *for any* $x \in E$;

(*iii*) $p(t+s,x;B) = \int_E p(s,y;B)p(t,x;dy)$ *for any* $s,t \geq 0$, *any* $x \in E$ *and any* $B \in \mathcal{B}(E)$.

A transition function is normal if $\lim_{t\to 0^+} p(t,x;E) = 1$ *for any* $x \in E$; *it is stochastically continuous if for any open set* $U \subset E$ *it holds that*

$$\lim_{t\to 0^+} p(t,x;U) = 1,$$

whenever $x \in U$.

Theorem 2.2.5 *There exists a semigroup of linear operators $\{T(t)\}$ defined in $C_b(\mathbb{R}^N)$ such that, for any $f \in C_b(\mathbb{R}^N)$, the solution of the problem (2.0.5), given by Theorem 2.2.1, is represented by*

$$u(t,x) = (T(t)f)(x), \qquad t \geq 0,\ x \in \mathbb{R}^N. \tag{2.2.6}$$

For any $t > 0$, $T(t)$ satisfies the estimate

$$\|T(t)f\|_\infty \leq \exp(c_0 t)\|f\|_\infty, \qquad f \in C_b(\mathbb{R}^N). \tag{2.2.7}$$

Moreover, there exist a family of Borel measures $p(t,x;dy)$ in $\mathbb{R}^N$ such that

$$(T(t)f)(x) = \int_{\mathbb{R}^N} f(y)p(t,x;dy), \qquad t \geq 0,\ x \in \mathbb{R}^N, \tag{2.2.8}$$

and a function $G : (0,+\infty) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ such that

$$p(t,x;dy) = G(t,x,y)dy, \qquad t > 0, \quad x,y \in \mathbb{R}^N. \tag{2.2.9}$$

The function G is strictly positive and the functions $G(t,\cdot,\cdot)$ and $G(t,x,\cdot)$ are measurable for any $t > 0$ and any $x \in \mathbb{R}^N$. Further, for almost any fixed $y \in \mathbb{R}^N$, the function $G(\cdot,\cdot,y)$ belongs to the space $C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\mathbb{R}^N)$, and it is a solution of the equation $D_t u - \mathcal{A}u = 0$.

Finally, if $c_0 \leq 0$ then $p(t,x;dy)$ is a stochastically continuous transition function.

Proof. Step 1: definition and properties of G. For any $k \in \mathbb{N}$ let $G_k \in C((0,+\infty)\times B(k)\times B(k))$ be the fundamental solution of the equation $D_t u - \mathcal{A}u = 0$ in $B(k)$, given by Proposition C.3.2. We extend the function G_k to $(0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^N$ with value zero for $x,y \notin B(k)$ and still denote by G_k the so obtained function. A straightforward computation shows that for any fixed $t \in (0,+\infty)$ and any $x,y \in \mathbb{R}^N$, the sequence $\{G_k(t,x,y)\}$ is increasing. Indeed, for any positive $f \in C(B(k))$, the function

$$w(t,x) = \int_{G(k)} f(y)(G_{k+1}(t,x,y) - G_k(t,x,y))dy, \quad t \geq 0,\ x \in \mathbb{R}^N,$$

is positive as well by virtue of Remark 2.2.2. Recalling that, for any $t > 0$ and any $x \in B(k)$, the function $G_{k+1}(t,x,\cdot)\ - G_k(t,x,\cdot)$ is continuous in $B(k)$, we easily deduce that $G_{k+1}(t,x,y) \geq G_k(t,x,y)$ for any $t > 0$ and any $x,y \in B(k)$, implying that the sequence $\{G_k(t,x,y)\}$ is increasing. Hence, we can define the function $G : (0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ by setting

$$G(t,x,y) := \lim_{k\to+\infty} G_k(t,x,y), \qquad t > 0,\ x,y \in \mathbb{R}^N.$$

The function G is finite almost everywhere. Indeed, the estimate (C.3.4) with $f = \mathbb{1}$ yields

$$\int_{B(k)} G_k(t,x,y)dy \leq \exp(c_0 t), \qquad t > 0,\ x \in B(k),\ k \in \mathbb{N},$$

and then, by monotone convergence,

$$\int_{\mathbb{R}^N} G(t,x,y)dy \le \exp(c_0 t), \qquad t>0,\ x\in\mathbb{R}^N, \tag{2.2.10}$$

so that $G(t,x,y)$ is finite for any $t>0$, any $x\in\mathbb{R}^N$ and almost any $y\in\mathbb{R}^N$. Since $G_k(t,x,\cdot)>0$ almost everywhere in $B(k)$ for any $t>0$ and any $x\in B(k)$, then G is strictly positive.

Of course, $G(t,\cdot,\cdot)$, $G(t,x,\cdot)$ and $G(t,\cdot,y)$ are measurable functions for any $t>0$ and any $x,y\in\mathbb{R}^N$ since they are the pointwise limit of measurable functions.

We now prove the regularity properties of G. Fix $R,T>0$, $x_0\in B(R)$ and let $y_0\in\mathbb{R}^N$ be such that $G(T,x_0,y_0)<+\infty$; actually we have seen that this holds for almost any $y_0\in\mathbb{R}^N$. If $h,k\in\mathbb{N}$ satisfy $R+1<h<k$, the functions $G_h(\cdot,\cdot,y_0)$ and $G_k(\cdot,\cdot,y_0)$ are solutions of the equation $D_t u-\mathcal{A}u=0$ in $(0,+\infty)\times B(R+1)$ (see Theorem C.1.4), and hence $G_k(\cdot,\cdot,y_0)-G_h(\cdot,\cdot,y_0)$ is as well. Moreover, $G_k(\cdot,\cdot,y_0)-G_h(\cdot,\cdot,y_0)$ is positive and, for any fixed $0<t_0<t_1<T$, it satisfies the following Harnack inequality (see [99]):

$$\begin{aligned}
&\sup\{G_k(t,x,y_0)-G_h(t,x,y_0),\ (t,x)\in[t_0,t_1]\times\overline{B}(R)\}\\
&\le C\inf\{G_k(T,x,y_0)-G_h(T,x,y_0),\ x\in\overline{B}(R)\}\\
&\le C(G_k(T,x_0,y_0)-G_h(T,x_0,y_0)),
\end{aligned}$$

where $C>0$ is a constant, independent of h and k. Since $G(T,x_0,y_0)\le+\infty$, then $\{G_n(\cdot,\cdot,y_0)\}$ turns out to be a Cauchy sequence in $C([t_0,t_1]\times\overline{B}(R))$. Since it converges pointwise to the function $G(t,x,y_0)$, we conclude that $G(\cdot,\cdot,y_0)\in C([t_0,t_1]\times\overline{B}(R))$. Moreover, from Theorem C.1.4 it follows that, for any $t_0'>t_0$, any $t_1'<t_1$ and any $R'<R$, the sequence $\{G_n(\cdot,\cdot,y_0)\}$ converges also in $C^{1+\beta/2,2+\beta}([t_0',t_1']\times\overline{B}(R'))$ for any $\beta\in(0,\alpha)$. Hence, $G(\cdot,\cdot,y_0)\in C^{1+\alpha/2,2+\alpha}([t_0',t_1']\times\overline{B}(R'))$. Since $T,R,R',t_0,t_0',t_1,t_1'>0$ are arbitrary, we get $G(\cdot,\cdot,y_0)\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\mathbb{R}^N)$.

Finally, since $D_tG_n-\mathcal{A}G_n=0$, as n goes to $+\infty$, it follows that $D_tG-\mathcal{A}G=0$.

Step 2: definition and properties of $p(t,x;dy)$ *and* $\{T(t)\}$. Now, for any $t>0$ and any $x\in\mathbb{R}^N$ we define the measure $p(t,x;dy)$ by (2.2.9), while for $t=0$ we set $p(t,x;dy)=\delta_x$. Then, for any $t>0$, we define the operator $T(t)$ by (2.2.8).

Let us prove that, for any $f\in C_b(\mathbb{R}^N)$, the solution u of the problem (2.0.5), found out in Theorem 2.2.1, is given by (2.2.6). Indeed,

$$u(t,x)=\lim_{k\to+\infty}\int_{\mathbb{R}^N} f(y)G_k(t,x,y)dy, \qquad t>0,\ x\in\mathbb{R}^N,$$

and we can split it as

$$u(t,x) = \lim_{k\to+\infty}\int_{\mathbb{R}^N} f^+(y)G_k(t,x,y)dy - \lim_{k\to+\infty}\int_{\mathbb{R}^N} f^-(y)G_k(t,x,y)dy.$$

By the monotone convergence theorem we immediately deduce that

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^N} f^+(y)G(t,x,y)dy - \int_{\mathbb{R}^N} f^-(y)G(t,x,y)dy \\ &= \int_{\mathbb{R}^N} f(y)G(t,x,y)dy, \end{aligned}$$

that is (2.2.6).

To show that $\{T(t)\}$ is a semigroup it suffices to observe that from (C.3.5), using the monotone convergence theorem, it follows that

$$G(t+s,x,y) = \int_{\mathbb{R}^N} G(s,x,z)G(t,z,y)dz, \qquad t,s>0,\ x,y\in\mathbb{R}^N. \tag{2.2.11}$$

Moreover, (2.2.7) is an immediate consequence of (2.2.1) and (2.2.6).

It remains to prove that, if $c_0 \le 0$, $\{p(t,x;dy) : t \ge 0,\ x \in \mathbb{R}^N\}$ is a stochastically continuous transition function. The condition (i) in Definition 2.2.4 follows immediately from (2.2.10), and the condition (ii) is obvious. The condition (iii) will follow from (2.2.11) once we have proved that the function $p(t,\cdot;B)$ is Borel measurable in $\mathbb{R}^N$, for any fixed $t \ge 0$ and any $B \in \mathcal{B}(\mathbb{R}^N)$. So, let us prove this property. If $t = 0$ the property is clear. If $t > 0$, consider a sequence of functions $\{f_n\} \subset C_b(\mathbb{R}^N)$ converging almost everywhere to $\chi_B(x)$ and such that $0 \le f_n \le 1$ for any $n \in \mathbb{N}$. For instance, if B has finite positive Lebesgue measure, one can take, up to a subsequence, $f_n = \rho_n \star \chi_B$ ($n \in \mathbb{N}$), where ρ is a standard mollifier and "$\star$" denotes convolution. If B has infinite Lebesgue measure, one can consider the sequence of bounded Borel sets $B_M = B \cap B(M)$ and a subsequence $\{f^M_{n_k}\}_{k\in\mathbb{N}}$ of the sequence $f^M_n = \rho_n \star \chi_{B_M}$ ($n \in \mathbb{N}$) pointwise converging a.e. in $\mathbb{R}^N$ to χ_{B_M} as n tends to $+\infty$, and assume that, for any $M \in \mathbb{N}$, $\{f^{M+1}_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{f^M_{n_k}\}_{k\in\mathbb{N}}$. Then, one can define the wished sequence by setting $f_n = f^n_n$. By the dominated convergence theorem we have

$$p(t,x;B) = \lim_{n\to+\infty}\int_{\mathbb{R}^N} f_n(y)p(t,x;dy) = \lim_{n\to+\infty}(T(t)f_n)(x), \qquad x\in\mathbb{R}^N.$$

Hence $p(t,\cdot;B)$ is Borel measurable, being the pointwise limit of a sequence of continuous functions.

Finally we show that $p(t,x;dy)$ is stochastically continuous; clearly this implies that it is also normal. For this purpose, for any $x_0 \in \mathbb{R}^N$ and any

$R > 0$, we introduce the function $f_{x_0,r} : \mathbb{R}^N \to \mathbb{R}$ defined by

$$f_{x_0,r}(x) = \begin{cases} 1 - r^{-1}|x - x_0|, & x \in x_0 + B(r), \\ 0, & x \notin x_0 + B(r). \end{cases}$$

As it is easily seen,

$$\begin{aligned} f_{x_0,r}(x_0) - (T(t)f_{x_0,r})(x_0) &= 1 - \int_{\mathbb{R}^N} f_{x_0,r}(y)p(t, x_0; dy) \\ &= 1 - \int_{x_0+B(r)} f_{x_0,r}(y)p(t, x_0; dy) \\ &\geq 1 - p(t, x_0; x_0 + B(r)). \end{aligned}$$

Hence, letting t go to 0^+, Theorem 2.2.1 yields

$$\lim_{t\to 0^+} p(t, x_0; x_0 + B(r)) = 1. \tag{2.2.12}$$

Now, we fix an arbitrary open set $U \subset \mathbb{R}^N$. Then, with any $x \in U$, we associate an open ball $x + B(r)$ contained in U. Since

$$1 \geq p(t, x; U) \geq p(t, x; x + B(r)),$$

from (2.2.12) we deduce that $p(t, x; U)$ tends to 1 as t tends to 0^+ and we are done. ∎

Theorem 2.2.6 *For any $\lambda > c_0$ we have*

$$K_\lambda(x, y) = \int_0^{+\infty} e^{-\lambda t} G(t, x, y)dt, \qquad x, y \in \mathbb{R}^N, \tag{2.2.13}$$

and, for any $f \in C_b(\mathbb{R}^N)$,

$$(R(\lambda)f)(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)dt, \qquad x \in \mathbb{R}^N. \tag{2.2.14}$$

Proof. The equality (2.2.13) is a consequence of (C.3.9) with $\Omega = B(n)$, (2.1.7) and the monotone convergence theorem. Then, (2.2.14) follows from (2.2.8), (2.2.13) and the Fubini theorem. ∎

In general, $\{T(t)\}$ is not a strongly continuous semigroup neither in $C_b(\mathbb{R}^N)$ nor in $BUC(\mathbb{R}^N)$ (see [148, Theorem 4.2] and Proposition 9.2.6). Nevertheless, as a straightforward consequence of Theorems 2.2.1 and 2.2.5, we deduce that, for a general $f \in C_b(\mathbb{R}^N)$, $T(t)f$ converges to f as t tends to 0, locally uniformly in $\mathbb{R}^N$. Actually, as next proposition shows, if f vanishes at infinity, then $T(t)f$ converges to f in $C_b(\mathbb{R}^N)$, as t tends to 0.

Proposition 2.2.7 ([116], Prop. 4.3) *For any function $f \in C_0(\mathbb{R}^N)$, $T(t)f$ tends to f in $C_b(\mathbb{R}^N)$, as t tends to 0^+.*

Proof. We prove the statement assuming that $f \in C_c^\infty(\mathbb{R}^N)$. The general case then will follow by density. So, let us fix $f \in C_c^\infty(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. Moreover, let $k \in \mathbb{N}$ be such that $B(k)$ contains both x and $\text{supp}(f)$. Then $(A_k f)(x) = (\mathcal{A}f)(x)$, where, as usual, A_k denotes the realization of the operator $\mathcal{A}$ in $C(\overline{B}(k))$ with homogeneous Dirichlet conditions. Let $u_k(t) = T_k(t)f$, where $\{T_k(t)\}$ is the analytic semigroup generated by A_k. For any $t > 0$ we have

$$\begin{aligned}
u_k(t,x) - f(x) &= \int_0^t \frac{\partial}{\partial s} u_k(s,x)ds \\
&= \int_0^t (A_k T_k(s)f)(x)ds \\
&= \int_0^t (T_k(s)\mathcal{A}f)(x)ds \\
&= \int_0^t ds \int_{\mathbb{R}^N} G_k(s,x,y)\mathcal{A}f(y)dy,
\end{aligned}$$

where we have extended $G_k(t,x,\cdot)$ to the whole of $\mathbb{R}^N$ by setting $G_k(t,x,y) = 0$ for any $y \notin B(k)$. Letting k go to $+\infty$ from the dominated convergence theorem it follows that

$$|(T(t)f)(x) - f(x)| = \left| \int_0^t (T(s)\mathcal{A}f)(x)ds \right| \le \|\mathcal{A}f\|_\infty \int_0^t \exp(c_0 s)ds.$$

Since $x \in \mathbb{R}^N$ is arbitrary, we conclude that

$$\|T(t)f - f\|_\infty \le \|Af\|_\infty \int_0^t \exp(c_0 s)ds,$$

which proves the proposition. ∎

Remark 2.2.8 The results of the previous proposition do not imply that the restriction of the semigroup to $C_0(\mathbb{R}^N)$ gives rise to a strongly continuous semigroup. Indeed, as it is shown in Section 5.3, in general, $\{T(t)\}$ does not map $C_0(\mathbb{R}^N)$ into itself.

Taking advantage of Theorem 2.2.5 we can prove some interesting properties of the semigroup $\{T(t)\}$.

Proposition 2.2.9 *Let $\{f_n\} \subset C_b(\mathbb{R}^N)$ be a bounded sequence of continuous functions converging pointwise to a function $f \in C_b(\mathbb{R}^N)$ as n tends to $+\infty$. Then, $T(\cdot)f_n$ tends to $T(\cdot)f$ locally uniformly in $(0, +\infty) \times \mathbb{R}^N$.*

Further, if f_n tends to f uniformly on compact subsets of $\mathbb{R}^N$, then $T(t)f_n$ converges to $T(t)f$ locally uniformly in $[0,+\infty)\times\mathbb{R}^N$ as n tends to $+\infty$.

Proof. To prove the first part of the proof, we fix $0<T_1<T_2$, $R>0$, a sequence $\{f_n\}\subset C_b(\mathbb{R}^N)$ converging pointwise to $f\in C_b(\mathbb{R}^N)$ and we prove that $T(\cdot)f_n$ converges to $T(\cdot)f$ in $[T_1,T_2]\times\overline{B}(R)$ as n tends to $+\infty$. As it is immediately seen from (2.2.8) and the dominated convergence theorem, $T(\cdot)f_n$ converges pointwise to $T(\cdot)f$ in $(0,+\infty)\times\mathbb{R}^N$ as n tends to $+\infty$.

Now let $K>0$ be such that $\sup_{n\in\mathbb{N}}\|f_n\|_\infty\le K$. Then, taking (2.2.7) into account, we easily deduce that $\sup_{n\in\mathbb{N}}\|T(t)f_n\|_\infty\le K(e^{c_0T}\vee 1)$ for any $t\in[0,T]$. The interior Schauder estimates in Theorem C.1.4 then imply that the sequence $\{T(\cdot)f_n\}$ is bounded in $C^{1+\alpha/2,2+\alpha}([T_1,T_2]\times\overline{B}(R))$. Hence, by the Ascoli-Arzelà Theorem, there exists a subsequence $\{T_{n_k}(\cdot)\}$ converging uniformly in $[T_1,T_2]\times\overline{B}(R)$ to a function $v\in C^{1+\alpha/2,2+\alpha}([T_1,T_2]\times\overline{B}(R))$. Since, $T(\cdot)f_n$ converges pointwise to $T(\cdot)f$ in $(0,+\infty)\times\mathbb{R}^N$, we deduce that $v=T(\cdot)f$ and the whole sequence $\{T(\cdot)f_n\}$ converges to $T(\cdot)f$ uniformly in $[T_1,T_2]\times\overline{B}(R)$.

Now, we suppose that the sequence $\{f_n\}\subset C_b(\mathbb{R}^N)$ converges uniformly to f on compact subsets of $\mathbb{R}^N$ and we show that, for any $R,T>0$, $T(\cdot)f_n$ tends to $T(\cdot)f$ uniformly in $[0,T]\times\overline{B}(R)$. Possibly replacing f_n with f_n-f, we can suppose that $f\equiv 0$. Moreover, without loss of generality, we can also assume that $\sup_{n\in\mathbb{N}}\|f_n\|\le 1$.

For any $n\in\mathbb{N}$, let $\varphi_n\in C_0(\mathbb{R}^N)$ be a nonnegative function such that $\chi_{B(n-1)}\le\varphi_n\le\chi_{B(n)}$. Moreover, for any $\varepsilon>0$, let $C_{\varepsilon,R}$ be the set defined by

$$C_{\varepsilon,R}=\left\{s\ge 0:\ \exists n\in\mathbb{N}\ \text{ s.t. }\inf_{(t,x)\in[0,s]\times B(R)}(T(t)(\varphi_n-\mathbb{1}))(x)\ge-\varepsilon\right\}.$$

Let us prove that $C_{\varepsilon,R}=[0,+\infty)$, for any $\varepsilon>0$. For this purpose, we will show that $C_{\varepsilon,R}$ is both an open and closed interval. Note that $C_{\varepsilon,R}$ is nonempty since it contains 0. To show that $C_{\varepsilon,R}$ is closed, we fix $s\in\overline{C_{\varepsilon,R}}$, $s\neq 0$. Then, there exists a sequence $\{s_n\}\subset C_{\varepsilon,R}$ converging to s as n tends to $+\infty$. Without loss of generality, we can assume that $\{s_n\}$ is either decreasing or increasing. Of course, if $\{s_n\}$ is decreasing, then $s\in C_{\varepsilon,R}$. So, let us consider the case when $\{s_n\}$ is increasing. Since $s_1\in C_{\varepsilon,R}$, there exists $n_1\in\mathbb{N}$ such that

$$(T(t)(\varphi_{n_1}-\mathbb{1}))(x)\ge-\varepsilon,\qquad t\in[0,s_1],\qquad x\in B(R).\tag{2.2.15}$$

Recalling that $\{\varphi_n\}$ is an increasing sequence, it turns out that (2.2.15) is satisfied by any $n\ge n_1$.

By the first part of the proof, we know that $T(\cdot)(\varphi_n-\mathbb{1})$ converges to 0 uniformly in $[s_1,s]\times B(R)$. Therefore, we can determine $n_0\in\mathbb{N}$ such that

$$(T(t)(\varphi_n-\mathbb{1}))(x)\ge-\varepsilon,\qquad t\in[s_1,s],\qquad \in B(R),\qquad n\ge n_0.$$

Now, if we take $\hat{n} = n_0 \vee n_1$, we deduce that

$$(T(t)(\varphi_{\hat{n}} - \mathbb{1}))(x) \geq -\varepsilon, \qquad t \in [0, s], \quad \in B(R).$$

Hence, $s \in C_{\varepsilon,R}$.

To show that C_ε is open in $[0, +\infty)$, we fix $s \in C_{\varepsilon,R}$ and prove that, for any $\delta > 0$, $[s, \delta] \subset C_{\varepsilon,R}$. For this purpose, it suffices to argue as above, observing that $T(\cdot)(\varphi_n - \mathbb{1})$ converges to 0, uniformly in $[s, \delta] \times B(R)$.

Now, since $p(t, x; B(m)) \geq (T(t)\varphi_m)(x)$ for any $t > 0$, any $x \in \mathbb{R}^N$ and any $m \in \mathbb{N}$, and $C_{\varepsilon,R} = [0, +\infty)$, we easily deduce that, for any arbitrarily fixed $T > 0$ and any $R > 0$, there exists $m \in \mathbb{N}$ such that

$$p(t, x; B(m)) \geq (T(t)\mathbb{1})(x) - \varepsilon = p(t, x; \mathbb{R}^N) - \varepsilon, \qquad t \in [0, T], \quad x \in B(R).$$

Therefore,

$$\begin{aligned} |(T(t)f_n)(x)| &\leq \int_{B(m)} |f_n(y)| p(t, x; dy) + \int_{\mathbb{R}^N \setminus B(m)} |f_n(y)| p(t, x; dy) dy \\ &\leq \sup_{y \in B(m)} |f_n(y)| + p(t, ; \mathbb{R}^N \setminus B(m)) \\ &\leq \sup_{y \in B(m)} |f_n(y)| + \varepsilon, \end{aligned}$$

for any $t \in [0, T]$ and any $x \in B(R)$. Now, the assertion follows. ■

Remark 2.2.10 Using the formula (2.2.8), the semigroup $\{T(t)\}$ can be extended to a semigroup (which we still denote by $\{T(t)\}$) in the space $B_b(\mathbb{R}^N)$ of all the bounded Borel measurable functions, and, for any $f \in B_b(\mathbb{R}^N)$ and any bounded sequence $\{f_n\} \in C_b(\mathbb{R}^N)$ converging pointwise to f, $(T(t)f_n)(x)$ converges to $(T(t)f)(x)$ for any $t > 0$ and any $x \in \mathbb{R}^N$. Moreover, by (2.2.9) and (2.2.10) it follows that the estimate (2.2.7) holds also for $f \in B_b(\mathbb{R}^N)$. Similarly, if $\{f_n\} \in B_b(\mathbb{R}^N)$ is a bounded sequence converging pointwise to $f \in B_b(\mathbb{R}^N)$, then $T(\cdot)f_n$ converges to $T(\cdot)f$ pointwise in $[0, +\infty) \times \mathbb{R}^N$. Actually, taking the forthcoming Proposition 2.2.12, into account, we can easily show that $T(\cdot)f_n$ converges locally uniformly in $(0, +\infty) \times \mathbb{R}^N$. Indeed, if $[a, b] \times K$ is a compact set in $(0, +\infty) \times \mathbb{R}^N$, we can split $T(t)f_n = T(t - a/2)T(a/2)f_n$ for any $t \geq a/2$. By the above result and Proposition 2.2.12, the sequence $\{T(a/2)f_n\}$ is contained in $C_b(\mathbb{R}^N)$, is bounded and it converges pointwise to $T(a/2)f$, as n tends to $+\infty$. Proposition 2.2.9 now implies that $T(\cdot)f_n$ converges to $T(\cdot)f$ in $[a, b] \times K$, as n tends to $+\infty$.

Let us now prove that $\{T(t)\}$ is irreducible and has the strong Feller property. For this purpose, we recall the following definition.

Definition 2.2.11 *A semigroup $\{S(t)\}$ in $B_b(\mathbb{R}^N)$ is irreducible if for any nonempty open set $U \subset \mathbb{R}^N$ it holds that*

$$(S(t)\chi_U)(x) > 0,$$

for any $t > 0$ and any $x \in \mathbb{R}^N$. It has the strong Feller property if $S(t)f \in C_b(\mathbb{R}^N)$ for any $f \in B_b(\mathbb{R}^N)$.

Proposition 2.2.12 *$\{T(t)\}$ is irreducible and has the strong Feller property.*

Proof. Showing that the semigroup is irreducible is easy due to the fact that G is strictly positive (see Theorem 2.2.5).

To prove that $\{T(t)\}$ is strong Feller, fix $f \in B_b(\mathbb{R}^N)$ and let $\{f_n\} \in C_b(\mathbb{R}^N)$ be a bounded sequence converging pointwise to f as n tends to $+\infty$. Applying the interior Schauder estimates in Theorem C.1.4 we deduce that for any compact set $F \subset (0, +\infty) \times \mathbb{R}^N$ there exists a positive constant $C = C(F)$ such that

$$\|T(\cdot)f_n\|_{C^{1+\alpha/2,2+\alpha}(F)} \leq C\|f_n\|_\infty, \qquad n \in \mathbb{N}.$$

Since $\sup_{n\in\mathbb{N}} \|f_n\|_\infty$ is finite, and $(T(t)f_n)(x)$ converges to $(T(t)f)(x)$ for any $t \in [0, +\infty)$ and any $x \in \mathbb{R}^N$ (this follows immediately applying the dominated convergence theorem to the formula (2.2.9)), we deduce that $T(t)f$ is continuous in $\mathbb{R}^N$ for any $t > 0$. ∎

Remark 2.2.13 The strictly positiveness of G actually implies that

$$(T(t)\chi_E)(x) > 0, \qquad t > 0,\; x \in \mathbb{R}^N,$$

for any Borel set $E \subset \mathbb{R}^N$ with positive Lebesgue measure.

2.3 The weak generator of $T(t)$

Since $\{T(t)\}$ is not strongly continuous in $C_b(\mathbb{R}^N)$ and in general is not strongly continuous in either $C_0(\mathbb{R}^N)$ or $BUC(\mathbb{R}^N)$, we cannot define the infinitesimal generator in the usual sense. Nevertheless, we can still associate a "generator" with $\{T(t)\}$, the so-called weak generator, which has properties similar to those of the infinitesimal generator.

We will provide three equivalent definitions of the weak generator. The first definition that we give was considered in [28] and in [77]: the resolvent operators $\{R(\lambda) : \lambda > c_0\}$ given by Theorem 2.1.3 satisfy the resolvent identity (2.1.5) and $R(\lambda)$ is injective in $C_b(\mathbb{R}^N)$ for any $\lambda > c_0$. Hence, by a classical result of functional analysis (see Proposition A.3.2), we can define the weak generator as the unique linear operator A_1 in $C_b(\mathbb{R}^N)$ such that

$$R(\lambda) = R(\lambda, A_1) \quad \text{and} \quad \mathrm{Im}(R(\lambda)) = D(A_1), \quad \lambda > c_0. \tag{2.3.1}$$

The second definition is based on the bounded pointwise convergence: a sequence $\{f_n\} \subset C_b(\mathbb{R}^N)$ is said to be boundedly and pointwise convergent to $f \in C_b(\mathbb{R}^N)$ if there exists a positive constant C such that $\|f_n\|_\infty \leq C$ for any $n \in \mathbb{N}$ and if $f_n(x)$ converges to $f(x)$ for any $x \in \mathbb{R}^N$ (see [53]). This notion of convergence leads to the following definition of the weak generator which was introduced in [125, 126]:

$$\begin{cases} D(A_2) = \Big\{ f \in C_b(\mathbb{R}^N) : \displaystyle\sup_{t\in(0,1)} \frac{\|T(t)f - f\|_\infty}{t} < +\infty \text{ and } \exists g \in C_b(\mathbb{R}^N) : \\ \qquad\qquad \displaystyle\lim_{t\to 0^+} \frac{(T(t)f)(x) - f(x)}{t} = g(x) \ \ \forall x \in \mathbb{R}^N \Big\}, \\ (A_2 f)(x) = \displaystyle\lim_{t\to 0^+} \frac{(T(t)f)(x) - f(x)}{t}, \qquad x \in \mathbb{R}^N, \qquad f \in D(A_2). \end{cases} \tag{2.3.2}$$

The third definition is based on the notion of mixed topology introduced in [145]. The mixed topology τ^M is the finest locally convex topology which agrees on every norm-bounded subset of $C_b(\mathbb{R}^N)$ with the topology of the uniform convergence on compact sets. Equivalently, it can be defined by the family of seminorms

$$p_{\{a_n\},\{K_n\}}(f) = \sup_{n\in\mathbb{N}} \left\{ a_n \sup_{x\in K_n} |f(x)| \right\}, \qquad f \in C_b(\mathbb{R}^N),$$

where $\{a_n\}$ is any sequence of positive numbers converging to zero and $\{K_n\}$ is any sequence of compact subsets of $\mathbb{R}^N$. Given a sequence $\{f_n\} \subset C_b(\mathbb{R}^N)$ and a function $f \in C_b(\mathbb{R}^N)$ we have

$$\tau^M\!-\lim_{n\to+\infty} f_n = f \Longleftrightarrow \begin{cases} \|f_n\|_\infty \leq C, & n \in \mathbb{N}, \\ f_n \to f & \text{locally uniformly.} \end{cases} \tag{2.3.3}$$

For results on transition semigroups and mixed topology we refer the reader to [68].

Thus, we can define the generator of the semigroup in the mixed topology, i.e., the operator $A_3 : D(A_3) \to C_b(\mathbb{R}^N)$ defined by

$$\begin{cases} D(A_3) = \Big\{ f \in C_b(\mathbb{R}^N) : \ \exists g \in C_b(\mathbb{R}^N) : \ \tau^M\!-\displaystyle\lim_{t\to 0^+} \frac{T(t)f - f}{t} = g \Big\}, \\ A_3 f = \tau^M\!-\displaystyle\lim_{t\to 0^+} \frac{T(t)f - f}{t}, \qquad f \in D(A_3). \end{cases} \tag{2.3.4}$$

Proposition 2.3.1 *The three operators A_1, A_2 and A_3 coincide.*

To prove Proposition 2.3.1, we need two preliminary lemmata.

Lemma 2.3.2 *For any $f \in C_b(\mathbb{R}^N)$, any $t > 0$, any $x \in \mathbb{R}^N$ and any $\lambda > c_0$ we have*

$$(T(t)R(\lambda)f)(x) = \int_0^{+\infty} e^{-\lambda s}(T(t+s)f)(x)ds. \tag{2.3.5}$$

Proof. We prove first that for any $\delta > 0$

$$T(t)\bigg(\int_0^\delta e^{-\lambda s}T(s)f ds\bigg)(x) = \int_0^\delta e^{-\lambda s}(T(t+s)f)(x)ds, \qquad t > 0 \;\; x \in \mathbb{R}^N. \tag{2.3.6}$$

To see it, it suffices to observe that, for any $x \in \mathbb{R}^N$, we have

$$\begin{aligned}\int_0^\delta e^{-\lambda s}(T(s)f)(x)ds &= \lim_{k\to+\infty} \frac{1}{k}\sum_{j=0}^{k-1} e^{-\lambda\delta j/k}(T(\delta j/k)f)(x)\\ &:= \lim_{k\to+\infty} \sigma_k(f)(x).\end{aligned}$$

As it is immediately seen, $(T(t)\sigma_k(f))(x)$ converges to $\int_0^\delta e^{-\lambda s}(T(s+t)f)(x)ds$ as k tends to $+\infty$. On the other hand, since $\{\sigma_k(f)\}_{k\in\mathbb{N}}$ is a bounded sequence in $C_b(\mathbb{R}^N)$, then $(T(t)\sigma_k(f))(x)$ converges to $T(t)\big(\int_0^\delta e^{-\lambda s}T(s)f\big)(x)ds$ as k tends to $+\infty$, by Proposition 2.2.9. Thus, (2.3.6) follows.

Now, (2.3.5) follows from (2.3.6) letting δ tend to $+\infty$ and using again Proposition 2.2.9. ■

Lemma 2.3.3 *For any $f \in D(A_2)$, any $t > 0$ and any $\lambda > c_0$ the functions $T(t)f$ and $R(\lambda)f$ belong to $D(A_2)$. Moreover, $A_2T(t)f = T(t)A_2f$, $A_2R(\lambda)f = R(\lambda)A_2f$.*

Proof. Fix $f \in D(A_2)$, $t > 0$ and $K > 0$ such that

$$\frac{\|T(h)f - f\|_\infty}{h} \le K, \qquad h \in (0,1). \tag{2.3.7}$$

Then, taking (2.2.7) into account, we get

$$\frac{\|T(h)T(t)f - T(t)f\|_\infty}{h} = \left\|T(t)\left(\frac{T(h)f - f}{h}\right)\right\|_\infty \le e^{c_0t}K, \qquad h \in (0,1).$$

Moreover, since $(T(h)f - f)/t$ converges to A_2f pointwise as h tends to 0^+, from (2.3.7) and Proposition 2.2.9 we deduce that $(T(h)(T(t)f - T(t)f)/h$ converges pointwise to $T(t)A_2f$ as h tends to 0^+. That is, $T(t)f \in D(A_2)$ and $A_2T(t)f = T(t)A_2f$.

Next, from (2.3.5) we deduce that

$$\frac{(T(h)R(\lambda)f)(x)-(R(\lambda)f)(x)}{h}=\int_0^{+\infty}e^{-\lambda s}\frac{(T(h+s)f)(x)-(T(s)f)(x)}{h}ds$$
$$=\int_0^{+\infty}e^{-\lambda s}T(s)\left(\frac{T(h)f-f}{h}\right)(x)ds,$$

for any $\lambda>c_0$ and any $x\in\mathbb{R}^N$. Thus, by (2.3.7) we get

$$\frac{\|T(h)R(\lambda)f-R(\lambda)f\|_\infty}{h}\le\frac{K}{\lambda-c_0},\qquad h\in(0,1).\tag{2.3.8}$$

Finally, by Proposition 2.2.9 and the dominated convergence theorem it follows that $(T(h)R(\lambda)f-R(\lambda)f)/h$ converges pointwise to $R(\lambda)A_2f$ as h tends to 0^+. This implies that $R(\lambda)f\in D(A_2)$ and $A_2R(\lambda)f=R(\lambda)A_2f$. ■

Proof of Proposition 2.3.1. Let us prove the inclusion $A_1\subset A_3$. For this purpose, fix $\lambda>c_0$ and $f\in C_b(\mathbb{R}^N)$. Taking (2.3.5) into account, we get

$$\begin{aligned}&\frac{(T(t)R(\lambda)f)(x)-(R(\lambda)f)(x)}{t}\\
&=\frac{1}{t}\left(\int_0^{+\infty}e^{-\lambda s}(T(s+t)f)(x)ds-\int_0^{+\infty}e^{-\lambda s}(T(s)f)(x)ds\right)\\
&=\frac{1}{t}\left(\int_t^{+\infty}e^{-\lambda(s-t)}(T(s)f)(x)ds-\int_0^{+\infty}e^{-\lambda s}(T(s)f)(x)ds\right)\\
&=\frac{e^{\lambda t}-1}{t}(R(\lambda)f)(x)-\frac{e^{\lambda t}}{t}\int_0^t e^{-\lambda s}(T(s)f)(x)ds,\end{aligned}\tag{2.3.9}$$

for any $t\in(0,+\infty)$ and any $x\in\mathbb{R}^N$. From (2.3.8), we deduce that the function $(t,x)\mapsto t^{-1}(T(t)R(\lambda)f-R(\lambda)f)(x)$ is bounded in $(0,1]\times\mathbb{R}^N$. Moreover, the right-hand side of (2.3.9) tends to $\lambda R(\lambda)f-f$ locally uniformly in $\mathbb{R}^N$ as t tends to 0^+. Indeed, for any $R>0$ and any $x\in\overline{B}(R)$ it holds that

$$\begin{aligned}&\left|\frac{e^{\lambda t}}{t}\int_0^t e^{-\lambda s}(T(s)f)(x)ds-f(x)\right|\\
&\le\left|\frac{e^{\lambda t}}{t}\int_0^t e^{-\lambda s}\left((T(s)f)(x)-f(x)\right)ds\right|+\frac{1}{t}\left|\int_0^t(e^{-\lambda(s-t)}-1)f(x)ds\right|\\
&\le e^{(\lambda\vee 0)t}\sup_{s\in[0,t]}\|T(s)f-f\|_{C(\overline{B}(R))}+\|f\|_\infty\frac{1}{t}\int_0^t(e^{-\lambda(s-t)}-1)ds,\end{aligned}\tag{2.3.10}$$

and the last side of (2.3.10) converges to 0 as t tends to 0^+. By the characterization (2.3.3), it follows that $(T(t)R(\lambda)f-R(\lambda)f)/t$ converges to $\lambda R(\lambda)f-f$

in the mixed topology. This implies that $R(\lambda)f \in D(A_3)$ and

$$(A_3R(\lambda)f)(x) = \lambda(R(\lambda)f)(x) - f(x) = (A_1R(\lambda)f)(x), \qquad x \in \mathbb{R}^N. \tag{2.3.11}$$

Therefore, we conclude that $A_1 \subset A_3$.

The inclusion $A_3 \subset A_2$ is clear from the characterization (2.3.3).

Thus, we conclude the proof by proving that $A_2 \subset A_1$. For this purpose, let $f \in D(A_2)$ and fix $\lambda > c_0$. From Lemma 2.3.3 and (2.3.11) we deduce that

$$f = (\lambda - A_1)R(\lambda)f = (\lambda - A_2)R(\lambda)f = R(\lambda)(\lambda f - A_2 f),$$

implying that $f \in D(A_1)$ and $A_1 f = A_2 f$. ∎

Definition 2.3.4 *The operator* $\widehat{A} := A_1 = A_2 = A_3$*, as defined in* (2.3.1), (2.3.2) *and* (2.3.4)*, is called the weak generator of* $\{T(t)\}$*.*

The weak generator fulfills the following further properties.

Proposition 2.3.5 *For any* $f \in D(\widehat{A})$ *and any fixed* $x \in \mathbb{R}^N$*, the function* $(T(\cdot)f)(x)$ *is continuously differentiable in* $[0,+\infty)$ *and*

$$\frac{d}{dt}(T(t)f)(x) = (T(t)\widehat{A}f)(x), \qquad t \geq 0. \tag{2.3.12}$$

For any sequence $\{f_n\} \subset D(\widehat{A})$ *such that* f_n *and* $\widehat{A}f_n$ *converge boundedly and pointwise to some functions* $f, g \in C_b(\mathbb{R}^N)$*, respectively, it holds that* $f \in D(\widehat{A})$ *and* $\widehat{A}f = g$*.*

Finally, $D(\widehat{A})$ *is dense in* $C_b(\mathbb{R}^N)$ *in the mixed topology.*

Proof. Fix $f \in D(\widehat{A})$ and $x \in \mathbb{R}^N$. By Lemma 2.3.3 the right derivative

$$\frac{d^+}{dt}(T(t)f)(x) := \lim_{h\to 0^+} \frac{(T(t+h)f)(x) - (T(t)f)(x)}{h}$$

exists at any $t \geq 0$ and

$$\frac{d^+}{dt}(T(t)f)(x) = (T(t)\widehat{A}f)(x).$$

Moreover, by Theorem 2.2.5 the function $t \mapsto (T(t)\widehat{A}f)(x)$ is continuous in $[0,+\infty)$. Hence, $(T(\cdot)f)(x)$ is differentiable in $[0,+\infty)$ and (2.3.12) holds.

Next, let $\{f_n\} \subset D(\widehat{A})$ be as in the statement. By the previous step, for any $x \in \mathbb{R}^N$ and any $n \in \mathbb{N}$, the function $(T(\cdot)f_n)(x)$ is differentiable in $[0,+\infty)$ and

$$\frac{d}{ds}(T(s)f_n)(x) = (T(s)\widehat{A}f_n)(x), \qquad s \geq 0.$$

Integrating such an equation with respect to $s \in [0,t]$ gives

$$\frac{(T(t)f_n)(x) - f_n(x)}{t} = \frac{1}{t}\int_0^t (T(s)\widehat{A}f_n)(x)ds, \qquad t \geq 0.$$

Letting n go to $+\infty$ and taking Proposition 2.2.9 into account, from the dominated convergence theorem we get

$$\frac{(T(t)f)(x) - f(x)}{t} = \frac{1}{t}\int_0^t (T(s)g)(x)ds.$$

From such an equality we immediately deduce that $f \in D(\widehat{A})$ and $\widehat{A}f = g$.

Finally, to prove that $D(\widehat{A})$ is dense in $C_b(\mathbb{R}^N)$ in the mixed topology it suffices to observe that any bounded and continuous function can be approximated by a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^N)$ $(n \in \mathbb{N})$, bounded in the sup-norm and converging locally uniformly to f as n tends to $+\infty$. ■

Next proposition shows the connections between $D_{\max}(\mathcal{A})$ and $D(\widehat{A})$.

Proposition 2.3.6 *We have*

$$D_{\max}(\mathcal{A}) \cap C_0(\mathbb{R}^N) \subseteq D(\widehat{A}) \subseteq D_{\max}(\mathcal{A}), \tag{2.3.13}$$

where $D_{\max}(\mathcal{A})$ is the set defined in (2.0.1).

Moreover, the following conditions are equivalent:

(i) $\lambda \in \rho(A)$ *for some* $\lambda > c_0$, *i.e., the equation* (2.0.2) *has a unique bounded solution* $u \in D_{\max}(\mathcal{A})$;

(ii) $(c_0, +\infty) \subset \rho(A)$;

(iii) $(A, D_{\max}(\mathcal{A})) = (\widehat{A}, D(\widehat{A}))$.

Proof. To prove the inclusion $D_{\max}(\mathcal{A}) \cap C_0(\mathbb{R}^N) \subseteq D(\widehat{A})$ we fix a function $u \in D_{\max}(\mathcal{A}) \cap C_0(\mathbb{R}^N)$, $\lambda > c_0$ and set $f = \lambda u - \mathcal{A}u$. As it is easily seen $f \in C_b(\mathbb{R}^N)$. For any $n \in \mathbb{N}$, let u_n be, as in the proof of Theorem 2.1.1, the solution of the elliptic problem

$$\begin{cases} \lambda u_n - \mathcal{A}u_n = f, & \text{in } B(n), \\ u_n = 0, & \text{on } \partial B(n). \end{cases}$$

The function $u_n - u$ solves the differential equation $\lambda v - \mathcal{A}v = 0$ and $u_n - u = -u$ on $\partial B(n)$. By the classical maximum principle, it follows that

$$\|u_n - u\|_{C(\overline{B}(n))} = \sup_{|x|=n} |u(x)|$$

and, consequently, u_n converges to u as n tends to $+\infty$, locally uniformly in $\mathbb{R}^N$. On the other hand, from the proof of Theorem 2.1.1 we know that u_n

converges to the function $R(\lambda)f$ as n tends to $+\infty$, locally uniformly in $\mathbb{R}^N$. It follows that $R(\lambda)f = u$ and, consequently, $u \in D(\widehat{A})$.

The inclusion $D(\widehat{A}) \subseteq D_{\max}(\mathcal{A})$ follows from Theorems 2.1.1 and 2.1.3.

Now, we prove the second part of the proposition. We limit ourselves to showing that "(i) $\Rightarrow$ (iii)" and "(iii) $\Rightarrow$ (ii)", since "(ii) $\Rightarrow$ (i)" is trivial.

"(i) $\Rightarrow$ (iii)". By Theorems 2.1.1 and 2.1.3 it follows immediately that $\widehat{A} \subset A$. Hence, we only need to prove that $A \subset \widehat{A}$. For this purpose, fix $u \in D_{\max}(\mathcal{A})$ and set $f = \lambda u - Au$ and $v = R(\lambda, \widehat{A})f$. Since $\widehat{A} \subset A$, we have $\lambda v - Av = f$. From the property (i), it follows that $u = v \in D(\widehat{A})$, and, therefore, the property (iii) follows.

"(iii) $\Rightarrow$ (ii)". We observe that the property (iii) implies that $\rho(A) = \rho(\widehat{A})$, which yields the property (ii) by virtue of Theorems 2.1.1 and 2.1.3. ∎

We will give some examples of situations in which $D_{\max}(\mathcal{A}) \neq D(\widehat{A})$, in Chapter 4, showing that, in general, the problem (2.0.5) is not uniquely solvable in $C_b(\mathbb{R}^N)$.

2.4 The Markov process

In this section we briefly consider the Markov process associated with the semigroup $\{T(t)\}$ and we show the Dynkin formula. In the whole section we assume that

$$c(x) \leq 0, \qquad x \in \mathbb{R}^N. \tag{2.4.1}$$

We introduce a few notations. Let E be a topological space and let $\mathcal{B}$ be the σ-algebra of Borel subsets of E. Moreover, let Ω be an arbitrary set, $\mathcal{F}$ be a σ-algebra on it and $\tau : \Omega \to [0, +\infty]$ be a $\mathcal{F}$-measurable function. For any $t \geq 0$, we denote by $\mathcal{F}_t$ a σ-algebra on the set $\Omega_t = \{\omega : \ t < \tau(\omega)\}$, such that $(\mathcal{F}_s)_{|\Omega_t} \subset \mathcal{F}_t \subset \mathcal{F}$ for any $0 < s < t$.

Next, we denote by $X = \{X_t : \Omega_t \to E, \ t \geq 0\}$ a family of functions defined in Ω such that, for any $\omega \in \Omega$, $X_t(\omega) \in E$ is a trajectory defined for $t \in [0, \tau(\omega))$ and such that X_t is $\mathcal{F}_t$-measurable on Ω_t. Finally, let $\{\mathbb{P}_x : \ x \in E\}$ be a family of probability measures on $(\Omega, \mathcal{F}_0)$ such that the function

$$x \mapsto p(t, x; B) = \mathbb{P}_x(X_t \in B) \tag{2.4.2}$$

is Borel measurable for any fixed $t > 0$ and $B \in \mathcal{B}$, and such that $\mathbb{P}_x(X_0 = x) = 1$.

The following definition of Markov process is taken from [49].

Definition 2.4.1 *X is a Markov process if for any $x \in E$, any $s, t \geq 0$ and* *any $B \in \mathcal{B}$ we have*

$$\mathbb{P}_x(X_{t+s} \in B|\mathcal{F}_t) = \mathbb{P}_{X_t}(X_s \in B), \qquad \mathbb{P}_x\text{-a.s.}, \tag{2.4.3}$$

and for any $\omega \in \Omega_t$ there exists $\omega' \in \Omega_t$ such that $\tau(\omega') = \tau(\omega) - t$ and $X_s(\omega') = X_{t+s}(\omega)$ for any $s \in [0, \tau(\omega'))$.

We say that X is continuous if all the trajectories are continuous. Moreover, we say that two Markov processes are equivalent if they have same transition probabilities $\{p(t, x; dy)\}$.

Definition 2.4.2 *A random variable τ' with values in $[0, +\infty]$ is a Markov time of a Markov process X if $\tau' \le \tau$ and $\{t < \tau'\} \in \mathcal{F}_t$ for any $t > 0$.*

The family of measures $\{p(t, x; dy)\}$ defined in (2.4.2) is a transition function (see Definition 2.2.4). Indeed, the conditions (i) and (ii) in Definition 2.2.9 are straightforward, while the condition (iii) follows from (2.4.3). Indeed, for any $B \in \mathcal{B}$ we have

$$\begin{aligned} p(t+s, x; B) &= \mathbb{P}_x(X_{t+s} \in B) \\ &= \mathbb{E}_x \mathbb{P}_x(X_{t+s} \in B|\mathcal{F}_t) \\ &= \mathbb{E}_x p(s, X_t; B) \\ &= \int_E p(s, y; B) p(t, x; dy). \end{aligned}$$

Conversely, given a transition function $\{p(t, x; dy)\}$, we say that a Markov process X is associated with it if (2.4.2) holds. It is known that for any normal transition function there exists an associated Markov process; see [49, Theorem 3.2]. In particular, as far as the semigroup $\{T(t)\}$ is concerned, we have the following result.

Theorem 2.4.3 *There exists a continuous Markov process X associated with the semigroup $\{T(t)\}$. We have*

$$(T(t)f)(x) = \mathbb{E}_x \chi_{t<\tau} f(X_t), \qquad t > 0, \;\; x \in \mathbb{R}^N, \tag{2.4.4}$$

and

$$(R(\lambda)f)(x) = \mathbb{E}_x \int_0^\tau e^{-\lambda s} f(X_s) ds, \qquad \lambda > 0, \;\; x \in \mathbb{R}^N, \tag{2.4.5}$$

for any $f \in B_b(\mathbb{R}^N)$.

Proof. The existence of X is proved in [49, Theorem 3.2], using the fact that the transition function of $\{T(t)\}$ is normal (see Theorem 2.2.5). See also [65, Theorem 1.6.3]. The continuity of X is proved in [10].

Then (2.4.4) is straightforward: if $f \equiv \chi_B$ is the characteristic function of a Borel set $B \subset \mathbb{R}^N$, then (2.4.4) is just (2.4.2). By linearity, (2.4.4) can be extended, first, to any simple function f and, then, to any $f \in B_b(\mathbb{R}^N)$, by approximating with simple functions.

Finally, (2.4.5) follows from the resolvent formula (2.2.14) and from (2.4.4), applying the Fubini theorem. ∎

For any set $U \subset \mathbb{R}^N$, we define the first exit time of X from U by

$$\tau_U = \inf\{t :\ X_t \notin U\}, \quad x \in U, \tag{2.4.6}$$

and we denote by X^U the process induced by X in U, that is

$$X_t^U = \begin{cases} X_t, & t < \tau_U, \\ \infty, & t \geq \tau_U, \end{cases}$$

and we recall the following result (see [10]).

Theorem 2.4.4 *Let $U \subset \mathbb{R}^N$ be a regular bounded domain. Then X^U is the Markov process associated with the semigroup $\{T^U(t)\}$.*

In the next theorem we state the Dynkin formula, in a slightly different form from the one in [49, Theorem 5.1].

Theorem 2.4.5 *Let $U \in \mathbb{R}^N$ be a regular bounded domain, let $\tau' \leq \tau_U$ be a Markov time, and let $\lambda \geq 0$. Let $u \in W^{2,p}_{\rm loc}(\mathbb{R}^N)$ for any $p \in [1, +\infty)$ be such that $\mathcal{A}u \in C(\mathbb{R}^N)$; then we have*

$$u(x) = \mathbb{E}_x e^{-\lambda\tau'} u(X_{\tau'}) + \mathbb{E}_x \int_0^{\tau'} e^{-\lambda s}(\lambda u - \mathcal{A}u)(X_s) ds, \quad x \in U. \tag{2.4.7}$$

Proof. Let $U_1 \subset \mathbb{R}^N$ be a regular bounded domain such that $\overline{U} \subset U_1$, and let $\vartheta \in C_c^\infty(U_1)$ be a function such that $\vartheta \equiv 1$ in U. Define $u' = \vartheta u$ and $f' = \lambda u' - \mathcal{A}u'$. Then we have $f' \in C_c(U_1)$ and $u' = R(\lambda, \mathcal{A}_{U_1})f'$, where $R(\lambda, \mathcal{A}_{U_1})$ is the resolvent operator of $\{T^{U_1}(t)\}$.

In [10, Theorem 1.6] the author proves that X is a strong Markov process; see also [65, Theorem 1.6.3]. Then also X^{U_1} is strong Markov. So we can apply [49, Theorem 5.1] to X^{U_1}, and we have the formula

$$u'(x) = \mathbb{E}_x e^{-\lambda\tau'} u'(X_{\tau'}) + \mathbb{E}_x \int_0^{\tau'} e^{-\lambda s} f'(X_s) ds, \qquad x \in U_1.$$

Since $\tau' \leq \tau_U$ and since $u = u'$ and $f' = \lambda u - \mathcal{A}u$ in U, the restriction of this formula to $x \in U$ gives (2.4.7). ∎

Notice that, taking $\tau' = \tau_U$ in (2.4.7), it follows that the solution of the boundary value problem

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) = f(x), & x \in U, \\ u(x) = h(x), & x \in \partial U, \end{cases}$$

with $h \in C(\partial U)$, $f \in C(\overline{U})$ and $\lambda \geq 0$, can be represented by the formula

$$u(x) = \mathbb{E}_x e^{-\lambda \tau_U} h(X_{\tau_U}) - \mathbb{E}_x \int_0^{\tau_U} e^{-\lambda s} f(X_s) ds, \quad x \in U. \tag{2.4.8}$$

2.5 The associated stochastic differential equation

In this section we consider the stochastic differential equation associated with the differential operator $\mathcal{A}$.

Let $\{W_t : t \geq 0\}$ be a N-dimensional Wiener process and let $\{\mathcal{F}_t^W : t \geq 0\}$ be the filtration generated by W_t. For any $x \in \mathbb{R}^N$, let $\sigma(x) \in \mathcal{L}(\mathbb{R}^N)$ be the unique positive definite matrix such that $Q(x) = \frac{1}{2}\sigma(x)\sigma(x)^*$. We consider the stochastic differential equation

$$\begin{cases} d\xi_t^x = b(\xi_t^x)dt + \sigma(\xi_t^x)dW_t, & t > 0, \\ \xi_0^x \equiv x, \end{cases} \tag{2.5.1}$$

where $x \in \mathbb{R}^N$ is fixed. This equation is a short writing of the integral equation

$$\xi_t^x = x + \int_0^t b(\xi_s^x)ds + \int_0^t \sigma(\xi_s^x)dW_s, \quad t > 0. \tag{2.5.2}$$

We say that ξ_t^x is a solution of the problem (2.5.1) if it is a continuous real process, defined for any $t \geq 0$, adapted to the filtration $\{\mathcal{F}_t^W\}$ and such that (2.5.2) holds almost surely. Moreover, the continuity of ξ_t^x ensures the existence of the stochastic integral in (2.5.2). Note that the continuity of ξ_t^x ensures the existence of the stochastic integral in (2.5.2).

Let us assume the following hypothesis.

Hypothesis 2.5.1 The functions b and σ are continuous and satisfy

$$\|\sigma(x) - \sigma(y)\|_2^2 + 2\langle b(x) - b(y), x - y\rangle \leq K_R|x - y|^2, \qquad x, y \in B(R), \tag{2.5.3}$$

$$\mathcal{A}(1 + |x|^2) = \|\sigma(x)\|_2^2 + 2\langle b(x), x\rangle \leq K(1 + |x|^2), \qquad x \in \mathbb{R}^N, \tag{2.5.4}$$

where $K, K_R > 0$ are constants and $\|\sigma\|_2^2 = \mathrm{Tr}\,(\sigma\sigma^*)$.

Theorem 2.5.2 *Assume Hypothesis* 2.5.1*. Then, there exists a unique* (*up to equivalence*) *solution* ξ_t^x *of the problem* (2.5.1)*, which is equivalent to the Markov process* X*.*

Proof. For the proof of the existence and uniqueness of the solution ξ_t^x of the equation (2.5.1) we refer the reader to [53, Theorems 3.7 & 3.11] or [88, Theorem V.1.1]. Then, using the Itô formula we can see that ξ_t^x is equivalent to X_t. Indeed, fix $t_0 > 0$ and define $v(t,x) = (T(t_0 - t)f)(x)$ for $t \in [0, t_0]$ and $x \in \mathbb{R}^N$. For any $n \in \mathbb{N}$, let $\tau_n(x)$ be the first exit time of ξ_t^x from the ball $B(n)$. Since ξ_t^x is continuous and defined for any $t > 0$, we have

$$\lim_{n\to+\infty} \tau_n(x) = +\infty, \qquad x \in \mathbb{R}^N, \tag{2.5.5}$$

almost surely. Since $D_t v + \mathcal{A} v = 0$ in $(0, t_0) \times \mathbb{R}^N$, the Itô formula gives

$$v(t \wedge \tau_n(x), \xi^x_{t\wedge\tau_n(x)}) = v(0,x) + \int_0^{t\wedge\tau_n(x)} \sigma(\xi_s^x) Dv(s, \xi_s^x) dW_s, \tag{2.5.6}$$

for any $t \in [0, t_0]$ and any $x \in B(n)$. If $t < t_0$, then, for any $s \in (0, t \wedge \tau_n(x))$, we have

$$|\sigma(\xi_s^x) Dv(s, \xi_s^x)| \le \sup_{B(n)} |\sigma| \sup_{[0,t]\times B(n)} |Dv| < +\infty$$

and, therefore,

$$\mathbb{E} \int_0^{t\wedge\tau_n(x)} |\sigma(\xi_s^x) Dv(s, \xi_s^x)|^2 ds < +\infty.$$

Then, the expectation of the stochastic integral in (2.5.6) is zero and we get

$$\mathbb{E}\, v(t \wedge \tau_n(x), \xi_{t\wedge\tau_n(x)}(x)) = v(0,x) = (T(t_0)f)(x),$$

for any $t < t_0$, $x \in B(n)$, $n \in \mathbb{N}$. Letting n tend to $+\infty$, from (2.5.5) and the dominated convergence theorem it follows that

$$\mathbb{E}\, v(t, \xi_t^x) = (T(t_0)f)(x), \qquad t < t_0,$$

and finally, letting t tend to t_0, we get $\mathbb{E}\, f(\xi_{t_0}^x) = (T(t_0)f)(x)$. This is equivalent to say that $\mathbb{P}(\xi_t^x \in B) = p(t,x;B)$ for any $t \ge 0$ and any $x \in \mathbb{R}^N$ and any Borel set B, where $\{p(t,x;dy)\}$ is the transition function of X. Therefore, ξ and X are equivalent. ∎

Chapter 3

One-dimensional theory

3.0 Introduction

The one-dimensional case is particular and easier. It has been studied by Feller ([57]). Here, we consider the case when the second-order differential operator $\mathcal{A}$ is defined by

$$\mathcal{A}\varphi(x) = q(x)\varphi''(x) + b(x)\varphi'(x), \qquad x \in \mathbb{R},$$

on smooth functions under the following assumptions on the coefficients q and b.

Hypothesis 3.0.1 *The coefficients q and b are continuous in $\mathbb{R}$. Moreover*

$$q(x) > 0, \qquad x \in \mathbb{R}.$$

Under such a hypothesis, the existence of a solution of the elliptic equation,

$$\lambda u - \mathcal{A}u = f, \tag{3.0.1}$$

in $D_{\max}(\mathcal{A}) = C_b(\mathbb{R}) \cap C_b^2(\mathbb{R})$ can be proved. Note that Hypothesis 3.0.1 is weaker than those in Chapter 2.

A deep analysis of the solutions to the homogeneous equation

$$\lambda u - \mathcal{A}u = 0$$

allows us to prove that, for any $f \in C_b(\mathbb{R})$, there exists at least a solution $u = R_\lambda f$ to the equation (3.0.1) which is in $C_b(\mathbb{R}) \cap C^2(\mathbb{R})$. We see that the boundedness at infinity of the solutions to the homogeneous equation is strictly connected to the integrability at infinity of some functions Q and R which depend on the coefficients q and b. This analysis will allow us to determine (integral) conditions on the coefficients which guarantee the uniqueness of the solution $u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ to the equation (3.0.1).

Then, we show that the arguments used in Chapter 2 to find out a solution $u = R(\lambda)f \in D_{\max}(\mathcal{A})$ to (3.0.1) apply also in this situation in which the coefficients are less regular. Moreover, we show that $R_\lambda f$ and $R(\lambda)f$ actually coincide.

Finally, under suitable additional hypotheses on q and b, we prove that

$$D_{\max}(\mathcal{A}) = \{u \in C_b^2(\mathbb{R}) : \ qu'', bu' \in C_b(\mathbb{R})\}.$$

3.1 The homogeneous equation

In this section, for any fixed $\lambda > 0$, we study the homogeneous equation

$$\lambda u - (qu'' + bu') = 0. \tag{3.1.1}$$

Remark 3.1.1 Let $u \in C^2(\mathbb{R})$ be a solution to the equation (3.1.1). Then, u can attain neither a positive maximum nor a negative minimum. Indeed, suppose for instance that $x_0 \in \mathbb{R}$ is a positive maximum of the function u. Then, $u'(x_0) = 0$ and $u''(x_0) \le 0$. From (3.1.1) we get $\lambda u(x_0) = q(x_0)u''(x_0) \le 0$, which leads us to a contradiction. If u has a negative minimum it suffices to apply the previous argument to the function $-u$. As a consequence, if u solves (3.1.1) and vanishes at two different points x_0 and x_1, then u vanishes identically in $\mathbb{R}$.

The following lemma is a crucial step in order to understand the behaviour of the solutions of (3.1.1) in a neighborhood of $-\infty$ and $+\infty$.

Lemma 3.1.2 *There exist a positive decreasing function $\overline{u}_1$ and a positive increasing function $\overline{u}_2$, which solve the equation* (3.1.1).

Proof. To prove the existence of a positive decreasing solution $\overline{u}_1$ to the equation (3.1.1), we introduce the set

$$\begin{aligned} B = \{b \in \mathbb{R} :\ \exists u_b \text{ solution to (3.1.1) with } u_b(0) = 1,\ u_b'(0) = b \\ \text{and } u_b(x) = 0 \text{ for some } x > 0\}. \end{aligned} \tag{3.1.2}$$

Let us prove that B is an interval. First of all, we observe that B is not empty. Indeed, denote by v_1 and v_2 two linearly independent solutions to the equation (3.1.1). Then, the more general solution to (3.1.1) is given by $v = c_1v_1 + c_2v_2$. By the last part of Remark 3.1.1 the matrix whose rows are $(v_1(0), v_2(0))$ and $(v_1(x_0), v_2(x_0))$ is invertible for any $x_0 > 0$, since $v \equiv 0$ is the unique solution to the equation (3.1.1) such that $v(0) = v(x_0) = 0$. This implies that for any $x_0 > 0$ there exists a unique solution v to (3.1.1) such that $v(0) = 1$, $v(x_0) = 0$. Hence, $a = v'(0)$ belongs to B.

To prove that B is an interval, we show that, if $b \in B$, then $(-\infty, b] \subset B$. For this purpose we observe that, if $c < b$, then $u_c < u_b$ in $(0, +\infty)$. Indeed, since $u_c'(0) < u_b'(0)$, there exists at least an interval $(0, x_1)$ in which $u_c < u_b$. Suppose that $x_1 < +\infty$. Then, $u_b(x_1) = u_c(x_1)$. This would imply that the function $v = u_b - u_c$, which solves the equation (3.1.1), should have two zeroes. Hence it should be constant in $\mathbb{R}$, which is a contradiction. Therefore, $u_c < u_b$ in $(0, +\infty)$ and, consequently, u_c vanishes at some point $x \in (0, +\infty)$, so that $c \in B$.

We now show that, if $b \in B$, then u_b is decreasing in $\mathbb{R}$. As a consequence we deduce that $B \subset (-\infty, 0]$. Let $b \in B$ and let $x > 0$ be the unique zero of the function u_b. Then, u_b is decreasing in $(-\infty, x)$, otherwise it should have a positive maximum. Similarly, u_b is decreasing in $[x, +\infty)$. To check it, we observe that $u_b'(x) < 0$. Indeed, if $u_b'(x) = 0$, then u should coincide with the null solution. Therefore, u_b is strictly decreasing in a neighborhood of x. If u_b were not decreasing in $(x, +\infty)$ it should have a negative minimum, which is a contradiction.

We now set $\overline{b} = \sup B$, and $\overline{u}_1 = u_{\overline{b}}$. Observe that $\overline{u}_1$ is decreasing in $\mathbb{R}$. Indeed, u_b converges to $\overline{u}_1$ in $C^2([-M, M])$ for any $M > 0$, as b tends to $\overline{b}$ from the left. Since $u_b' \le 0$ for any $b < \overline{b}$, it follows that $\overline{u}_1' \le 0$ as well. Let us now prove that $\overline{u}_1$ is positive in $\mathbb{R}$. This is equivalent to proving that $\overline{b} \notin B$. By contradiction, suppose that $\overline{b} \in B$. Let $x > 0$ be a positive zero of $\overline{u}_1$ and let u be the solution to (3.1.1) such that $u(0) = 1$ and $u(2x) = 0$. Arguing as above, we can show that $u \ge \overline{u}_1$. Hence, $u'(0) > \overline{b}$, and this, of course, leads us to a contradiction.

Finally, to prove the existence of a positive increasing solution to (3.1.1) it suffices to set $\overline{u}_2(x) = v(-x)$ for any $x \in \mathbb{R}$, where v is the positive solution to the equation $\lambda v(x) - q(-x)v''(x) + b(-x)v'(x) = 0$, provided by the previous arguments. ■

The following proposition describes the behaviour of the solutions to the equation (3.1.1). For this purpose, we introduce the functions

$$W(x) = \exp\left(-\int_0^x \frac{b(s)}{q(s)}ds\right), \qquad x \in \mathbb{R}, \tag{3.1.3}$$

$$Q(x) = \frac{1}{q(x)W(x)}\int_0^x W(s)ds, \qquad x \in \mathbb{R}, \tag{3.1.4}$$

$$R(x) = W(x)\int_0^x \frac{1}{q(s)W(s)}ds, \qquad x \in \mathbb{R}. \tag{3.1.5}$$

Moreover, we observe that $u \in C^2(\mathbb{R})$ is a solution to the equation (3.1.1) if and only if u solves the differential equation

$$\left(\frac{u'}{W}\right)' = \lambda\frac{u}{qW}. \tag{3.1.6}$$

Therefore, any solution u of (3.1.1) satisfies

$$u'(x) = W(x)\left(u'(0) + \lambda\int_0^x \frac{u(s)}{q(s)W(s)}ds\right), \qquad x \in \mathbb{R}. \tag{3.1.7}$$

Remark 3.1.3 Notice that

$$\overline{u}_1(x)\overline{u}_2'(x) - \overline{u}_1'(x)\overline{u}_2(x) = w_0 W(x), \qquad x \in \mathbb{R}, \tag{3.1.8}$$

for some positive constant w_0. The formula (3.1.8) is immediately checked since the function $W^{-1}(\overline{u}_1\overline{u}_2' - \overline{u}_1'\overline{u}_2)$ is positive (by virtue of Lemma 3.1.2) and its first-order derivative identically vanishes in $\mathbb{R}$.

Proposition 3.1.4 *The following properties are met:*

(i) *all the solutions to* (3.1.1) *admit finite limit at* $+\infty$ *if and only if the function* R *belongs to* $L^1(0,+\infty)$*;*

(ii) *if* $Q \in L^1(0,+\infty)$ *and* $R \notin L^1(0,+\infty)$ *then any positive decreasing solution to* (3.1.1) *satisfies*

$$\lim_{x\to+\infty} \frac{u'(x)}{W(x)} = 0;$$

(iii) *if* Q, R *belong to* $L^1(0,+\infty)$ *then, for any solution* u *to* (3.1.1), *the limits* $\lim_{x\to+\infty} u(x)$ *and* $\lim_{x\to+\infty} u'(x)/W(x)$ *exist and are finite. Moreover, there exist two decreasing solutions* u *and* v *of* (3.1.1) *such that*

$$\begin{aligned} &(i) \quad \lim_{x\to+\infty} u(x) = 0, \qquad \lim_{x\to+\infty} \frac{u'(x)}{W(x)} = -1, \\ &(ii) \quad \lim_{x\to+\infty} v(x) = 1, \qquad \lim_{x\to+\infty} \frac{v'(x)}{W(x)} = 0; \end{aligned} \tag{3.1.9}$$

(iv) *the equation* (3.1.1) *admits a decreasing solution with* $\lim_{x\to+\infty} u(x) > 0$ *if and only if* $Q \in L^1(0,+\infty)$.

Proof. (i). Since any solution to (3.1.1) is given by a linear combination of the functions $\overline{u}_1$ and $\overline{u}_2$, and $\overline{u}_1$ is decreasing (see Lemma 3.1.2), it suffices to show that $\lim_{x\to+\infty} \overline{u}_2(x) \in \mathbb{R}$ if and only if $R \in L^1(0,+\infty)$. For this purpose, we observe that, since $\overline{u}_2$ is increasing and $\overline{u}_2(0) = 1$, then

$$R(x) \le W(x) \int_0^x \frac{\overline{u}_2(s)}{q(s)W(s)} ds \le \overline{u}_2(x)R(x), \qquad x > 0. \tag{3.1.10}$$

Moreover, by (3.1.7) we can write

$$\overline{u}_2'(x) = W(x) \left(\overline{u}_2'(0) + \lambda \int_0^x \frac{\overline{u}_2(s)}{q(s)W(s)} ds \right), \qquad x > 0. \tag{3.1.11}$$

Suppose that $\overline{u}_2$ is bounded in a neighborhood of $+\infty$. Then, the two terms in the right-hand side of (3.1.11) are in $L^1(0,+\infty)$, since they are both positive. Therefore, from (3.1.10) it follows that $R \in L^1(0,+\infty)$.

Conversely, suppose that $R \in L^1(0,+\infty)$. Then, plugging (3.1.10) into (3.1.11) we see that $\overline{u}_2$ satisfies the differential inequality

$$\overline{u}_2'(x) \le \overline{u}_2'(0)W(x) + \lambda R(x)\overline{u}_2(x), \qquad x > 0.$$

Therefore, the Gronwall Lemma yields

$$\overline{u}_2(x) \le \exp\left(\lambda \int_0^x R(t)dt\right) \left[1 + \overline{u}_2'(0) \int_0^x W(t) \exp\left(-\lambda \int_0^t R(s)ds\right) dt\right], \tag{3.1.12}$$

for any $x > 0$. Since R is integrable in $(0,+\infty)$ and

$$W(x) \le \left(\int_0^1 \frac{1}{q(s)W(s)}ds\right)^{-1} R(x), \qquad x \ge 1,$$

we easily deduce that $W \in L^1(0,+\infty)$. Therefore, from (3.1.12) it easily follows that $\overline{u}_2$ is bounded in $(0,+\infty)$.

(*ii*). Let u be a positive decreasing solution to (3.1.1). Then, the equation (3.1.6) implies that the function u'/W is negative and increasing in $\mathbb{R}$. Therefore, there exists $k \le 0$ such that

$$k = \lim_{x\to+\infty} \frac{u'(x)}{W(x)}.$$

Let us prove that $k = 0$. For this purpose, we observe that, integrating (3.1.6) from x to c and, then, letting c go to $+\infty$, gives

$$u'(x) = W(x)\left(k - \lambda\int_x^{+\infty} \frac{u(s)}{q(s)W(s)}ds\right), \qquad x \in \mathbb{R}. \tag{3.1.13}$$

Since $Q \in L^1(0,+\infty)$, the Fubini theorem implies that the function $x \mapsto W(x)\int_x^{+\infty}(q(s)W(s))^{-1}ds$ is integrable in $(0,+\infty)$. The boundedness of u in $(0,+\infty)$ yields the integrability of the function $x \mapsto \int_x^{+\infty} u(s)/(q(s)W(s))ds$ in $(0,+\infty)$. Therefore, from (3.1.13) it follows that, if $k \ne 0$, the function W is integrable in $(0,+\infty)$. Since

$$\frac{1}{q(x)W(x)} \le \left(\int_0^1 W(s)ds\right)^{-1} Q(x), \qquad x \ge 1, \tag{3.1.14}$$

then the function $1/(qW)$ is integrable in $(0,+\infty)$. But this implies that R is integrable in $(0,+\infty)$ as well. Hence, we get a contradiction.

(*iii*). Let us prove that, if $Q, R \in L^1(0,+\infty)$, then any solution to (3.1.1) is such that u and u'/W admit finite limits at $+\infty$. By the property (i) we can limit ourselves to showing that the limit $\lim_{x\to+\infty} u'/W$ is finite for any solution u of (3.1.1). For this purpose we observe that, from (3.1.14) and the boundedness of u at $+\infty$, it follows that $u/(qW)$ is integrable in $(0,+\infty)$. Therefore, dividing both the sides of (3.1.7) by $W(x)$ and letting x go to $+\infty$, we obtain that the limit $\lim_{x\to+\infty} u'(x)/W(x)$ is finite.

Let us now set $\overline{u} = \overline{u}_1 - c\overline{u}_2$ where c is a constant such that $\lim_{x\to+\infty}\overline{u}(x) = 0$. Let us observe that $c \in [0,1)$. Of course, $c > 0$ since $\overline{u}_j$ $(j = 1,2)$ is positive in $\mathbb{R}$. Moreover, $\overline{u}_2 \ge \overline{u}_1$ in $[0,+\infty)$ and $\overline{u}_1(0) = \overline{u}_2(0) = 1$. If $c > 1$, then $\tilde{u}$ should have a negative minimum and if $c = 1$ it should have a positive maximum or a negative minimum. Of course, these are all contradictions by Remark 3.1.1. Therefore, $c \in [0,1)$. It follows that $\overline{u}(0) > 0$ and, since $\lim_{x\to+\infty}\overline{u}(x) = 0$, Remark 3.1.1 implies that $\overline{u}$ is decreasing in $\mathbb{R}$.

Arguing as in the proof of the property (ii) we see that $\overline{u}'(x)/W(x)$ tends to a nonpositive limit k as x tends to $+\infty$. If $k = 0$, from (3.1.13) we would get

$$\frac{\overline{u}'(x)}{\overline{u}(x)} = -\lambda\frac{W(x)}{\overline{u}(x)}\int_x^{+\infty}\frac{\overline{u}(s)}{q(s)W(s)}ds \geq -\lambda W(x)\int_x^{+\infty}\frac{1}{q(s)W(s)}ds, \tag{3.1.15}$$

for any $x \in \mathbb{R}$. As it has been shown in the proof of the property (ii) the last side of (3.1.15) is integrable in $(0, +\infty)$ since $Q \in L^1(0, +\infty)$. It follows that the function $\log(\overline{u})$ is bounded from below in $(0, +\infty)$ and, consequently, $\lim_{x\to+\infty}\overline{u}(x) = l > 0$. Since this is a contradiction, k should be strictly negative. Therefore, setting $u = -\tilde{u}/k$, we obtain a solution to (3.1.1) satisfying (3.1.9)(i).

Now, let us show that the problem (3.1.1) admits a positive decreasing solution v satisfying (3.1.9)(ii). For this purpose, let w be the solution to (3.1.1) satisfying $w(0) = 0$ and $w'(0) = 1$. According to Remark 3.1.1, w is increasing in $\mathbb{R}$. Therefore, from (3.1.6) it follows that w'/W is increasing as well and, consequently, $\lim_{x\to+\infty} w'(x)/W(x) \in (1/W(0), +\infty)$.

Let us set $\overline{v} = u + dw$ where u satisfies (3.1.9)(i) and d is a positive constant such that $\lim_{x\to+\infty}\overline{v}'(x)/W(x) = 0$. Since $\overline{v}$ is nonnegative, by (3.1.6) we now easily see that $\overline{v}$ is decreasing in $\mathbb{R}$. Moreover, since $\lim_{x\to+\infty}\overline{v}'(x)/W(x) = 0$, the previous arguments show that $l := \lim_{x\to+\infty}\overline{v}(x)$ is strictly greater than 0. Thus, the function $v = \overline{v}/l$ is a solution to (3.1.1) satisfying (3.1.9)(ii).

(iv). Let u be a decreasing solution such that $\lim_{x\to+\infty} u(x) = l > 0$. Then, arguing as in the proof of the property (ii), we can show that there exists $k \leq 0$ such that

$$k = \lim_{x\to+\infty}\frac{u'(x)}{W(x)}$$

and u' is given by (3.1.13). It follows that the function $u/(qW)$ is integrable in $(0, +\infty)$. Since all the terms in the right-hand side of (3.1.13) are nonpositive, and the left-hand side is integrable in $(0, +\infty)$, then the function $x \mapsto W(x)\int_x^{+\infty} u(s)/(q(s)W(s))ds$ is integrable in $(0, +\infty)$ as well. Since u is positive and $u \geq l$ in $(0, +\infty)$, it follows that also the function $x \mapsto W(x)\int_x^{+\infty}(q(s)W(s))^{-1}ds$ is integrable in $(0, +\infty)$. Using the Fubini Theorem, we see that this implies that $Q \in L^1(0, +\infty)$.

Conversely, let us suppose that $Q \in L^1(0, +\infty)$ and let us prove that the equation (3.1.1) admits a positive decreasing solution u with $\lim_{x\to+\infty} u(x) = l > 0$. If $R \in L^1(0, +\infty)$, the property (iii) gives us the wished function u. If $R \notin L^1(0, +\infty)$, then any positive and decreasing solution u to (3.1.1) is given by (3.1.13) and $k = 0$ by the property (ii). Hence u'/u satisfies (3.1.15) and, consequently, arguing as in the proof of the property (iii), we see that $\lim_{x\to+\infty} u(x) = l > 0$. ■

Concerning the behaviour of the solution to (3.1.1) in a neighborhood of $-\infty$, we have the following result.

Proposition 3.1.5 *The following properties are met:*

(*i*) *all the solutions to* (3.1.1) *admit finite limit at* $-\infty$ *if and only if the function* R *belongs to* $L^1(-\infty, 0)$*;*

(*ii*) *if* $Q \in L^1(-\infty, 0)$ *and* $R \notin L^1(-\infty, 0)$ *then any positive increasing solution to* (3.1.1) *satisfies*

$$\lim_{x \to -\infty} \frac{u'(x)}{W(x)} = 0;$$

(*iii*) *if* Q*,* R *belong to* $L^1(-\infty, 0)$ *then any solution to* (3.1.1) *is such that the limits* $\lim_{x \to -\infty} u(x)$ *and* $\lim_{x \to -\infty} u'(x)/W(x)$ *exist and are finite. Moreover, there exist two increasing solutions* u *and* v *of* (3.1.1) *such that*

$$(i) \quad \lim_{x \to -\infty} u(x) = 0, \qquad \lim_{x \to -\infty} \frac{u'(x)}{W(x)} = 1,$$
$$(ii) \quad \lim_{x \to -\infty} v(x) = 1, \qquad \lim_{x \to -\infty} \frac{v'(x)}{W(x)} = 0;$$

(iv) *the equation* (3.1.1) *admits an increasing solution with* $\lim_{x \to -\infty} u(x) > 0$ *if and only if* $Q \in L^1(-\infty, 0)$.

Proof. Let $\tilde{\mathcal{A}}$ be the second order differential operator defined by $\tilde{\mathcal{A}}u = \tilde{q}u'' + \tilde{b}u'$ where $\tilde{q}(x) = q(-x)$, and $\tilde{b}(x) = -b(-x)$ for any $x \in \mathbb{R}$. Let $\tilde{W}$, $\tilde{Q}$ and $\tilde{R}$ be defined according to (3.1.3)-(3.1.5) with q, b being replaced with $\tilde{q}$ and $\tilde{b}$. As it is easily seen, $\tilde{W}(x) = W(-x)$, $\tilde{Q}(x) = -Q(-x)$, $\tilde{R}(x) = -R(-x)$. Therefore, $\tilde{W}$, $\tilde{Q}$ and $\tilde{R}$ are integrable in a neighborhood of $-\infty$ if and only if W, Q and R are integrable in a neighborhood of $+\infty$. Moreover, u solves $\lambda u - \mathcal{A}u = 0$ if and only if the function $x \mapsto v(x) := u(-x)$ solves the differential equation $\lambda v - \tilde{\mathcal{A}}v = 0$. Now the assertion follows from Proposition 3.1.4. ∎

We now give the following definitions.

Definition 3.1.6 *The point* $+\infty$ *is said*

$$\textit{accessible if} \quad \begin{cases} \textit{regular, i.e.,} & Q \in L^1(0, +\infty), \ R \in L^1(0, +\infty), \\ \textit{exit, i.e.,} & Q \notin L^1(0, +\infty), \ R \in L^1(0, +\infty), \end{cases}$$

$$\textit{unaccessible if} \quad \begin{cases} \textit{entrance, i.e.,} & Q \in L^1(0, +\infty), \ R \notin L^1(0, +\infty), \\ \textit{natural, i.e.,} & Q \notin L^1(0, +\infty), \ R \notin L^1(0, +\infty). \end{cases}$$

Similarly, the point $-\infty$ is said

$$\textit{accessible if}\quad \begin{cases} \textit{regular, i.e.,}\ \ Q\in L^1(-\infty,0),\ \ R\in L^1(-\infty,0), \\ \textit{exit, i.e.,}\qquad Q\notin L^1(-\infty,0),\ \ R\in L^1(-\infty,0), \end{cases}$$

$$\textit{unaccessible if}\quad \begin{cases} \textit{entrance, i.e.,}\ \ Q\in L^1(-\infty,0),\ \ R\notin L^1(-\infty,0), \\ \textit{natural, i.e.,}\quad Q\notin L^1(-\infty,0),\ \ R\notin L^1(-\infty,0). \end{cases}$$

The results in Propositions 3.1.4 and 3.1.5 can be rephrased as follows.

Proposition 3.1.7 *The following properties are met:*

(*i*) $+\infty$ (*resp.* $-\infty$) *is regular if and only if the differential equation* (3.1.1) *admits two positive decreasing* (*resp. increasing*) *solutions* u_1 *and* u_2 *such that*

$$\lim_{x\to+\infty} u_j(x)=j-1,\qquad \lim_{x\to+\infty}\frac{u_j'(x)}{W(x)}=-2+j,\qquad j=1,2,$$

$$\left(\textit{resp.}\ \lim_{x\to-\infty} u_j(x)=j-1,\qquad \lim_{x\to-\infty}\frac{u_j'(x)}{W(x)}=2-j,\qquad j=1,2\right).$$

In this case all the solutions to (3.1.1) *are bounded in* $(0,+\infty)$ (*resp. in* $(-\infty,0)$);

(*ii*) $+\infty$ (*resp.* $-\infty$) *is an exit if and only if all the solutions of* (3.1.1) *are bounded in* $(0,+\infty)$ (*resp. in* $(-\infty,0)$) *and any positive decreasing* (*resp. increasing*) *solution* u *vanishes at* $+\infty$ (*resp. at* $-\infty$);

(*iii*) $+\infty$ (*resp.* $-\infty$) *is an entrance if and only if the differential equation* (3.1.1) *admits a positive decreasing* (*resp. increasing*) *solution* u *such that*

$$\lim_{x\to+\infty} u(x)=1,\qquad \lim_{x\to+\infty}\frac{u'(x)}{W(x)}=0,$$

$$\left(\textit{resp.}\ \lim_{x\to-\infty} u(x)=1,\qquad \lim_{x\to-\infty}\frac{u'(x)}{W(x)}=0\right),$$

and any other solution of (3.1.1), *which is independent of* u, *is unbounded in* $(0,+\infty)$ (*resp. in* $(-\infty,0)$);

(*iv*) $+\infty$ (*resp.* $-\infty$) *is natural if and only the differential equation* (3.1.1) *admits a positive decreasing* (*resp. increasing*) *solution* u *such that*

$$\lim_{x\to+\infty} u(x)=0,\qquad \lim_{x\to+\infty}\frac{u'(x)}{W(x)}=0,$$

$$\left(\lim_{x\to-\infty} u(x)=0, \qquad \lim_{x\to-\infty}\frac{u'(x)}{W(x)}=0\right),$$

and any other solution of (3.1.1)*, which is independent of* u*, is unbounded in* $(0,+\infty)$ (*resp. in* $(-\infty,0)$).

Proof. We just prove the property (iv) since the other properties follow easily from Propositions 3.1.4 and 3.1.5. Moreover, we limit ourselves to dealing with the case when the point $+\infty$ is natural, since the case when $-\infty$ is natural can be then deduced from this one using Proposition 3.1.5 instead of Proposition 3.1.4.

As a first step we observe that, according to Proposition 3.1.4(iv), $\overline{u}_1$ vanishes at $+\infty$.

Let us now prove that $\overline{u}_1'/W$ vanishes at $+\infty$ as well. For this purpose, we begin by observing that, from (3.1.6) written with $\overline{u}_1$ instead of u, we easily deduce that $\overline{u}_1'/W$ admits finite (and nonpositive) limit as x tends to $+\infty$ since it is negative and increasing. By contradiction, we assume that the previous limit is negative and we denote it by k. Since any solution of (3.1.1) is a linear combination of $\overline{u}_1$ and $\overline{u}_2$ and, according to Proposition 3.1.4(i), the equation (3.1.1) admits solutions which are unbounded in a neighborhood of $+\infty$, it follows that $\overline{u}_2(x)$ tends to $+\infty$ as x tends to $+\infty$. Taking Remark 3.1.3 into account, we can write

$$1=\frac{W(x)}{W(x)}=\frac{1}{w_0}\left(\frac{\overline{u}_2'(x)}{W(x)}\overline{u}_1(x)-\frac{\overline{u}_1'(x)}{W(x)}\overline{u}_2(x)\right), \qquad x>0.$$

Recalling that $\overline{u}_2'$ and $\overline{u}_1$ are positive in $\mathbb{R}$ and taking the limit as x tends to $+\infty$, we are led to a contradiction.

Conversely, let us assume that there exists a positive decreasing solution $\widetilde{u}$ to the problem (3.1.1) vanishing at $+\infty$ together with the function $\widetilde{u}'/W$ and that any other solution to (3.1.1), independent of $\widetilde{u}$, is unbounded at $+\infty$. According to Proposition 3.1.4(i), it is clear that $R\notin L^1(0,+\infty)$. To show that $Q\notin L^1(0,+\infty)$, we observe that, denoting by v a solution to (3.1.1) linearly independent of $\widetilde{u}$, then any solution u to the problem (3.1.1) is given by $u=c_1\widetilde{u}+c_2v$, for some $c_1,c_2\in\mathbb{R}$. Therefore, u is bounded at $+\infty$ if and only if $c_2=0$. But in such a case, u vanishes at $+\infty$. Therefore, Proposition 3.1.4(iv) implies that $Q\notin L^1(0,+\infty)$ and we are done. ∎

3.2 The nonhomogeneous equation

In this section we study the solutions $u\in C^2(\mathbb{R})$ of the nonhomogeneous equation

$$\lambda u-\mathcal{A}u=f, \tag{3.2.1}$$

when $f \in C_b(\mathbb{R})$.

By the classical theory of ordinary differential equations, it is easy to check that the more general solution to the differential equation (3.2.1) is given by

$$u(x) = -\int_0^x \langle \mathcal{W}(x)\mathcal{W}^{-1}(t)e_2, e_1\rangle \frac{f(t)}{q(t)}dt + c_1\overline{u}_1(x) + c_2\overline{u}_2(x), \qquad x \in \mathbb{R}, \tag{3.2.2}$$

where c_1, c_2 are arbitrary real constants, $e_1 = (1, 0)$, $e_2 = (0, 1)$, $\overline{u}_1$, $\overline{u}_2$ are as in Lemma 3.1.2 and $\mathcal{W}$ is the wronskian matrix

$$\mathcal{W}(x) = \begin{pmatrix} \overline{u}_1(x) & \overline{u}_2(x) \\ \overline{u}_1'(x) & \overline{u}_2'(x) \end{pmatrix}, \qquad x \in \mathbb{R}.$$

Taking (3.1.8) into account, from (3.2.2) we can write

$$\begin{aligned} u(x) = &\left(c_1 + \frac{1}{w_0}\int_0^x \frac{f(t)}{q(t)W(t)}\overline{u}_2(t)dt\right)\overline{u}_1(x) \\ &+ \left(c_2 - \frac{1}{w_0}\int_0^x \frac{f(t)}{q(t)W(t)}\overline{u}_1(t)dt\right)\overline{u}_2(x), \end{aligned} \tag{3.2.3}$$

for any $x \in \mathbb{R}$.

We can now prove the following result.

Proposition 3.2.1 *For any $f \in C_b(\mathbb{R})$ and any $\lambda > 0$, the function $u = R_\lambda f$ defined by*

$$(R_\lambda f)(x) = \int_{-\infty}^{+\infty} G_\lambda(x, s)f(s)ds, \qquad x \in \mathbb{R}, \tag{3.2.4}$$

where $G_\lambda : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$G_\lambda(x, s) = \begin{cases} \dfrac{\overline{u}_1(s)\overline{u}_2(x)}{w_0 q(s)W(s)}, & x < s, \\[2ex] \dfrac{\overline{u}_1(x)\overline{u}_2(s)}{w_0 q(s)W(s)}, & x \geq s, \end{cases}$$

belongs to $C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ and solves the differential equation (3.2.1)*. Moreover, the operator $R_\lambda : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ is bounded and $\|R_\lambda\|_{L(C_b(\mathbb{R}^N))} \leq 1/\lambda$.*

Proof. Observe that a formal computation shows that

$$(R_\lambda f)(x) = \frac{\overline{u}_1(x)}{w_0}\int_{-\infty}^x \frac{\overline{u}_2(s)}{q(s)W(s)}f(s)ds + \frac{\overline{u}_2(x)}{w_0}\int_x^{+\infty} \frac{\overline{u}_1(s)}{q(s)W(s)}f(s)ds, \tag{3.2.5}$$

for any $x \in \mathbb{R}$. So, let us prove that the integral terms in the right-hand side of (3.2.5) are well defined for any $x \in \mathbb{R}$.

Since the function $W^{-1}\overline{u}_2'$ is positive and increasing in $\mathbb{R}$ (as it easily follows from (3.1.6)), then, for any $c < x$, we have

$$\begin{aligned}\int_c^x \frac{\overline{u}_2(s)}{q(s)W(s)}|f(s)|ds &\le \|f\|_\infty \int_c^x \frac{\overline{u}_2(s)}{q(s)W(s)}ds \\ &= \frac{1}{\lambda}\|f\|_\infty \int_c^x \left(\frac{\overline{u}_2'}{W}\right)'(s)ds \\ &= \frac{1}{\lambda}\|f\|_\infty \left(\frac{\overline{u}_2'(x)}{W(x)} - \frac{\overline{u}_2'(c)}{W(c)}\right) \\ &\le \frac{1}{\lambda}\|f\|_\infty \left(\frac{\overline{u}_2'(x)}{W(x)} - \lim_{c\to-\infty}\frac{\overline{u}_2'(c)}{W(c)}\right) \\ &\le \frac{1}{\lambda}\|f\|_\infty \frac{\overline{u}_2'(x)}{W(x)}. \end{aligned} \tag{3.2.6}$$

Hence, the first integral in the right-hand side of (3.2.5) is well defined. A similar argument shows that, for any $c > x$,

$$\int_x^c \frac{\overline{u}_1(s)}{q(s)W(s)}|f(s)|ds \le -\frac{1}{\lambda}\|f\|_\infty \frac{\overline{u}_1'(x)}{W(x)}. \tag{3.2.7}$$

Hence, also the other integral is well defined. Of course, they both define continuous functions in $\mathbb{R}$. Therefore, $R_\lambda f$ is a continuous function in $\mathbb{R}$ and it is bounded since (3.2.6) gives

$$|(R_\lambda f)(x)| \le \|f\|_\infty \left(\frac{\overline{u}_1(x)}{w_0\lambda}\frac{\overline{u}_2'(x)}{W(x)} - \frac{\overline{u}_1'(x)}{w_0\lambda}\frac{\overline{u}_2(x)}{W(x)}\right) = \frac{1}{\lambda}\|f\|_\infty,$$

for any $x \in \mathbb{R}$. Finally, a straightforward computation shows that $R_\lambda f$ solves the differential equation (3.2.1). This finishes the proof. ∎

Thanks to Propositions 3.1.7 and 3.2.1, it is now easy to solve the problem of the uniqueness of the solution $u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ to the equation (3.2.1).

Theorem 3.2.2 *The elliptic equation* (3.2.1) *is uniquely solvable in* $C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ *for any* $\lambda > 0$ *and any* $f \in C_b(\mathbb{R})$ *if and only if* $-\infty$ *and* $+\infty$ *are unaccessible. If both* $-\infty$ *and* $+\infty$ *are accessible, then any solution* $u \in C^2(\mathbb{R})$ *to* (3.2.1) *is bounded in* $\mathbb{R}$.

In Chapter 2 we have seen that if the coefficients q and b belong to $C^\alpha_{\rm loc}(\mathbb{R})$, then a solution $u \in C^{2+\alpha}_{\rm loc}(\mathbb{R})$ can be obtained by approximating the equation (3.1.1) with the Dirichlet problems

$$\begin{cases} \lambda u(x) - q(x)u''(x) - b(x)u'(x) = f(x), & x \in (-n,n), \\ u(-n) = u(n) = 0, \end{cases} \tag{3.2.8}$$

and, then, letting n go to 0. Now two questions arise naturally. Does this approximation argument work also in the case when the coefficients are merely continuous in $\mathbb{R}$ and, if this is the case, what are the relations between the solution provided by this method and the solution given by (3.2.4)? The answers to the previous two questions are contained in the following proposition.

Proposition 3.2.3 *For any $f \in C_b(\mathbb{R})$, let u_n be the solution to the problem (3.2.8). Then, u_n converges to the function $R_\lambda f$ (see (3.2.4)) in $C^2([-a,a])$ for any $a > 0$, as n tends to $+\infty$.*

Proof. Throughout the proof, $\overline{u}_1$ and $\overline{u}_2$ will denote the functions in Lemma 3.1.2. Moreover, to simplify the notation we write $u(+\infty)$, respectively $u(-\infty)$, to denote the limit of u at $+\infty$, respectively at $-\infty$, whenever these limits exist. Finally, for any fixed function $f \in C_b(\mathbb{R})$, we denote by g_f and h_f the functions defined by

$$g_f(x) = \frac{1}{w_0}\int_0^x \frac{f(t)}{q(t)W(t)}\overline{u}_1(t)dt, \qquad h_f(x) = \frac{1}{w_0}\int_0^x \frac{f(t)}{q(t)W(t)}\overline{u}_2(t)dt,$$

for any $x \in \mathbb{R}$. First of all, we prove that the sequence $\{u_n\}$ converge in $C^2([-a,a])$, for any $a > 0$, to a function $u \in C^2(\mathbb{R}) \cap C_b(\mathbb{R})$. For this purpose, we observe that a simple computation shows that u_n is given by (3.2.3), with $c = c_1(n)$ and $c_2 = c_2(n)$ satisfying

$$\begin{cases} (c_1(n) + h_f(n))\overline{u}_1(n) + (c_2(n) - g_f(n))\overline{u}_2(n) = 0, \\ (c_1(n) + h_f(-n))\overline{u}_1(-n) + (c_2(n) - g_f(-n))\overline{u}_2(-n) = 0. \end{cases} \tag{3.2.9}$$

Moreover, by the classical maximum principle (see Theorem C.2.2(i)) it follows that

$$\sup_{x\in[-n,n]} |u_n(x)| \le \frac{1}{\lambda}\|f\|_\infty, \qquad n \in \mathbb{N}. \tag{3.2.10}$$

Thus, writing (3.2.3) at $x = \pm 1$, we deduce that $|c_1(n)\overline{u}_1(1) + c_1(n)\overline{u}_2(1)|$ and $|c_1(n)\overline{u}_1(-1) + c_2(n)\overline{u}_2(-1)|$ are bounded by a positive constant, independent of n. Since the matrix A, whose rows are $(\overline{u}_1(-1), \overline{u}_2(-1))$ and $(\overline{u}_1(1), \overline{u}_2(1))$, is invertible, it follows that $|c_1(n)|$ and $|c_2(n)|$ are bounded uniformly with respect to $n \in \mathbb{N}$. Therefore, up to a subsequence, we can assume that $c_1(n)$ and $c_2(n)$ converge to some real numbers $c_1(+\infty)$ and $c_2(+\infty)$. Now, from (3.2.3) is immediate to check that u_n converges in $C^2([-a,a])$ to some function $u \in C^2(\mathbb{R})$ which, of course, satisfies the differential equation (3.2.1). Moreover, the estimate (3.2.10) implies that u is bounded in $\mathbb{R}$. Therefore, the function $R(\lambda)f$ is well defined for any $f \in C_b(\mathbb{R})$.

Let us now prove that $R(\lambda)f = R_\lambda f$, where $R_\lambda f$ is given by (3.2.4). For this purpose, it suffices to show that $c_1(n)$ and $c_2(n)$ converge as n tends to $+\infty$ and

$$c_1(+\infty) = -h_f(-\infty), \qquad c_2(+\infty) = g_f(+\infty). \tag{3.2.11}$$

Of course, according to Theorem 3.2.2 it suffices to consider the case when at least one between $-\infty$ and $+\infty$ is accessible.

To begin with we observe that, from (3.2.6) and (3.2.7), we easily deduce that the function $\overline{u}_1/(qW)$ is integrable in $(0,+\infty)$, whereas the function $\overline{u}_2/(qW)$ is integrable in $(-\infty,0)$. Therefore, $g_f(+\infty)$ and $h_f(-\infty)$, as well as $g_1(+\infty)$ and $h_1(-\infty)$, exist and are finite. Moreover, if $\overline{u}_1(+\infty)=0$ and $\overline{u}_2(+\infty)$ is finite, then $(h_f\overline{u}_1)(+\infty)=0$. Indeed,

$$\begin{aligned}
|h_f(n)\overline{u}_1(n)| &= \frac{\overline{u}_1(n)}{w_0}\left|\int_0^n \frac{f(t)}{q(t)W(t)}\overline{u}_2(t)dt\right| \\
&\le \|f\|_\infty \frac{\overline{u}_1(n)}{w_0}\int_0^n \frac{\overline{u}_2(t)}{q(t)W(t)}dt \\
&\le \|f\|_\infty \frac{\overline{u}_1(n)\overline{u}_2(n)}{w_0}\left(\int_0^x \frac{1}{q(t)W(t)}dt + \int_x^n \frac{1}{q(t)W(t)}dt\right) \\
&\le \|f\|_\infty \left\{\frac{\overline{u}_1(n)\overline{u}_2(n)}{w_0}\int_0^x \frac{1}{q(t)W(t)}dt + \overline{u}_2(n)\left(g_1(n)-g_1(x)\right)\right\},
\end{aligned}$$

for any $x>0$. Therefore, letting n go to $+\infty$ we get

$$\limsup_{n\to+\infty} |h_f(n)\overline{u}_1(n)| \le \|f\|_\infty u_2(+\infty)\left(g_1(+\infty)-g_1(x)\right).$$

Then, letting x go to $+\infty$, we get $\limsup_{n\to+\infty} h_f(n)\overline{u}_1(n) = 0$, namely $(h_f\overline{u}_1)(+\infty)=0$. Similarly, if $\overline{u}_1(-\infty)$ is real and $\overline{u}_2(-\infty)=0$, then

$$\lim_{n\to+\infty} \overline{u}_2(-n)g_f(-n)=0.$$

To complete the proof we need a deeper analysis of the functions $\overline{u}_1$ and $\overline{u}_2$. For this purpose we split the remainder of the proof into several steps.

The case when $+\infty$ and $-\infty$ are both accessible. In such a situation, the functions $\overline{u}_1$ and $\overline{u}_2$ are bounded in $\mathbb{R}$ and $\overline{u}_1(+\infty)=\overline{u}_2(-\infty)=0$. The boundedness of $\overline{u}_1$ and $\overline{u}_2$ follows immediately from the properties (i) and (ii) in Proposition 3.1.7. Let us now show that $\overline{u}_1(+\infty)=0$. This is clear by Proposition 3.1.7(ii) if $+\infty$ is an exit point. So, let us assume that $+\infty$ is regular. A similar argument then can be used to show that $\overline{u}_2(-\infty)=0$ when $-\infty$ is natural. For this purpose, let u be a decreasing solution of (3.1.1) with $u(+\infty)=0$, provided by Proposition 3.1.7. Then, we set $v=u/u(0)$. As it is immediately seen, v vanishes at $+\infty$ and $v(0)=1$. We claim that $v\ge\overline{u}_1$ in $(0,+\infty)$. Since $\overline{u}_1>0$ in $\mathbb{R}$, it then follows that $\overline{u}_1(+\infty)=0$. By contradiction suppose that there exists $x>0$ such that $\overline{u}_1(x)>v(x)$. Then, $\overline{u}_1>v$ in $(0,+\infty)$. Indeed if this were not the case, then there should exist a point $x_1>0$ such that $\overline{u}_1(x_1)=v(x_1)$. By the last part of Remark 3.1.1 it would follow that $\overline{u}_1\equiv v$: a contradiction. Therefore, if $\overline{u}_1(x)>v(x)$ for some $x>0$, then $\overline{u}_1>v$ in $(0,+\infty)$. But this implies that $\overline{u}_1'(0)>v'(0)$. Since B is an interval, this would imply that $\overline{u}_1'(0) \neq \sup B$: a contradiction. Therefore, $\overline{u}_1 \leq v$ in $(0,+\infty)$ and $v'(0) \notin B$ (see (3.1.2)).

Now, from (3.2.9) it easily follows that $c_1(+\infty)$ and $c_2(+\infty)$ exist finite and are given by (3.2.11).

The case when $+\infty$ *is accessible and* $-\infty$ *is unaccessible.* According to the properties (i) and (ii) in Proposition 3.1.7, and arguing as in the previous step, it follows that $\overline{u}_1(+\infty) = 0$, $\overline{u}_1(-\infty) = +\infty$, $\overline{u}_2(+\infty) \in (0,+\infty)$. Hence, letting n go to $+\infty$ in the first equation in (3.2.9) we deduce that $c_2(+\infty)$ is given by (3.2.11). To prove that $c_1(+\infty)$ is given by (3.2.11), we begin by studying the behaviour of u_2 at $-\infty$. For this purpose, we observe that we can write $\overline{u}_2 = d_1\overline{u}_1 + d_2 u$ where u is an increasing solution to (3.1.1) satisfying $\lim_{x\to-\infty} u'(x)/W(x) = 0$ and $u(-\infty) = 0$, if $-\infty$ is natural, and $u(-\infty) = 1$, if $-\infty$ is an entrance. Since $\overline{u}_2(-\infty)$ is real, we easily see that $d_1 = 0$ and consequently $\overline{u}_2 = u/u(0)$. Therefore, $\overline{u}_2(-\infty) = 0$ if $-\infty$ is natural whereas $\overline{u}_2(-\infty) > 0$ if $-\infty$ is an entrance. In any case, $\overline{u}_2'/W$ tends to 0 as x tends to $-\infty$. Therefore,

$$
\begin{aligned}
|g_f(-n)|\overline{u}_2(-n) &\leq \frac{\|f\|_\infty}{w_0}\overline{u}_2(-n)\int_{-n}^0 \frac{\overline{u}_1(t)}{q(t)W(t)}dt \\
&= \frac{\|f\|_\infty}{w_0}\frac{\overline{u}_2(-n)}{\lambda}\left(\overline{u}_1'(0) - \frac{\overline{u}_1'(-n)}{W(-n)}\right) \\
&= \frac{\|f\|_\infty}{w_0}\frac{u_2(-n)}{\lambda}\overline{u}_1'(0) - \frac{\|f\|_\infty}{w_0}\frac{\overline{u}_2'(-n)}{\lambda W(-n)}\overline{u}_1(-n) + \frac{1}{\lambda}\|f\|_\infty.
\end{aligned}
$$

Therefore,

$$
\lim_{n\to+\infty} \frac{g_f(-n)\overline{u}_2(-n)}{\overline{u}_1(-n)} = 0.
$$

Thus, letting n go to $+\infty$ in the second equation in (3.2.9), we now easily see that $c_1(+\infty)$ is given by (3.2.11).

The case when $+\infty$ *is unaccessible and* $-\infty$ *is accessible.* By changing x to $-x$, we go back again to the previous case (see the proof of Proposition 3.1.5). Hence, also in this case $c_1(+\infty)$ and $c_2(+\infty)$ are given by (3.2.11). This finishes the proof. ∎

Now, we consider some examples.

Example 3.2.4 Let $\mathcal{A}$ be the operator defined by

$$
\mathcal{A}\varphi(x) = \varphi''(x) - x^3\varphi(x), \quad x \in \mathbb{R},
$$

on smooth functions φ. Let us show that $+\infty$ and $-\infty$ are both unaccessible. By Theorem 3.2.2 this will imply that the equation $\lambda u - \mathcal{A}u = f$ is uniquely solvable in $C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ for any $f \in C_b(\mathbb{R})$. A straightforward computation shows that

$$W(x) = e^{x^4/4}, \qquad Q(x) = e^{-x^4/4} \int_0^x e^{s^4/4} ds, \qquad R(x) = e^{x^4/4} \int_0^x e^{-s^4/4} ds,$$

for any $x \in \mathbb{R}$. It is readily seen that $R \notin L^1(-\infty, 0) \cup L^1(0, +\infty)$. Moreover, using De L'Hôpital rule it can be seen that

$$\lim_{x \to \pm\infty} x^3 Q(x) = 1.$$

Hence $Q \in L^1(-\infty, 0) \cap L^1(0, +\infty)$. We conclude that $+\infty$ and $-\infty$ are entrance points.

We will generalize this example to the N-dimensional setting in Chapter 4 (see Example 4.1.13).

Example 3.2.5 Let $\mathcal{A}$ be the one-dimensional differential operator defined by

$$\mathcal{A}\varphi(x) = \varphi''(x) + x^3 \varphi'(x), \qquad x \in \mathbb{R},$$

on smooth functions φ. In this case

$$W(x) = e^{-\frac{x^4}{4}}, \qquad Q(x) = e^{\frac{x^4}{4}} \int_0^x e^{-\frac{s^4}{4}} ds, \qquad R(x) = e^{-\frac{x^4}{4}} \int_0^x e^{\frac{s^4}{4}} ds.$$

Arguing as in the previous example, it follows that $Q \notin L^1(-\infty, 0) \cup L^1(0, +\infty)$ and $R \in L^1(-\infty, 0) \cap L^1(0, +\infty)$. We conclude that $+\infty$ and $-\infty$ are both exit points. Therefore, according to Theorem 3.2.2, for any $f \in C_b(\mathbb{R})$ the equation $\lambda u - \mathcal{A}u = f$ admits more than one solution belonging to $C_b(\mathbb{R}) \cap C^2(\mathbb{R})$.

We will generalize this example to the N-dimensional setting in Chapter 5 (see Example 5.2.5).

Remark 3.2.6 These two examples show us that the uniqueness of the solution $u \in D_{\max}(\mathcal{A})$ to the elliptic equation $\lambda u - \mathcal{A}u = f$ does not depend merely on the growth at infinity of the coefficients of the operator $\mathcal{A}$. In fact, the operators defined in Examples 3.2.4 and 3.2.5 differ only in the sign of the drift term, but this difference is crucial. Indeed, using the notation of Chapter 2, we have $D(\widehat{A}) = D_{\max}(A)$, if $\mathcal{A}$ is the operator in Example 3.2.4 whereas $D(\widehat{A})$ is properly contained in $D_{\max}(A)$ if $\mathcal{A}$ is the operator in Example 3.2.5.

See also Example 5.2.5 and [88, Section 5.2] for a discussion about these two previous examples in the probabilistic framework.

We study the problem of the uniqueness of the solution $u \in D_{\max}(\mathcal{A})$ to the elliptic equation $\lambda u - \mathcal{A}u = f$, and of the classical solution to the homogeneous Cauchy problem associated with the operator $\mathcal{A}$ (in the N-dimensional setting), in Chapter 4.

To conclude this section we characterize $D_{\max}(\mathcal{A})$ under some additional assumptions of the coefficients q and b. The following theorem has been proved in [58].

Theorem 3.2.7 *Suppose that $q \in C(\mathbb{R})$ and $q(x) \geq \kappa_0$ for any $x \in \mathbb{R}$ and some positive constant κ_0. Further assume that $b \in C^1(\mathbb{R})$ satisfies*

$$q(x)b'(x) \leq c_1 + c_2(b(x))^2, \qquad x \in \mathbb{R}, \tag{3.2.12}$$

for some constants $c_1 \in \mathbb{R}$ and $c_2 < 1$. Finally, assume that $+\infty$ and $-\infty$ are both unaccessible. Then,

$$D_{\max}(\mathcal{A}) = \{u \in C_b^2(\mathbb{R}) : \; qu'', bu' \in C_b(\mathbb{R})\}. \tag{3.2.13}$$

Proof. According to Theorem 3.2.2 and Proposition 3.2.3, for any $f \in C_b(\mathbb{R})$, the elliptic equation

$$u - \mathcal{A}u = f \tag{3.2.14}$$

admits a unique solution u in $D_{\max}(\mathcal{A})$. Therefore, to prove the assertion it suffices to show that u actually belongs to the space defined in the right-hand side of (3.2.13). For this purpose, for any $n \in \mathbb{N}$, we denote by $u_n \in C^2([-n,n])$ the solution to the Cauchy problem

$$\begin{cases} u(x) - \mathcal{A}u(x) = f(x), & x \in (-n,n), \\ u'(-n) = u'(n) = 0. \end{cases}$$

According to Theorem C.2.2(ii), we have

$$\|u_n\|_{C([-n,n])} \leq \|f\|_{C([-n,n])}, \qquad \|\mathcal{A}u_n\|_{C([-n,n])} \leq 2\|f\|_{C([-n,n])}, \tag{3.2.15}$$

for any $n \in \mathbb{N}$.

The main step of the proof consists in proving that the C^2-norm of u_n and the sup-norm of qu_n'' and bu_n' are uniformly bounded with respect to n. Then, a compactness argument will allow us to show that u_n converges to a solution of the equation (3.2.14) belonging to the space defined by the right-hand side of (3.2.13).

To begin with, let us prove that there exists a positive constant C, independent of n, such that

$$\|qu_n''\|_{C([-n,n])} + \|bu_n'\|_{C([-n,n])} \leq C\|f\|_{C([-n,n])} \leq C\|f\|_\infty, \qquad n \in \mathbb{N}. \tag{3.2.16}$$

For this purpose, let $x_0 = x_0(n)$ be a point in $[-n,n]$ such that $|b(x_0)u_n'(x_0)| = \|bu_n'\|_{C([-n,n])}$. Up to replacing u_n with $-u_n$ and f with $-f$, we can assume that x_0 is a maximum of the function bu_n'. Moreover, we can also assume that $x_0 \in (-n,n)$ and $b(x_0) \neq 0$, otherwise $b(x_0)u_n'(x_0) = 0$ and

(3.2.16) would follow immediately. Since $(bu_n')'(x_0) = 0$, we have $u_n''(x_0) = -b'(x_0)(b(x_0))^{-1}u_n'(x_0)$. Hence, (3.2.12) implies that

$$q(x_0)u_n''(x_0) = -q(x_0)b'(x_0)\frac{u_n'(x_0)}{b(x_0)} \geq -c_1\frac{u_n'(x_0)}{b(x_0)} - c_2 b(x_0)u_n'(x_0).$$

Therefore, taking (3.2.15) into account, we deduce that

$$\begin{aligned} 2\|f\|_{C([-n,n])} &\geq -f(x_0) + u_n(x_0) \\ &= q(x_0)u_n''(x_0) + b(x_0)u_n'(x_0) \\ &\geq (1-c_2)b(x_0)u_n'(x_0) - c_1\frac{u_n'(x_0)}{b(x_0)}. \end{aligned} \tag{3.2.17}$$

Multiplying the first and the last sides of (3.2.17) by $b(x_0)u_n'(x_0)$ we get

$$\begin{aligned} 2\|bu_n'\|_{C([-n,n])}\|f\|_{C([-n,n])} &\geq (1-c_2)\|bu_n'\|^2_{C([-n,n])} - c_1(u_n'(x_0))^2 \\ &\geq (1-c_2)\|bu_n'\|^2_{C([-n,n])} - c_1^+\|u_n'\|^2_{C([-n,n])}. \end{aligned}$$

If we set

$$x = \|bu_n'\|_{C([-n,n])}, \qquad \alpha = \frac{2}{1-c_2}\|f\|_{C([-n,n])}, \qquad \beta = \frac{c_1^+}{1-c_2}\|u_n'\|^2_{C([-n,n])},$$

we obtain that x satisfies the inequality $x^2 \leq \alpha x + \beta$, which implies that $x \leq \alpha + \sqrt{\beta}$ or, equivalently,

$$\|bu_n'\|_{C([-n,n])} \leq \frac{2}{1-c_2}\|f\|_{C([-n,n])} + \left(\frac{c_1^+}{1-c_2}\right)^{\frac{1}{2}}\|u_n'\|_{C([-n,n])}. \tag{3.2.18}$$

Now, we observe that there exists a positive constant C, independent of n, such that

$$\|v'\|_{C([-n,n])} \leq C\|v\|^{\frac{1}{2}}_{C([-n,n])}\|v''\|^{\frac{1}{2}}_{C([-n,n])}, \tag{3.2.19}$$

for any $v \in C^2([-n,n])$ and any $n \in \mathbb{N}$. Indeed, if $n = 1$ the estimate (3.2.19) follows from the Landau inequality applied to any extension $w \in C_b^2(\mathbb{R})$ of v. To get (3.2.19) for a general n, with a constant being independent of n, it suffices to apply the Landau inequality with $n = 1$ to the function $w : [-1,1] \to \mathbb{R}$ defined by $w(x) = v(nx)$ for any $x \in [-1,1]$.

Now, from (3.2.19) it follows immediately that, for any $\varepsilon > 0$, there exists a positive constant C_ε, independent of n, such that

$$\begin{aligned} \|u_n'\|_{C([-n,n])} &\leq \varepsilon\|u_n''\|_{C([-n,n])} + C_\varepsilon\|u_n\|_{C([-n,n])} \\ &\leq \varepsilon\|u_n''\|_{C([-n,n])} + C_\varepsilon\|f_n\|_{C([-n,n])}, \end{aligned} \tag{3.2.20}$$

for any $n \in \mathbb{N}$. Next we observe that, taking (3.2.15) into account, we can write

$$\begin{aligned}\|u_n''\|_{C([-n,n])} &\le \frac{1}{\kappa_0}\|qu_n''\|_{C([-n,n])}\\ &\le \frac{1}{\kappa_0}\left(\|\mathcal{A}u_n\|_{C([-n,n])} + \|bu_n'\|_{C([-n,n])}\right)\\ &\le \frac{1}{\kappa_0}\left(\|bu_n'\|_{C([-n,n])} + 2\|f_n\|_{C([-n,n])}\right). \end{aligned} \tag{3.2.21}$$

Therefore, from (3.2.18), (3.2.20) (with ε small enough) and (3.2.21), we easily get (3.2.16).

Now we observe that from (3.2.21) we obtain

$$\|u_n''\|_{C([-n,n])} \le \frac{C}{\kappa_0}\|f\|_\infty, \qquad n \in \mathbb{N}.$$

Therefore, according to (3.2.15) and (3.2.19) it follows that $\|u_n\|_{C^2([-n,n])}$ is bounded by a positive constant independent of n. Now, using a compactness argument similar to that used in the proof of Theorem 2.1.1, we can easily show that, up to a subsequence, u_n converges to a function $u \in C^1(\mathbb{R}^N)$ as n tends to $+\infty$, locally uniformly in $\mathbb{R}$. Since $qu_n'' = u_n - f - bu_n'$ and $q \ge \kappa_0 > 0$, we then deduce that u_n'' converges locally uniformly in $\mathbb{R}$ as well. Therefore, $u \in C^2(\mathbb{R})$ and, since the sequence $\{u_n\}$ is bounded in $C^2_b(\mathbb{R})$, u belongs to $C^2_b(\mathbb{R})$ as well. Moreover, qu'' and bu' are bounded in $\mathbb{R}$, thanks to (3.2.16). Therefore, u belongs to the space defined by the right-hand side of (3.2.13).

To complete the proof it suffices to observe that since u_n solves, for any $n \in \mathbb{N}$, the differential equation (3.2.14) in $[-n,n]$, then u satisfies such a differential equation in $\mathbb{R}$. Therefore, $u = R_1 f$ and we are done. ■

Chapter 4

Uniqueness results, conservation of probability and maximum principles

4.0 Introduction

In this chapter we deal with the problem of the uniqueness of the solution of the elliptic equation

$$\lambda v(x) - \mathcal{A}v(x) = f(x), \qquad x \in \mathbb{R}^N, \tag{4.0.1}$$

($f \in C_b(\mathbb{R}^N)$) which belongs to

$$D_{\max}(\mathcal{A}) = \left\{ u \in C_b(\mathbb{R}^N) \cap \bigcap_{1 \le p < +\infty} W^{2,p}_{\rm loc}(\mathbb{R}^N) : \ \mathcal{A}u \in C_b(\mathbb{R}^N) \right\}, \tag{4.0.2}$$

and of the solution to the parabolic problem

$$\begin{cases} D_t u(t,x) - \mathcal{A}u(t,x) = 0, & t > 0,\, x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{4.0.3}$$

which belongs to $C([0,+\infty) \times \mathbb{R}^N) \cap C^{1,2}((0,+\infty) \times \mathbb{R}^N)$ and it is bounded in $[0,T] \times \mathbb{R}^N$ for any $T > 0$.

Throughout the chapter, we assume that the hypotheses of Chapter 2 are satisfied. For the reader's convenience, we state them again.

Hypotheses 4.0.1 (i) $q_{ij} \equiv q_{ji}$,

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \ge \kappa(x)|\xi|^2, \quad \kappa(x) > 0, \qquad \xi, x \in \mathbb{R}^N;$$

(ii) q_{ij}, b_i $(i,j = 1, \ldots, N)$ and c belong to $C^\alpha_{\rm loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$;

(iii) there exists $c_0 \in \mathbb{R}$ such that

$$c(x) \le c_0, \quad x \in \mathbb{R}^N.$$

Under the previous hypotheses, in Chapter 2 we have proved the existence of solutions to the problems (4.0.1) and (4.0.3), with the regularity properties claimed above, for any $f \in C_b(\mathbb{R}^N)$ and any $\lambda > 0$ (see Theorems 2.1.1 and 2.2.1).

The two problems of uniqueness are strictly connected. We exploit this connection in Proposition 4.1.1 (in the general case) and in Proposition 4.1.10 (in the case when $c \equiv 0$).

The following assumption is often considered to get uniqueness results (see, e.g., [83, 104, 139]).

Hypothesis 4.0.2 There exists a positive function $\varphi \in C^2(\mathbb{R}^N)$ satisfying

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty, \qquad \mathcal{A}\varphi(x) - \lambda_0\varphi(x) \le 0, \quad x \in \mathbb{R}^N, \tag{4.0.4}$$

for some $\lambda_0 > c_0$.

Remark 4.0.3 Observe that one can equivalently assume that there exists $\varphi \in C^2(\mathbb{R}^N)$ such that

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty, \qquad \mathcal{A}\varphi(x) - \lambda_0\varphi(x) \le C, \quad x \in \mathbb{R}^N, \tag{4.0.5}$$

for some $C \in \mathbb{R}$. Indeed, if φ satisfies (4.0.5), then the function $\varphi + M$ satisfies Hypothesis 4.0.2 provided the constant M is sufficiently large.

A function $\varphi \in C^2(\mathbb{R}^N)$ satisfying (4.0.4) or (4.0.5) is usually called a Lyapunov function for the operator $\mathcal{A}$.

The condition (4.0.5) can be made clearer with a particular choice of φ. For instance, if we take $\varphi(x) = \log(m + |x|^2)$, where m is a positive constant, then (4.0.5) reads as follows:

$$\begin{aligned}
&(m + |x|^2)\mathrm{Tr}\, Q(x) - 2\langle Q(x)x, x\rangle + (m + |x|^2)\langle b(x), x\rangle \\
&+\frac{c(x)}{2}(m + |x|^2)^2 \log(m + |x|^2) \\
\le\ & \frac{\lambda_0}{2}(m + |x|^2)^2 \log(m + |x|^2) + \frac{C}{2}(m + |x|^2)^2, \qquad x \in \mathbb{R}^N.
\end{aligned} \tag{4.0.6}$$

Assuming Hypotheses 4.0.1 and 4.0.2 we prove some maximum principles for the elliptic equation and for the parabolic problem. The maximum principles yield the uniqueness of the bounded and continuous solution of the elliptic equation (4.0.1) and the uniqueness of the classical solution of the parabolic problem (4.0.3), which is bounded in $[0, T] \times \mathbb{R}^N$ for any $T > 0$.

Afterwards, we focus our attention on the case when $c \equiv 0$. In such a case the function $u_1 \equiv \mathbb{1}$ is clearly a bounded solution of the problem (4.0.3) with initial value $f \equiv \mathbb{1}$. Also the function $u_2 = T(\cdot)\mathbb{1}$ is a bounded solution of the same problem. Therefore, if the problem (4.0.3) admits a unique bounded solution for any $f \in C_b(\mathbb{R}^N)$, then, necessarily,

$$T(t)\mathbb{1} \equiv \mathbb{1}, \qquad t > 0. \tag{4.0.7}$$

In general, we only have $0 < T(t)\mathbb{1} \leq \mathbb{1}$. In Proposition 4.1.10 we prove that, actually, (4.0.7) is also a sufficient condition guaranteeing the uniqueness of the classical solution to the problem (4.0.3), which is bounded in $[0,T] \times \mathbb{R}^N$ for any $T > 0$. In this case also the equation (4.0.1) has a unique solution in $D_{\max}(\mathcal{A})$, for any $f \in C_b(\mathbb{R}^N)$.

When (4.0.7) holds, we say that $\{T(t)\}$ is conservative or that conservation of probability holds.

Finally, in Section 4.2 we give a nonuniqueness result. In the case when $c \equiv 0$, some other sufficient conditions implying that the problem (4.0.3) is not uniquely solvable are given in Section 5.2.

4.1 Conservation of probability and uniqueness

The following proposition describes the relation between the uniqueness of the elliptic equation (4.0.1) and the parabolic problem (4.0.3).

Proposition 4.1.1 *Consider the following conditions:*

(i) for any $\lambda > c_0$ and any $f \in C_b(\mathbb{R}^N)$, the function $u = R(\lambda)f$ is the unique solution of the elliptic equation (4.0.1) *in $D_{\max}(\mathcal{A})$;*

(ii) for any $f \in C_b(\mathbb{R}^N)$, the function $u = T(\cdot)f$ is the unique solution of the parabolic problem (4.0.3) *which belongs to $C_b([0,T]\times\mathbb{R}^N)\cap C^{1,2}((0,T)\times\mathbb{R}^N)$ for any $T > 0$;*

(iii) for any $f \in C_b(\mathbb{R}^N)$, the function $u = T(\cdot)f$ is the unique solution of the parabolic problem (4.0.3) *in $C([0,+\infty)\times\mathbb{R}^N)\cap C^{1,2}((0,+\infty)\times\mathbb{R}^N)$ satisfying $|u(t,x)| \leq M\exp(c_0 t)$ for some $M > 0$, any $t > 0$ and any $x \in \mathbb{R}^N$.*

Then "(i) ⇒ (iii)" and "(ii) ⇒ (i)".

Proof. "(i) ⇒ (iii)". Let $u \in C([0,+\infty) \times \mathbb{R}^N) \cap C^{1,2}((0,+\infty) \times \mathbb{R}^N)$ be a solution of the parabolic problem (4.0.3) with $u(0,\cdot) = 0$, satisfying

$$|u(t,x)| \leq M\exp(c_0 t), \quad t > 0, \; x \in \mathbb{R}^N, \tag{4.1.1}$$

for some $M > 0$. Let us prove that $u(t,x) = 0$ for any $t > 0$ and any $x \in \mathbb{R}^N$. For this purpose, fix $\lambda > c_0$, and consider the functions

$$v_\infty(x) = \int_0^{+\infty} e^{-\lambda t} u(t,x)dt, \qquad x \in \mathbb{R}^N,$$
$$v_n(x) = \int_{1/n}^{n} e^{-\lambda t} u(t,x)dt, \qquad x \in \mathbb{R}^N, \ n \in \mathbb{N}.$$

Taking (4.1.1) into account, we easily deduce that $v_n \in C_b(\mathbb{R}^N)$ for any $n \in \mathbb{N} \cup \{0\}$ and

$$\|v_n\|_\infty \le \frac{M}{\lambda - c_0}, \qquad n \in \mathbb{N} \cup \{0\}. \tag{4.1.2}$$

We now observe that, for any $x \in \mathbb{R}^N$ and any $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{A}v_n(x) &= \int_{1/n}^{n} e^{-\lambda t} \mathcal{A}u(t,x)dt \\ &= \int_{1/n}^{n} e^{-\lambda t} D_t u(t,x)dt \\ &= e^{-\lambda n} u(n,x) - e^{-\lambda/n} u(1/n,x) + \lambda v_n(x), \end{aligned} \tag{4.1.3}$$

which implies that $\mathcal{A}v_n \in C_b(\mathbb{R}^N)$, so that $v_n \in D_{\max}(\mathcal{A})$. Moreover, by (4.1.1), (4.1.2) and (4.1.3) it follows that there exists a constant $C > 0$ such that $\|\mathcal{A}v_n\|_\infty \le C$ for any $n \in \mathbb{N}$, and

$$\lim_{n \to +\infty} \mathcal{A}v_n(x) = \lambda v_\infty(x), \tag{4.1.4}$$

for any $x \in \mathbb{R}^N$. We now define, for any $n \in \mathbb{N}$, the function $\varphi_n : \mathbb{R}^N \to \mathbb{R}$ by setting $\varphi_n = \lambda v_n - \mathcal{A}v_n$. By the above results we deduce that $\varphi_n \in C_b(\mathbb{R}^N)$ and, by (i),

$$v_n(x) = (R(\lambda)\varphi_n)(x) = \int_{\mathbb{R}^N} K_\lambda(x,y)\varphi_n(y)dy, \qquad x \in \mathbb{R}^N, \tag{4.1.5}$$

where $R(\lambda)$ and K_λ are given by Theorem 2.1.3. Moreover, as we can easily see, there exists a constant $C' > 0$ such that $\|\varphi_n\|_\infty \le C'$ for any $n \in \mathbb{N}$. Thus, letting n go to $+\infty$ in (4.1.5), from (4.1.4) and the dominated convergence theorem, we get

$$\lim_{n \to +\infty} v_n(x) = 0, \qquad x \in \mathbb{R}^N,$$

that is $v_\infty(x) = 0$ for any $x \in \mathbb{R}^N$. Since $\lambda > c_0$ is arbitrary, by the uniqueness of the Laplace transform, we conclude that $u(t,x) = 0$ for any $t > 0$ and any $x \in \mathbb{R}^N$.

"(ii) $\Rightarrow$ (i)". Let $v \in D_{\max}(\mathcal{A})$ be a solution of the equation $\lambda v - \mathcal{A}v = 0$; let us prove that $v = 0$. By local regularity results for elliptic equations in bounded domains, we know that $v \in C^{2+\alpha}_{\rm loc}(\mathbb{R}^N)$. Then, the function

$$u(t,x) = e^{\lambda t} v(x), \qquad t > 0, \; x \in \mathbb{R}^N,$$

belongs to $u \in C([0,+\infty) \times \mathbb{R}^N) \cap C^{1,2}((0,+\infty) \times \mathbb{R}^N)$, and it is immediate to check that it is a solution of the problem (4.0.3), with $u(0,\cdot) = v$; besides u is bounded in $[0,T] \times \mathbb{R}^N$ for any $T > 0$. By (ii), we have $u(t,x) = (T(t)v)(x)$ for any $t > 0$ and any $x \in \mathbb{R}^N$, and, then, by Theorem 2.2.5,

$$|u(t,x)| \le \exp(c_0 t)\|v\|_\infty, \qquad t > 0, \; x \in \mathbb{R}^N.$$

Since $\sup_{x\in\mathbb{R}^N} |u(t,x)| = \exp(\lambda t)\|v\|_\infty$ and $\lambda > c_0$, we conclude that $\|v\|_\infty = 0$. ∎

4.1.1 Maximum principles

In this subsection we prove two maximum principles which provide us with uniqueness results for the solution to the equation (4.0.1) and the Cauchy problem (4.0.3). First we prove the following lemma which is a local maximum principle for functions in $W^{2,p}_{\rm loc}(\mathbb{R}^N)$.

Lemma 4.1.2 *Assume Hypotheses* 4.0.1 *and suppose that* $u \in W^{2,p}_{\rm loc}(\mathbb{R}^N)$, *for any* $p \in [1,+\infty)$, *and that* $\mathcal{A}u \in C(\mathbb{R}^N)$. *If* x_0 *is a local maximum (resp. minimum) of* u, *then*

$$\mathcal{A}u(x_0) - c(x_0)u(x_0) \le 0, \qquad (\textit{resp.}\;\; \mathcal{A}u(x_0) - c(x_0)u(x_0) \ge 0).$$

Proof. We limit ourselves to considering the case when x_0 is a local maximum point for u, since the case when x_0 is a local minimum follows easily from this one replacing u with $-u$. Moreover, without loss of generality, we can suppose that $c(x) = 0$ for any $x \in \mathbb{R}^N$. The general case follows from this one replacing the operator $\mathcal{A}$ with the operator $\mathcal{A}_0 = \mathcal{A} - c$. Possibly replacing u with $u(\cdot - x_0) + C$ for a suitable constant $C > 0$, we can also assume that $x_0 = 0$ and $u(0) > 0$. Let $r > 0$ be such that $u_{|\overline{B}(r)}$ attains its maximum value at 0. Moreover, let $\psi \in C^\infty_c(\mathbb{R}^N)$ be such that $\chi_{B(r/2)} \le \psi \le \chi_{B(r)}$. Thus, 0 is a global maximum point of the function $v = \psi u$. Moreover, the function

$$\mathcal{A}v = \psi \mathcal{A}u + u\mathcal{A}\psi + 2\langle QDu, D\psi\rangle$$

belongs to $C_b(\mathbb{R}^N)$ since, by the Sobolev embedding theorems (see [2, Theorem 5.4]), $u \in C^1(\mathbb{R}^N)$. Therefore, $v \in D_{\max}(\mathcal{A}) \cap C_0(\mathbb{R}^N)$ and, consequently, Proposition 2.3.6 implies that $v \in D(\widehat{A})$. Now, since $c \equiv 0$, the family of measures $\{p(t,x;dy) : \; t > 0, \; x \in \mathbb{R}^N\}$, introduced in Theorem 2.2.5, defines a transition function and, by (2.2.8), we have

$$(T(t)v)(0) - v(0) \le \int_{B(r)} (v(y) - v(0))p(t,0;dy) \le 0.$$

Taking Proposition 2.3.6 into account, we now deduce that

$$\mathcal{A}u(0) = \mathcal{A}v(0) = \widehat{A}v(0) = \lim_{t\to 0^+} \frac{(T(t)v)(0) - v(0)}{t} \le 0.$$

∎

The proof of Lemma 4.1.2 strongly relies on the representation formula (2.2.8) and on the characterization of $D(\widehat{A})$. We now give an alternative proof of Lemma 4.1.2 which has the advantage that it can be easier extended to the case when $\mathbb{R}^N$ is replaced with an open set Ω and $u \in \bigcap_{1\le p<+\infty} W^{2,p}_{\rm loc}(\Omega)$ satisfies $\lambda u - \mathcal{A}u \le 0$ in Ω.

A second proof of Lemma 4.1.2. As in the other proof, we deal only with the case when x_0 is a relative maximum. Moreover, we assume that $c \equiv 0$, $x_0 = 0$ and $u(x) \ge 0$ for any $x \in B(r)$ and some $r > 0$.

Let us fix a function $\psi \in C^\infty_c(\mathbb{R}^N)$ compactly supported in $B(r)$ and such that $\psi(0) = 1$, $D\psi(0) = 0$, $D^2\psi(0) = 0$ and $0 \le \psi(x) < 1$ for any $x \in B(r) \setminus \{0\}$. As it is easily seen, $x = 0$ is the unique maximum point of the function $v := \psi u$, and $\mathcal{A}v \in C(\mathbb{R}^N)$. Moreover, since $D\psi(0) = 0$ and $D^2\psi(0) = 0$, we deduce that $\mathcal{A}u(0) = \mathcal{A}v(0)$.

Now let $\varphi \in C^\infty_c(\mathbb{R}^N)$ be a smooth function such that $0 \le \varphi(x) \le 1$ for any $x \in \mathbb{R}^N$, $\operatorname{supp}(\varphi) \subset B(1)$, and with $\|\varphi\|_{L^1(\mathbb{R}^N)} = 1$. For any $n \in \mathbb{N}$ we set

$$\varphi_n(x) = n^N\varphi(nx), \qquad v_n = \varphi_n \star v \in C^\infty_c(\mathbb{R}^N),$$

where "$\star$" denotes the convolution operator. Since $v \in C_0(\mathbb{R}^N)$, then v_n converges to v, uniformly in $\mathbb{R}^N$, as n tends to $+\infty$. Moreover, for any $n \in \mathbb{N}$, v_n has an absolute maximum at some point $x_n \in \overline{B}(r+1/n)$ so that $\mathcal{A}v_n(x_n) \le 0$. Since $D_iv_n = \varphi_n \star D_iv$ and $D_{ij}v_n = \varphi_n \star D_{ij}v$ for any $i, j = 1, \ldots, N$, it follows that

$$\begin{aligned}
\mathcal{A}v_n(x) &= \sum_{i,j=1}^N q_{ij}(x)D_{ij}v_n(x) + \sum_{j=1}^N b_j(x)D_jv_n(x)\\
&= \int_{\mathbb{R}^N} \varphi_n(y) \sum_{i,j=1}^N q_{ij}(x-y)D_{ij}v(x-y)dy\\
&\quad + \int_{\mathbb{R}^N} \varphi_n(y) \sum_{j=1}^N b_j(x-y)D_jv(x-y)dy\\
&\quad + \int_{\mathbb{R}^N} \varphi_n(y) \sum_{i,j=1}^N (q_{ij}(x) - q_{ij}(x-y))D_{ij}v(x-y)dy\\
&\quad + \int_{\mathbb{R}^N} \varphi_n(y) \sum_{j=1}^N (b_j(x) - b_j(x-y))D_jv(x-y)dy
\end{aligned}$$

$$
\begin{aligned}
&= (\varphi_n \star \mathcal{A}v)(x)\\
&\qquad + \int_{B(1/n)} \varphi_n(y) \sum_{i,j=1}^N (q_{ij}(x) - q_{ij}(x-y))D_{ij}v(x-y)dy\\
&\qquad + \int_{B(1/n)} \varphi_n(y) \sum_{j=1}^N (b_j(x) - b_j(x-y))D_jv(x-y)dy,
\end{aligned}
\tag{4.1.6}
$$

for any $x \in \mathbb{R}^N$. Since $\mathcal{A}v \in C_0(\mathbb{R}^N)$, it follows that $\varphi_n \star \mathcal{A}v$ tends to $\mathcal{A}v$ uniformly in $\mathbb{R}^N$ as n tends to $+\infty$. As far as the remaining terms in the last side of (4.1.6) are concerned, we observe that they vanish as n tends to $+\infty$, uniformly with respect to $x \in \mathbb{R}^N$. Indeed, if $p > N/\alpha$ (α being as in Hypothesis 4.0.1(ii)), then

$$
\begin{aligned}
&\int_{B(1/n)} \varphi_n(y) \sum_{i,j=1}^N (q_{ij}(x) - q_{ij}(x-y))D_{ij}v(x-y)dy\\
&+ \int_{B(1/n)} \varphi_n(y) \sum_{j=1}^N (b_j(x) - b_j(x-y))D_jv(x-y)dy\\
&\le C_1 n^{-\alpha}\|\varphi_n\|_{L^{p'}(\mathbb{R}^N)}\|D^2v\|_{L^p(\mathbb{R}^N)} + C_2 n^{-\alpha}\|\varphi_n\|_{L^{p'}(\mathbb{R}^N)}\|Dv\|_{L^p(\mathbb{R}^N)}\\
&= C_1 n^{-\alpha+N/p}\|\varphi\|_{L^{p'}(\mathbb{R}^N)}\|D^2v\|_{L^p(\mathbb{R}^N)} + C_2 n^{-\alpha+N/p}\|\varphi\|_{L^{p'}(\mathbb{R}^N)}\|Dv\|_{L^p(\mathbb{R}^N)}.
\end{aligned}
$$

Here $1/p + 1/p' = 1$,

$$
C_1^2 = \sum_{i,j=1}^N [q_{ij}]^2_{C^\alpha(B(r+2))}, \qquad C_2^2 = \sum_{j=1}^N [b_j]^2_{C^\alpha(B(r+2))},
$$

and $\|Du\|_p$, $\|D^2u\|_p$ denote the L^p-norms of the functions

$$
x \mapsto \left(\sum_{j=1}^N |D_ju(x)|^2\right)^{1/2}, \qquad \text{and} \qquad x \mapsto \left(\sum_{i,j=1}^N |D_{ij}u(x)|^2\right)^{1/2},
$$

respectively.

Summing up, we have proved that v_n and $\mathcal{A}v_n$ converge, respectively, to v and $\mathcal{A}v$ uniformly in $\mathbb{R}^N$ as n tends to $+\infty$.

Since $\{x_n\}$ is a bounded sequence, using a compactness argument we easily see that, up to a subsequence, we can assume that $\{x_n\}$ converges to some point $\hat{x} \in \overline{B}(r)$ as n tends to $+\infty$. By continuity, $v(x_n)$ tends to $v(\hat{x})$ as n tends to $+\infty$. Hence,

$$
v(\hat{x}) = \lim_{n\to+\infty} v(x_n) = \lim_{n\to+\infty} v_n(x_n) = \lim_{n\to+\infty} \|v_n\|_\infty = \|v\|_\infty = v(0),
$$

where the second equality follows from the fact that v_n converges uniformly in $\mathbb{R}^N$ to v. Recalling that 0 is the unique point where v attains its maximum value, it follows that $\hat{x}=0$. Since $\mathcal{A}v_n \le 0$ for any $n\in\mathbb{N}$ and it converges uniformly to $\mathcal{A}v$, we immediately deduce that $\mathcal{A}v(0)\le 0$, and the assertion follows. ∎

Theorem 4.1.3 *Assume Hypotheses* 4.0.1 *and* 4.0.2. *Fix* $T>0$, $f\in C(\mathbb{R}^N)$, $g:(0,T]\times\mathbb{R}^N\to\mathbb{R}$. *Further suppose that the function* $u\in C([0,T]\times\mathbb{R}^N)$ *is such that* $D_tu\in C((0,T]\times\mathbb{R}^N)$, $u(t,\cdot)\in W^{2,p}_{\rm loc}(\mathbb{R}^N)$, $\mathcal{A}u(t,\cdot)\in C(\mathbb{R}^N)$ *for any* $t\in(0,T]$ *and any* $p\in[1,+\infty)$ *and it solves the Cauchy problem*

$$\begin{cases} D_tu(t,x)-\mathcal{A}u(t,x)=g(t,x), & t\in(0,T],\ x\in\mathbb{R}^N,\\ u(0,x)=f(x), & x\in\mathbb{R}^N.\end{cases}\tag{4.1.7}$$

Then, the following properties are met:

(*i*) *If*

$$\sup_{x\in\mathbb{R}^N} f(x)<+\infty,\qquad g\le 0\ \text{ in }\ (0,T]\times\mathbb{R}^N,$$

and u *satisfies*

$$\limsup_{|x|\to+\infty}\left(\sup_{t\in[0,T]}\frac{u(t,x)}{\varphi(x)}\right)\le 0,\tag{4.1.8}$$

then

$$u(t,x)\le e^{c_0t}\max\left\{0,\sup_{\mathbb{R}^N} f\right\},\qquad t\in[0,T],\quad x\in\mathbb{R}^N.\tag{4.1.9}$$

(*ii*) *If*

$$\inf_{x\in\mathbb{R}^N} f(x)>-\infty,\qquad g\ge 0\ \text{ in }\ (0,T]\times\mathbb{R}^N,$$

u *satisfies*

$$\liminf_{|x|\to+\infty}\left(\inf_{t\in[0,T]}\frac{u(t,x)}{\varphi(x)}\right)\ge 0,$$

then

$$u(t,x)\ge e^{c_0t}\min\left\{0,\inf_{\mathbb{R}^N} f\right\},\qquad t\in[0,T],\quad x\in\mathbb{R}^N.\tag{4.1.10}$$

(*iii*) *In particular, for any* $f\in C(\mathbb{R}^N)$ *such that*

$$\lim_{|x|\to+\infty}\frac{f(x)}{\varphi(x)}=0,$$

there exists at most one solution of the parabolic problem (4.1.7) (*with* $g\equiv 0$) *in* $C([0,+\infty)\times\mathbb{R}^N)$ *such that* $D_tu\in C((0,T]\times\mathbb{R}^N)$, $u(t,\cdot)\in W^{2,p}_{\rm loc}(\mathbb{R}^N)$, $\mathcal{A}u(t,\cdot)\in C(\mathbb{R}^N)$ *for any* $p\in[1,+\infty)$ *and any* $t\in(0,T]$, *and such that* (4.1.8) *holds for any* $T>0$.

Proof. We limit ourselves to proving (4.1.9), since (4.1.10) can be obtained applying (4.1.9) to the function $-u$, whereas the last part of the statement is a straightforward consequence of (4.1.9) and (4.1.10).

We make some reductions. First, we assume that $c_0 \le 0$. Indeed, in the case when $c_0 > 0$, it suffices to consider the function $\tilde{u} : [0,T] \times \mathbb{R}^N$ defined by $\tilde{u}(t,x) = e^{-c_0 t}u(t,x)$ for any $t \in [0,T]$ and any $x \in \mathbb{R}^N$, which satisfies (4.1.8) as well as the equation

$$D_t\tilde{u}(t,x) - \mathcal{A}_0\tilde{u}(t,x) = e^{-c_0 t}g(t,x), \qquad t \in (0,T], \ \ x \in \mathbb{R}^N,$$

where $\mathcal{A}_0 u = \mathcal{A}u - c_0 u$ has nonpositive zero-order coefficient and it satisfies Hypothesis 4.0.2 with the same Lyapunov function φ and the same λ_0. Thus, from the case $c_0 = 0$ it follows that

$$\tilde{u}(t,x) \le \max\left\{0, \sup_{\mathbb{R}^N} f\right\}, \qquad t \in [0,T], \ x \in \mathbb{R}^N,$$

which yields (4.1.9).

Besides, we can also assume that $\sup_{\mathbb{R}^N} f \le 0$. Indeed, in the case when $\sup_{\mathbb{R}^N} f > 0$, it suffices to consider the function $\overline{u} = u - \sup_{\mathbb{R}^N} f$ that has a nonpositive initial value and it satisfies the equation

$$D_t\overline{u}(t,x) - \mathcal{A}\overline{u}(t,x) = c(x)\sup_{\mathbb{R}^N} f + g(t,x), \qquad t \in (0,T], \ x \in \mathbb{R}^N,$$

where the right-hand side is nonpositive. Then, from the case $\sup_{\mathbb{R}^N} f \le 0$ we get $\overline{u} \le 0$, that is (4.1.9).

Taking these reductions into account, we introduce the function

$$v(t,x) = e^{-\lambda_0 t}u(t,x), \qquad t \in [0,T], \ \ x \in \mathbb{R}^N,$$

where λ_0 is as in Hypothesis 4.0.2, and, for any $k \in \mathbb{N}$, we introduce the functions $v_k : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$v_k(t,x) = v(t,x) - \frac{1}{k}\varphi(x), \qquad t \in [0,T], \ \ x \in \mathbb{R}^N.$$

We prove that

$$v_k(t,x) \le 0, \qquad t \in [0,T], \ \ x \in \mathbb{R}^N. \tag{4.1.11}$$

First we observe that, by Hypothesis 4.0.2,

$$D_t v_k(t,x) - (\mathcal{A} - \lambda_0)v_k(t,x) = e^{-\lambda_0 t}g(t,x) + \frac{1}{k}(\mathcal{A}\varphi(x) - \lambda_0\varphi(x)) \le 0, \tag{4.1.12}$$

for any $t \in (0,T]$ and any $x \in \mathbb{R}^N$. Moreover, by Hypothesis 4.0.2 and assumption (4.1.8) we deduce that for any $k \in \mathbb{N}$ the function v_k has a maximum point (t_k, x_k) in $[0,T] \times \mathbb{R}^N$. If $t_k = 0$ then (4.1.11) follows, since $f \le 0$ and $\varphi \ge 0$. If instead $t_k \in (0,T]$, then by Lemma 4.1.2 we have

$$(\mathcal{A} - c(x_k))v_k(t_k,x_k) \le 0, \qquad 0 \le D_t v_k(t_k,x_k),$$

which, combined with (4.1.12), yields

$$(\mathcal{A} - c(x_k))v_k(t_k, x_k) \le (\mathcal{A} - \lambda_0)v_k(t_k, x_k).$$

Since $\lambda_0 > c(x_k)$, we conclude that $v_k(t_k, x_k) \le 0$, and (4.1.11) follows.

Now, (4.1.11) implies that $v(t,x) \le k^{-1}\varphi(x)$ for any $t \in [0,T]$, any $x \in \mathbb{R}^N$ and any $k \in \mathbb{N}$. Letting k go to $+\infty$, we get $v \le 0$, and then also $u \le 0$, that is (4.1.9). ∎

Remark 4.1.4 The maximum principle in Theorem 4.1.3 implies, in particular, that, under Hypotheses 4.0.1 and 4.0.2, the function $u = T(\cdot)f$ is the unique classical solution of the Cauchy problem (4.0.3).

Next, we prove two maximum principles for the elliptic equation (4.0.1). The former holds for any $\lambda > c_0$ and gives uniqueness of the bounded solution, while the latter holds only in the case when $\lambda > \lambda_0$ (recall that $\lambda_0 > c_0$, see Hypothesis 4.0.2) but it ensures uniqueness also for unbounded solutions.

Theorem 4.1.5 *Let Hypotheses* 4.0.1, 4.0.2 *be satisfied and let* $u \in C(\mathbb{R}^N) \cap W^{2,p}_{\rm loc}(\mathbb{R}^N)$*, for any* $p \in [1, +\infty)$*, be such that* $\mathcal{A}u \in C(\mathbb{R}^N)$*. Further, fix* $\lambda > c_0$*. If* u *satisfies*

$$\sup_{x\in\mathbb{R}^N} u(x) < +\infty \tag{4.1.13}$$

and

$$\lambda u(x) - \mathcal{A}u(x) \le 0, \qquad x \in \mathbb{R}^N, \tag{4.1.14}$$

then $u \ge 0$*.*

Similarly, if u *satisfies*

$$\inf_{x\in\mathbb{R}^N} u(x) > -\infty \tag{4.1.15}$$

and

$$\lambda u(x) - \mathcal{A}u(x) \ge 0, \qquad x \in \mathbb{R}^N, \tag{4.1.16}$$

then $u \ge 0$*.*

In particular, for any $f \in C_b(\mathbb{R}^N)$ *and any* $\lambda > c_0$*, the function* $R(\lambda)f$ *(see Theorem* 2.1.3*) is the unique bounded solution to the elliptic equation* (4.0.1) *in* $D_{\max}(\mathcal{A})$*.*

Proof. We prove the first part of the statement since the second part follows applying the first one to the function $-u$, while the last part is a straightforward consequence of the two previous parts. Thus, we assume (4.1.13) and (4.1.14).

Consider the function $\tilde{u} : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$\tilde{u}(t,x) = e^{\lambda t}u(x), \qquad t \ge 0, \quad x \in \mathbb{R}^N.$$

As it is easily seen, $\tilde{u}$ satisfies the regularity assumptions in Theorem 4.1.3, the condition (4.1.8) and the equation

$$D_t\tilde{u}(t,x) - \mathcal{A}\tilde{u}(t,x) = e^{\lambda t}(\lambda u(x) - \mathcal{A}u(x)) \le 0, \qquad t>0, \quad x\in\mathbb{R}^N.$$

Therefore,

$$u(x) \le e^{(c_0-\lambda)t}\max\left\{0, \sup_{\mathbb{R}^N} u\right\}, \qquad t>0, \quad x\in\mathbb{R}^N.$$

Letting t go to $+\infty$, we conclude that $u\le 0$. ■

Theorem 4.1.6 *Assume Hypotheses* 4.0.1 *and* 4.0.2. *Further, assume that* $u\in C(\mathbb{R}^N)\cap W^{2,p}_{\rm loc}(\mathbb{R}^N)$, *for any* $p\in[1,+\infty)$, *is such that* $\mathcal{A}u\in C(\mathbb{R}^N)$,

$$\limsup_{|x|\to+\infty}\frac{u(x)}{\varphi(x)} \le 0 \tag{4.1.17}$$

and

$$\lambda u(x) - \mathcal{A}u(x) \le 0, \qquad x\in\mathbb{R}^N, \tag{4.1.18}$$

for some $\lambda\ge\lambda_0$. *Then* $u\le 0$.

If u *satisfies*

$$\liminf_{|x|\to+\infty}\frac{u(x)}{\varphi(x)} \ge 0 \tag{4.1.19}$$

and

$$\lambda u(x) - \mathcal{A}u(x) \ge 0, \quad x\in\mathbb{R}^N, \tag{4.1.20}$$

for some $\lambda\ge\lambda_0$, *then* $u\ge 0$.

Proof. We prove the first part of the statement; then the second part follows applying the first one to the function $-u$.

Without loss of generality we can assume that $c_0\le 0$. Indeed, in the case when $c_0>0$, it suffices to replace the operator $\mathcal{A}$ with the operator $\mathcal{A}_0=\mathcal{A}-c_0$ which has a nonpositive zero order coefficient. Then, the Lyapunov function φ satisfies

$$\mathcal{A}_0\varphi(x) - (\lambda_0 - c_0)\varphi(x) \le 0, \quad x\in\mathbb{R}^N,$$

and the function u satisfies

$$(\lambda - c_0)u(x) - \mathcal{A}_0u(x) \le 0, \quad x\in\mathbb{R}^N.$$

Therefore, from the case $c_0\le 0$ it follows that $u\le 0$.

So, let us assume that $c_0\le 0$. Then, Hypothesis 4.0.2 implies that, for any $\lambda\ge\lambda_0$,

$$\mathcal{A}\varphi(x) - \lambda\varphi(x) \le 0, \quad x\in\mathbb{R}^N. \tag{4.1.21}$$

Now, for any $k \in \mathbb{N}$, we consider the function

$$u_k(x) = u(x) - \frac{1}{k}\varphi(x), \qquad x \in \mathbb{R}^N.$$

By (4.1.18) and (4.1.21) we have

$$\lambda u_k(x) - \mathcal{A}u_k(x) \le 0, \qquad x \in \mathbb{R}^N. \tag{4.1.22}$$

Moreover, by (4.1.17) and Hypothesis 4.0.2, the function u_k has a maximum point x_k in $\mathbb{R}^N$. By Lemma 4.1.2,

$$(\mathcal{A} - c(x_k))u_k(x_k) \le 0,$$

which, jointly with (4.1.22), yields

$$\lambda u_k(x_k) \le c(x_k)u_k(x_k).$$

Since $\lambda > c(x_k)$, it follows that $u_k(x_k) \le 0$. This means that $u_k(x) \le 0$ for any $x \in \mathbb{R}^N$ and any $k \in \mathbb{N}$ and, therefore, $u \le 0$. ■

4.1.2 The case when $c \equiv 0$

We now consider the case when $c \equiv 0$. Note that in this case both the functions $u, v : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ defined by $u(t, x) = (T(\cdot)\mathbb{1})(x)$ and $v(t, x) = 1$ for any $t \in [0, +\infty)$ and any $x \in \mathbb{R}^N$ are bounded solutions of the parabolic problem (4.0.3) with initial value $f \equiv \mathbb{1}$. In general, they can differ at some point $x \in \mathbb{R}^N$. In any case, we have $0 \le T(t)\mathbb{1} \le \mathbb{1}$ for any $t > 0$, as follows from Remark 2.2.3.

With this remark in mind we give the following definition.

Definition 4.1.7 *We say that $\{T(t)\}$ is conservative, or that the conservation of probability holds, if $T(t)\mathbb{1} \equiv \mathbb{1}$ for any $t > 0$.*

Let us prove the following preliminary result.

Lemma 4.1.8 *Assume Hypotheses 4.0.1 and let $c \equiv 0$. For any fixed $x \in \mathbb{R}^N$ the functions $t \mapsto (T(t)\mathbb{1})(x)$ and $\lambda \mapsto \lambda(R(\lambda)\mathbb{1})(x)$ are, respectively, decreasing and increasing in $(0, +\infty)$.*

Proof. To prove the first part of the lemma, we observe that for any $t, s > 0$ and any $x \in \mathbb{R}^N$, we have $(T(t + s)\mathbb{1})(x) = (T(t)T(s)\mathbb{1})(x) \le (T(t)\mathbb{1})(x)$. The inequality follows easily observing that, since $T(t)$ maps nonnegative functions into nonnegative functions for any $t > 0$, then, for any pair of functions f_1 and f_2 with $f_1 \le f_2$, it holds that $T(t)f_1 \le T(t)f_2$ for any $t > 0$.

To prove the last part, we begin by observing that, according to Theorem 2.1.3, $0 \le \lambda R(\lambda)\mathbb{1} \le \mathbb{1}$ for any $\lambda > 0$. Let us now fix $0 < \lambda < \mu$. From the resolvent identity (2.1.5) we have

$$\begin{aligned}\lambda R(\lambda)\mathbb{1} - \mu R(\mu)\mathbb{1} &= \lambda(R(\mu)\mathbb{1} + (\mu-\lambda)R(\mu)R(\lambda)\mathbb{1}) - \mu R(\mu)\mathbb{1}\\ &= (\lambda-\mu)R(\mu)\mathbb{1} + \lambda(\mu-\lambda)R(\mu)R(\lambda)\mathbb{1}\\ &\le (\lambda-\mu)R(\mu)\mathbb{1} + (\mu-\lambda)R(\mu)\mathbb{1} = 0.\end{aligned}$$

∎

Remark 4.1.9 As the proof shows, the lemma holds also in the case when $c \le 0$. We will use this generalization of Lemma 4.1.8 in Chapter 5.

Now, we can prove the following proposition.

Proposition 4.1.10 *Assume Hypotheses* 4.0.1 *and suppose that* $c \equiv 0$. *Then, the following conditions are equivalent:*

(*i*) $T(t)\mathbb{1} = \mathbb{1}$ *for some* $t > 0$;

(*ii*) $\{T(t)\}$ *is conservative;*

(*iii*) $R(\lambda)\mathbb{1} = \mathbb{1}/\lambda$ *for some* $\lambda > 0$;

(*iv*) $R(\lambda)\mathbb{1} = \mathbb{1}/\lambda$ *for any* $\lambda > 0$;

(*v*) *for any* $f \in C_b(\mathbb{R}^N)$, *the function* $u = T(\cdot)f$ *is the unique solution of the parabolic problem* (4.0.3) *which belongs to* $C_b([0,T]\times\mathbb{R}^N)\cap C^{1,2}((0,T)\times\mathbb{R}^N)$ *for any* $T > 0$;

(*vi*) *for any* $\lambda > 0$ *and any* $f \in C_b(\mathbb{R}^N)$, *the function* $u = R(\lambda)f$ *is the unique solution of the elliptic equation* (4.0.1) *in* $D_{\max}(\mathcal{A})$.

Proof. Showing that "(ii) ⇒ (i)", "(iv) ⇒ (iii)" and "(vi) ⇒ (iv)" is trivial, while the implication "(v) ⇒ (vi)" has been shown in Proposition 4.1.1. Hence, we can limit ourselves to showing that "(i) ⇒ (ii)", "(iii) ⇒ (ii)" and "(ii) ⇒ (v)".

"(i) ⇒ (ii)". Fix $x \in \mathbb{R}^N$. If $T(t_0)\mathbb{1} \equiv \mathbb{1}$ for some $t_0 > 0$, then, according to the semigroup law, the function $t \mapsto (T(t)\mathbb{1})(x)$ is periodic, with period t_0. By Lemma 4.1.8 it is also decreasing. Hence, it is constant.

"(iii) ⇒ (ii)" follows from (2.2.14). Indeed, if $(T(t)\mathbb{1})(x)$ were less than 1 for some $t > 0$ and $x \in \mathbb{R}^N$, from (2.2.14) it would follow that $(R(\lambda)\mathbb{1})(x) < 1/\lambda$.

"(ii) ⇒ (v)". Let u be a solution of the parabolic problem (4.0.3), with $f \equiv 0$, which belongs to $C_b([0,T]\times\mathbb{R}^N)\cap C^{1,2}((0,T)\times\mathbb{R}^N)$ for any $T > 0$. We prove that $u \equiv 0$. On the contrary, suppose that $u \not\equiv 0$. By changing sign and multiplying u by a suitable constant if needed, we can assume that

$$\sup_{(0,T)\times\mathbb{R}^N} u > 0, \qquad \sup_{(0,T)\times\mathbb{R}^N} |u| < 1.$$

Set $v = \mathbb{1} - u$. Then $v(t,x) > 0$ for any $t > 0$ and any $x \in \mathbb{R}^N$, and

$$\inf_{(0,T)\times\mathbb{R}^N} v < 1.$$

Moreover, v is a bounded solution of the problem (4.0.3) with initial value $v(0,\cdot) = \mathbb{1}$. Then, by Remark 2.2.3, we have $T(\cdot)\mathbb{1} \le v$. This implies that $\inf_{\mathbb{R}^N} T(t)\mathbb{1} < \mathbb{1}$ for some $t \in (0,T)$, which is in contradiction with statement (ii). ∎

The following proposition is an immediate consequence of the formula (2.2.8) with $f \equiv \mathbb{1}$.

Proposition 4.1.11 *Let $\{p(t,x;dy) : \ t > 0,\ x \in \mathbb{R}^N\}$ be the transition function associated with the semigroup $\{T(t)\}$ (see Theorem 2.2.5). Then, $\{T(t)\}$ is conservative if and only if $p(t,x;\mathbb{R}^N) = 1$ for any $t > 0$ and any $x \in \mathbb{R}^N$.*

Similarly, taking Theorem 4.1.3 into account, the following sufficient condition, ensuring that $\{T(t)\}$ is conservative, follows.

Proposition 4.1.12 *Suppose that $c \equiv 0$ and assume Hypotheses 4.0.1 and 4.0.2. Then, the conservation of probability holds.*

Example 4.1.13 Consider the operator $\mathcal{A}$ defined on smooth functions by

$$\mathcal{A}f = \Delta f + \langle b(\cdot), Df\rangle,$$

where $b \in C^\alpha_{\rm loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Choosing $\varphi(x) = \log(m + |x|^2)$ for some $m > 1$, the condition (4.0.5) becomes

$$\langle b(x), x\rangle \le M(m + |x|^2)\log(m + |x|^2), \qquad x \in \mathbb{R}^N, \tag{4.1.23}$$

for some $M > 0$ (see (4.0.6)). Therefore, if (4.1.23) holds the corresponding semigroup $\{T(t)\}$ is conservative.

For instance, this is the case when $b(x) = -x|x|^2$ for any $x \in \mathbb{R}^N$. Note that the one-dimensional version of the operator $\mathcal{A}$, with $b(x) = -x^3$ for any $x \in \mathbb{R}$, has been already considered in Chapter 3 (see Example 3.2.4), where, in fact, we proved, with a different argument, that the associated semigroup is conservative.

Remark 4.1.14 Let us notice that the papers [9, 19, 50] contain some uniqueness results for the classical solution to the problem (4.0.3), with a prescribed growth rate at infinity. In particular such results apply to prove the uniqueness of the bounded solution to the problem (4.0.3). We stress that the results in [9, 19, 50] apply to operators with diffusion coefficients with an at most quadratic growth at infinity.

4.2 Nonuniqueness

By the results in Chapter 3 (see Theorem 3.2.2), in the one-dimensional setting, one can give necessary and sufficient conditions for the elliptic equation (4.0.1) to be not uniquely solvable in $D_{\max}(\mathcal{A})$. Such conditions can be expressed in terms of integrability properties at $\pm\infty$ of the functions Q and R in (3.1.4) and (3.1.5).

In this section we prove a sufficient condition for the nonuniqueness of the solution $v \in D_{\max}(\mathcal{A})$ of the elliptic equation (4.0.1) in the N-dimensional setting (see [83, Section 3.4], or [116, Remark 3.16]). According to Proposition 4.1.1, this result will imply that the parabolic problem (4.0.3) admits several solutions belonging to $C([0,+\infty)\times\mathbb{R}^N)\cap C^{1,2}((0,+\infty)\times\mathbb{R}^N)$ which are bounded in $[0,T]\times\mathbb{R}^N$ for any $T>0$. Here, we do not assume that $c\equiv 0$.

Proposition 4.2.1 *Suppose that there exists a nonidentically vanishing function $\varphi \in C^2(\mathbb{R}^N)$, such that*

$$0 \le \varphi \le C, \qquad \lambda\varphi - \mathcal{A}\varphi \le 0, \tag{4.2.1}$$

for some $\lambda > c_0$ and some $C > 0$. Then, there exists a bounded, continuous and nonidentically vanishing solution of the equation $\lambda u - \mathcal{A}u = 0$.

Proof. For any $n \in \mathbb{N}$ consider the solution $u_n \in C(\overline{B}(n))$ of the Dirichlet problem

$$\begin{cases} \lambda u_n(x) - \mathcal{A}u_n(x) = 0, & x \in B(n), \\ u_n(x) = \|\varphi\|_\infty, & x \in \partial B(n). \end{cases}$$

From (4.2.1) and the maximum principle it follows that

$$\varphi \le u_n \le \|\varphi\|_\infty. \tag{4.2.2}$$

Moreover, $\{u_n\}$ is a decreasing sequence. Indeed, the function $v = u_{n+1} - u_n$ solves the problem

$$\begin{cases} \lambda v(x) - \mathcal{A}v(x) = 0, & x \in B(n), \\ v(x) = u_{n+1}(x) - \|\varphi\|_\infty \le 0, & x \in \partial B(n) \end{cases}$$

and, then, it is negative, still by virtue of the maximum principle. Thus, we can define the function $u : \mathbb{R}^N \to \mathbb{R}$ by setting

$$u(x) := \lim_{n\to+\infty} u_n(x), \qquad x \in \mathbb{R}^N.$$

By (4.2.2), u is bounded. Taking the *a priori* interior estimates of Theorem C.1.1 into account and arguing as in the proof of Theorem 2.1.1, it follows that u is a continuous and nonidentically vanishing solution of the equation $\lambda u - \mathcal{A}u = 0$ in $\mathbb{R}^N$.

Remark 4.2.2 In Section 5.2 we will see other conditions implying nonuniqueness of the bounded solutions. Actually, we will see sufficient conditions ensuring that $T(t)(C_b(\mathbb{R}^N)) \subset C_0(\mathbb{R}^N)$ for any $t > 0$. It is clear that, in such a situation when $c \equiv 0$, uniqueness does not hold, since we have $T(t)\mathbb{1} \not\equiv \mathbb{1}$.

Chapter 5

Properties of $\{T(t)\}$ in spaces of continuous functions

5.0 Introduction

In this chapter we study several properties of the semigroup $\{T(t)\}$ in $C_b(\mathbb{R}^N)$. More precisely, we deal with: the compactness of the semigroup in $C_b(\mathbb{R}^N)$, the inclusion $T(t)(C_b(\mathbb{R}^N)) \subset C_0(\mathbb{R}^N)$ and the invariance of $C_0(\mathbb{R}^N)$ under the action of the semigroup.

Compactness. In Section 5.1 we study the compactness of $\{T(t)\}$ in $C_b(\mathbb{R}^N)$. First, we consider the conservative case and, then, the nonconservative one. In the conservative case $T(t)$ is a compact operator, for any $t > 0$, provided that the family $\{p(t,x;dy) : x \in \mathbb{R}^N\}$ is tight for any $t > 0$ (see Definition 5.1.2). Here, $\{p(t,x;dy) : t > 0,\ x \in \mathbb{R}^N\}$ is the transition family defined in Theorem 2.2.5. In such a case the function $t \mapsto T(t)$ is continuous in $(0,+\infty)$ with respect to the operator topology.

We show that the family $\{p(t,x;dy) : x \in \mathbb{R}^N\}$ is tight for any $t > 0$ if there exist a strictly positive function $\varphi \in C^2(\mathbb{R}^N)$ and a convex function $g : [0,+\infty) \to \mathbb{R}$ such that

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty, \quad 1/g \text{ is integrable at } +\infty, \ \ \mathcal{A}\varphi(x) \le -g(\varphi(x)), \ \ x \in \mathbb{R}^N.$$

It is not difficult to give sufficient conditions on the coefficients of the operator $\mathcal{A}$ guaranteeing that the previous conditions are satisfied. For instance, this is the case of the semigroup generated by the operator

$$\mathcal{A}\varphi(x) = \Delta\varphi(x) - |x|^2\langle x, D\varphi(x)\rangle, \qquad x \in \mathbb{R}^N.$$

In the nonconservative case, we show that there are some connections between the compactness of $T(t)$ $(t > 0)$ and the fact that $T(t)$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$. More precisely, we show that $T(t)$ is compact and maps $C_0(\mathbb{R}^N)$ into itself, for some $t > 0$, if and only if $T(t)\mathbb{1}$ belongs to $C_0(\mathbb{R}^N)$. In such a case (see Proposition 5.1.12), the resolvent operator $R(\lambda, \widehat{A})$ (see Sections 2.1 and 2.3 for the definitions of $\widehat{A}$ and the resolvent operator $R(\lambda, \widehat{A})$) is compact for any $\lambda > 0$ and the semigroup is norm-continuous.

The inclusion $T(t)(C_b(\mathbb{R}^N)) \subset C_0(\mathbb{R}^N)$. It is not always true that $T(t)$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ for any $t > 0$, since it is not always true that

$C_0(\mathbb{R}^N)$ is invariant under $\{T(t)\}$. For instance, if, for some $t > 0$, $T(t)$ is conservative and compact, then $C_0(\mathbb{R}^N)$ is not invariant under $T(t)$. In fact, $T(t)$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$, for some $t > 0$, if and only if $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$.

As it is shown in Proposition 5.2.2, $T(t)$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ for any $t > 0$, if and only if the resolvent operator $R(\lambda) = R(\lambda, \widehat{A})$ does, for some (and, hence, for any) $\lambda > 0$ or, equivalently, if and only if $C_0(\mathbb{R}^N) \cap D_{\max}(\mathcal{A})$ is the domain of the weak generator $\widehat{A}$ of the semigroup $\{T(t)\}$.

A sufficient condition implying that $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$ is given in Proposition 5.2.4 in the case when there exists a function $\varphi \in C^2(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, which is positive outside a compact subset K and satisfies

$$\inf_{x \in \mathbb{R}^N \setminus K} (\lambda\varphi(x) - \mathcal{A}\varphi(x)) = a > 0,$$

for some $\lambda > c_0$, where, as usual, $c_0 = \sup_{x \in \mathbb{R}^N} c(x)$.

Invariance of $C_0(\mathbb{R}^N)$. As it has already been remarked above, it is not always true that $T(t)(C_0(\mathbb{R}^N)) \subset C_0(\mathbb{R}^N)$ for any $t > 0$. This occurs, for instance, when there exists a strictly positive function $\varphi \in C^2(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ such that

$$\lambda\varphi - \mathcal{A}\varphi \geq 0,$$

for some $\lambda > 0$. In such a case the operator A_0, defined by $A_0 u = \mathcal{A}u$ for any $u \in D(A_0) = \{u \in D_{\max}(\mathcal{A}) \cap C_0(\mathbb{R}^N) : \mathcal{A}u \in C_0(\mathbb{R}^N)\}$ (i.e., the part of $\widehat{A}$ in $C_0(\mathbb{R}^N)$), is the generator of a strongly continuous semigroup.

5.1 Compactness of $\{T(t)\}$

In this section we study the compactness of the semigroup $\{T(t)\}$ in the space $C_b(\mathbb{R}^N)$ of bounded and continuous functions. The results that we present here are taken from [44] and [115]. In this and in the subsequent sections we assume that the coefficients of the operator $\mathcal{A}$, defined on smooth functions by

$$\mathcal{A}u = \sum_{i,j=1}^N q_{ij} D_{ij} u + \sum_{j=1}^N b_j D_j u + cu,$$

satisfy the same hypotheses of Chapter 2. For the reader's convenience we state them again.

Hypotheses 5.1.1 (i) $q_{ij} \equiv q_{ji}$,

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \geq \kappa(x)|\xi|^2, \qquad \kappa(x) > 0, \qquad \xi, x \in \mathbb{R}^N;$$

(ii) q_{ij}, b_i $(i,j=1,\ldots,N)$ and c belong to $C^{\alpha}_{\rm loc}(\mathbb{R}^N)$ for some $\alpha\in(0,1)$;

(iii) there exists $c_0\in\mathbb{R}$ such that

$$c(x)\le c_0, \qquad x\in\mathbb{R}^N.$$

We consider both the cases when the semigroup is conservative and non-conservative.

5.1.1 The conservative case

We begin this subsection recalling that the semigroup $\{T(t)\}$ is conservative whenever $T(t)\mathbb{1}=\mathbb{1}$ for any $t>0$ or, equivalently, whenever the elliptic equation

$$\lambda u(x)-\mathcal{A}u(x)=f(x),\quad x\in\mathbb{R}^N, \tag{5.1.1}$$

and the Cauchy problem

$$\begin{cases} D_t u(t,x)-\mathcal{A}u(t,x)=0, & t>0,\ x\in\mathbb{R}^N,\\ u(0,x)=f(x), & x\in\mathbb{R}^N, \end{cases} \tag{5.1.2}$$

are uniquely solvable, respectively, in

$$D_{\max}(\mathcal{A})=\left\{u\in C_b(\mathbb{R}^N)\cap\bigcap_{1\le p<+\infty}W^{2,p}_{\rm loc}(\mathbb{R}^N):\ \mathcal{A}u\in C_b(\mathbb{R}^N)\right\}$$

and in $C_b([0,T]\times\mathbb{R}^N)\cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\mathbb{R}^N)$, for any $T>0$ (see Proposition 4.1.10). Of course, if $\{T(t)\}$ is conservative then, necessarily, $c\equiv 0$.

We also recall that we can associate a transition family $\{p(t,x;dy):t>0,\ x\in\mathbb{R}^N\}$ with the semigroup (see Definition 2.2.4 and Theorem 2.2.5). The connection between the semigroup and the transition family is explained by the formula (2.2.8) which allows us to write $T(t)f$ for any $t>0$ and any $f\in C_b(\mathbb{R}^N)$ in terms of the Borel measures $p(t,x;dy)$. For the reader's convenience we rewrite here such a formula:

$$(T(t)f)(x)=\int_{\mathbb{R}^N}f(y)p(t,x;dy),\qquad t>0,\ x\in\mathbb{R}^N,\qquad f\in C_b(\mathbb{R}^N). \tag{5.1.3}$$

With all these remarks in mind we can now show that the compactness of the semigroup $\{T(t)\}$ is equivalent to the tightness of the family of measures $\{p(t,x;\cdot):\ x\in\mathbb{R}^N\}$ for any $t>0$, and it implies that the semigroup is norm-continuous in $(0,+\infty)$ (see Definition B.2.4).

Definition 5.1.2 *A family of Borel probability measures $\{\mu_\alpha\}_{\alpha\in\mathcal{F}}$ is said to be tight if for any $\varepsilon>0$ there exists $\rho>0$ such that $\mu_\alpha(B(\rho))\ge 1-\varepsilon$ for any $\alpha\in\mathcal{F}$.*

Proposition 5.1.3 *Assume that the semigroup $\{T(t)\}$ is conservative. Then, $T(t)$ is compact in $C_b(\mathbb{R}^N)$, for any $t>0$, if and only if the family of measures $\{p(t,x;\cdot) : x\in\mathbb{R}^N\}$ is tight for any $t>0$. In this case the semigroup is norm-continuous in $(0,+\infty)$, and the resolvent $R(\lambda,\widehat{A})$ is compact for any $\lambda>0$.*

Proof. Fix $t,\varepsilon>0$ and suppose that $T(t)$ is compact. Let $\{f_n\}$ be a sequence of bounded and continuous functions in $\mathbb{R}^N$ such that $\chi_{B(n-1)}\le f_n\le\chi_{B(n)}$ for any $n\in\mathbb{N}$. By Proposition 2.2.9 we deduce that $T(t)f_n$ tends to $T(t)\mathbb{1}=1$ as n tends to $+\infty$, locally uniformly in $\mathbb{R}^N$. Since $T(t)$ is compact, then $T(t)f_n$ tends to $\mathbb{1}$ uniformly in $\mathbb{R}^N$. Indeed, if this were not the case one could find out a positive number ε_0 and a subsequence $\{f_{n_k}\}$ such that

$$\|T(t)f_{n_k}-\mathbb{1}\|_\infty\ge\varepsilon_0.$$

Hence, no subsequences uniformly converging in $\mathbb{R}^N$ could be extracted from the sequence $\{T(t)f_{n_k}\}$, contradicting the compactness of $T(t)$. It follows that the family of measures $\{p(t,x;\cdot) : x\in\mathbb{R}^N\}$ is tight for any $t>0$, since (see (5.1.3))

$$p(t,x;B(n))\ge(T(t)f_n)(x),\qquad x\in\mathbb{R}^N.$$

Conversely, assume that, for any $t>0$, the family $\{p(t,x;\cdot) : x\in\mathbb{R}^N\}$ is tight. Fix $t>0$ and let $\rho_n>0$ be such that

$$p(t/2,x;B(\rho_n))\ge 1-n^{-1},\qquad x\in\mathbb{R}^N. \tag{5.1.4}$$

Define the operator $S_{t,n}$ in $C_b(\mathbb{R}^N)$ by

$$(S_{t,n}f)(x)=\int_{B(\rho_n)}(T(t/2)f)(y)p(t/2,x;dy),$$

for any $x\in\mathbb{R}^N$ and any $f\in C_b(\mathbb{R}^N)$. By (5.1.4), we have

$$|(T(t)f)(x)-(S_{t,n}f)(x)|\le\int_{\mathbb{R}^N\setminus B(\rho_n)}|(T(t/2)f)(y)|p(t/2,x;dy)\le n^{-1}\|f\|_\infty, \tag{5.1.5}$$

for any $f\in C_b(\mathbb{R}^N)$ and any $x\in\mathbb{R}^N$, since $\{T(t)\}$ is a semigroup of contractions (see (2.2.7)). Moreover, $S_{t,n}$ is compact for any $n\in\mathbb{N}$. Indeed, we have

$$S_{t,n}=G_n\circ R_n\circ T(t/2),\qquad n\in\mathbb{N},$$

where $R_n : C_b(\mathbb{R}^N)\to C(\overline{B}(\rho_n))$ is the restriction operator (i.e., $R_nf(x)=f(x)$ for any $x\in\overline{B}(\rho_n)$ and any $f\in C_b(\mathbb{R}^N)$) and $G_n : C(\overline{B}(\rho_n))\to C_b(\mathbb{R}^N)$ is the bounded operator defined by

$$(G_nh)(x)=\int_{B(\rho_n)}h(y)p(t/2,x;dy)=T(t/2)(h\chi_{B(n)}),\qquad x\in\mathbb{R}^N,$$

for any $h \in C(\overline{B}(\rho_n))$. We observe that $G_n h \in C_b(\mathbb{R}^N)$ for any $h \in C(\overline{B}(\rho_n))$, since $\{T(t)\}$ has the strong Feller property (see Proposition 2.2.12). By the interior Schauder estimates in Theorem C.1.4, we deduce that $R_n \circ T(t/2)$ is bounded from $C_b(\mathbb{R}^N)$ into $C^{2+\alpha}(\overline{B}(\rho_n))$, and the Ascoli-Arzelà Theorem implies that $R_n \circ T(t/2)$ is compact. It follows that $S_{t,n}$ is compact as well. Since $\{S_{t,n}\}$ is a sequence of compact operators converging to $T(t)$ in the operator topology (see (5.1.5)), we deduce that $T(t)$ is, itself, a compact operator. Due to the arbitrariness of $t > 0$, it follows that $T(t)$ is compact for any $t > 0$.

Let us now prove that $T(t)$ is norm-continuous for any $t > 0$. We begin by showing that $T(t)f$ is continuous in $(0, +\infty)$ for any $f \in C_b(\mathbb{R}^N)$ with $\|f\|_\infty \le 1$. For this purpose, we fix $\varepsilon > 0$, $t_0 > 0$ and let ρ be sufficiently large such that

$$p(t_0, x; \mathbb{R}^N \setminus B(\rho)) \le \varepsilon, \qquad x \in \mathbb{R}^N.$$

Then, for any $h > 0$ it holds that

$$\begin{aligned}
&(T(t_0 + h)f)(x) - (T(t_0)f)(x) \\
&= \int_{\mathbb{R}^N} f(y) p(t_0 + h, x; dy) - \int_{\mathbb{R}^N} f(y) p(t_0, x; dy) \\
&= \int_{B(\rho)} ((T(h)f)(y) - f(y))\, p(t_0, x; dy) \\
&\quad + \int_{\mathbb{R}^N \setminus B(\rho)} ((T(h)f)(y) - f(y))\, p(t_0, x; dy) \\
&\le \|T(h)f - f\|_{C(\overline{B}(\rho))} + 2\varepsilon
\end{aligned} \tag{5.1.6}$$

and the right-hand side of (5.1.6) is less than 3ε provided h is sufficiently small since $T(\cdot)f \in C([0, +\infty) \times \mathbb{R}^N)$ and $T(0)f = f$ (see Theorems 2.2.1, 2.2.5). Hence,

$$\lim_{h \to 0^+} \|T(t_0 + h)f - T(t_0)f\|_\infty = 0.$$

For negative values of h we fix $\delta \in (0, t_0)$. Then, for any $h \in (\delta - t_0, 0)$ we split

$$T(t_0 + h)f - T(t_0)f = T(t_0 + h - \delta)(T(\delta)f - T(\delta - h)f).$$

Hence,

$$\begin{aligned}
\|T(t_0 + h)f - T(t_0)f\|_\infty &= \|T(t_0 + h - \delta)\,(T(\delta)f - T(\delta - h)f)\,\|_\infty \\
&\le \|T(\delta)f - T(\delta - h)f\|_\infty
\end{aligned}$$

and letting h go to 0^-, by the previous step, we get

$$\lim_{h \to 0^-} T(t_0 + h)f = T(t_0)f. \tag{5.1.7}$$

Hence,

$$\lim_{h \to 0} \|T(t_0 + h)f - T(t_0)f\|_\infty = 0, \qquad t > 0. \tag{5.1.8}$$

We can now show that (5.1.8) and the compactness of the semigroup imply that $t \mapsto T(t)$ is norm-continuous in $(0,+\infty)$. Suppose by contradiction that $T(t)$ is not norm-continuous in $(0,+\infty)$. Then, we can find out $t_0, \varepsilon > 0$ two sequences $\{f_n\} \subset C_b(\mathbb{R}^N)$, with $\|f_n\|_\infty = 1$, for any $n \in \mathbb{N}$, and $\{t_n\} \subset (0,+\infty)$, tending to t_0 as n tends to $+\infty$, such that

$$\|(T(t_n) - T(t_0))f_n\|_\infty \geq \varepsilon, \qquad n \in \mathbb{N}. \tag{5.1.9}$$

Without loss of generality, we can assume that $\{t_n\}$ is a monotone sequence. We first assume that it is decreasing and we set $a_n = t_n - t_0$ for any $n \in \mathbb{N}$. Since $T(t_0)$ is a compact operator, up to a subsequence we can assume that $T(t_0)f_n$ converges uniformly to a function $g \in C_b(\mathbb{R}^N)$ as n tends to $+\infty$. Hence, from (5.1.9), we get

$$\begin{aligned}\|(T(a_n) - I)g\|_\infty &\geq \|(T(a_n) - I)T(t_0)f_n\|_\infty - \|(T(a_n) - I)(g - T(t_0)f_n)\|_\infty\\ &\geq \varepsilon - 2\|g - T(t_0)f_n\|_\infty.\end{aligned}$$

Letting n go to $+\infty$, we get

$$\limsup_{n\to+\infty} \|(T(a_n) - I)g\|_\infty \geq \varepsilon,$$

which contradicts (5.1.8). Therefore, $\lim_{t\to t_0^+} T(t) = T(t_0)$ in $L(C_b(\mathbb{R}^N))$. Now to prove that $T(t)$ is norm continuous from the left in $(0,+\infty)$, it suffices to argue as in the proof of (5.1.7).

To conclude the proof, we observe that, since $T(t)$ is norm-continuous and bounded in $(0,+\infty)$, the integral

$$\int_0^{+\infty} e^{-\lambda t} T(t)dt$$

converges in the operator topology to $R(\lambda)$, for any $\lambda > 0$. Since $T(t)$ is compact for any $t > 0$, the operator $R(\lambda)$ is compact as well. ∎

Now, we prove a lemma.

Lemma 5.1.4 *Suppose that $\{T(t)\}$ is conservative. Let $\varphi \in C^2(\mathbb{R}^N)$ be any function such that*

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty, \qquad \mathcal{A}\varphi(x) \leq K, \qquad x \in \mathbb{R}^N,$$

for some $K > 0$. Then, we have

$$(T(t)\varphi)(x) := \int_{\mathbb{R}^N} \varphi(y)p(t,x;dy) \in \mathbb{R}, \tag{5.1.10}$$

$$(T(t)\mathcal{A}\varphi)(x) := \int_{\mathbb{R}^N} \mathcal{A}\varphi(y)p(t,x;dy) \in \mathbb{R}, \tag{5.1.11}$$

for any $t>0$ and any $x\in\mathbb{R}^N$. Moreover, the function $T(\cdot)\varphi$ belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\mathbb{R}^N)\cap C([0,+\infty)\times\mathbb{R}^N)$ and

$$\lim_{t\to 0^+}(T(t)\varphi)(x)=\varphi(x),\qquad \left(\frac{\partial}{\partial t}T(t)\varphi\right)(x)\le (T(t)\mathcal{A}\varphi)(x),\qquad t>0,\ x\in\mathbb{R}^N. \tag{5.1.12}$$

Proof. Without loss of generality, we can suppose that $\varphi\ge 0$. Indeed, if this is not the case, we replace φ with $\varphi-\inf_{x\in\mathbb{R}^N}\varphi(x)$. Observe that $\inf_{x\in\mathbb{R}^N}\varphi(x)$ is a real number, since φ goes to $+\infty$ as $|x|$ goes to $+\infty$. Moreover, $\mathcal{A}\varphi=\mathcal{A}(\varphi-\inf_{x\in\mathbb{R}^N}\varphi(x))$, since the zero-order coefficient of $\mathcal{A}$ identically vanishes in $\mathbb{R}^N$.

For any $n\in\mathbb{N}$, let $\psi_n\in C^2_b([0,+\infty))$ be any function such that

$$\psi_n(s)=s,\quad s\in[0,n],\qquad \psi_n(s)=n+1,\quad s\ge n+2,$$
$$0\le\psi_n'(s)\le 1,\quad \psi_n''(s)\le 0\quad s\in[0,+\infty).$$

Then $\psi_n\circ\varphi$ is bounded and, in particular, it is constant outside some compact set of $\mathbb{R}^N$. Consequently, the function $\mathcal{A}(\psi_n\circ\varphi)$ has compact support. This implies that $\psi_n\circ\varphi\in D_{\max}(\mathcal{A})$. By Propositions 2.3.6 and 4.1.10, we know that $D_{\max}(\mathcal{A})=D(\widehat{A})$ (see Section 2.3). Hence, by Lemma 2.3.3 we deduce that

$$T(t)\mathcal{A}(\psi_n\circ\varphi)=T(t)\widehat{A}(\psi_n\circ\varphi)=\widehat{A}T(t)(\psi_n\circ\varphi)=\mathcal{A}T(t)(\psi_n\circ\varphi),\quad t\ge 0. \tag{5.1.13}$$

Now, let

$$u_n(t,x)=(T(t)(\psi_n\circ\varphi))(x):=\int_{\mathbb{R}^N}\psi_n(\varphi(y))p(t,x;dy),\qquad t>0,\ x\in\mathbb{R}^N.$$

Thanks to our assumptions on the functions ψ_n ($n\in\mathbb{N}$), we easily deduce that the sequence $\{\psi_n(s)\}$ is increasing to $\varphi(s)$ for any $s\in\mathbb{R}$. Hence, the monotone convergence theorem yields

$$\lim_{n\to\infty}u_n(t,x)=\lim_{n\to+\infty}\int_{\mathbb{R}^N}\psi_n(\varphi(y))p(t,x;dy)=u(t,x):=(T(t)\varphi)(x), \tag{5.1.14}$$

for any $t>0$ and any $x\in\mathbb{R}^N$. Now, from (5.1.13) we deduce that, for any $s>0$ and any $x\in\mathbb{R}^N$,

$$\begin{aligned}D_tu_n(s,x)&=\mathcal{A}u_n(s,x)=(T(s)\mathcal{A}(\psi_n\circ\varphi))(x)\\&=\int_{\mathbb{R}^N}\{(\psi_n'\circ\varphi)\mathcal{A}\varphi+(\psi_n''\circ\varphi)\langle QD\varphi,D\varphi\rangle\}\,p(s,x;dy)\\&\le\int_{\mathbb{R}^N}(\psi_n'\circ\varphi)\mathcal{A}\varphi p(s,x;dy)\le K,\end{aligned} \tag{5.1.15}$$

where $Q(y) = (q_{ij}(y))$. Integrating (5.1.15) with respect to $s \in [0,t]$, and recalling that the semigroup is positivity preserving, gives

$$0 \le u_n(t,x) \le \psi_n(\varphi(x)) + Kt \le \varphi(x) + Kt, \qquad t>0, \quad x \in \mathbb{R}^N, \quad (5.1.16)$$

for any $n \in \mathbb{N}$. Then, letting n go to $+\infty$, we get

$$0 \le u(t,x) \le \varphi(x) + Kt < +\infty, \qquad t>0, \quad x \in \mathbb{R}^N,$$

and (5.1.10) follows.

We now prove (5.1.11) and (5.1.12) and, at the same time, we show that $u \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\mathbb{R}^N)$. By (5.1.16) and Theorem C.1.4, the sequence $\{u_n\}$ is bounded in $C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times F)$ for any $\varepsilon, T>0$ and any compact set $F \subset \mathbb{R}^N$. From (5.1.14) and the Ascoli-Arzelà Theorem, we deduce that $u \in C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times F)$ and u_n converges to u in $C^{1+\beta/2,2+\beta}([\varepsilon,T]\times F)$ for any $\beta<\alpha$. In particular,

$$\lim_{n\to+\infty} D_t u_n(t,x) = D_t u(t,x), \tag{5.1.17}$$

for any $t>0$ and any $x \in \mathbb{R}^N$. From (5.1.15) it follows that

$$\begin{aligned} D_t u_n(t,x) \le & \int_{\{\mathcal{A}\varphi\ge 0\}} \psi_n'(\varphi(y))\mathcal{A}\varphi(y)p(t,x;dy) \\ & + \int_{\{\mathcal{A}\varphi< 0\}} \psi_n'(\varphi(y))\mathcal{A}\varphi(y)p(t,x;dy). \end{aligned} \tag{5.1.18}$$

As n tends to $+\infty$, by (5.1.17) and the monotone convergence theorem, we get (5.1.11) and (5.1.12). Note that the first integral in the right-hand side of (5.1.18) converges for any $x \in \mathbb{R}$, since $0 \le \psi_n' \le 1$ for any $n \in \mathbb{N}$, and $\mathcal{A}\varphi$ is bounded from above. Therefore, the second integral in the right-hand side of (5.1.18) converges as well for any $x \in \mathbb{R}^N$ and the second formula in (5.1.12) follows.

To show that u is continuous up to $t=0$ we argue as in the proof of Theorem 2.2.1. For this purpose, we fix $m \in \mathbb{N}$ and introduce a cut-off function $\vartheta = \vartheta_m \in C^\infty_c(\mathbb{R}^N)$ such that $\chi_{B(m-1)} \le \vartheta \le \chi_{B(m)}$. Then, we consider the function $v_n = u_n\vartheta_m$ and we take n sufficiently large such that

$$n \ge \sup_{x\in B(m)} |\varphi(x)|.$$

This choice of n implies that $\vartheta\psi_n(\varphi) = \vartheta\varphi$. As it is easily seen, v_n solves the Cauchy-Dirichlet problem

$$\begin{cases} D_t v_n(t,x) = \mathcal{A}v_n(t,x) + g_n(t,x), & t>0,\, x \in B(m), \\ v_n(t,x) = 0, & t>0,\, x\in \partial B(m), \\ v_n(0,x) = \vartheta(x)\varphi(x), & x \in \overline{B}(m), \end{cases}$$

for any $n \geq m$, where

$$g_n = -\sum_{i,j=1}^{N} q_{ij}(2D_i u_n D_j\vartheta + u_n D_{ij}\vartheta) - u_n \sum_{i=1}^{N} b_i D_i\vartheta$$

satisfies

$$|g_n(t,x)| \leq C_1 \left(\|\varphi\|_{L^\infty(B(m))} + \|u_n\|_{L^\infty(B(m))} + \sum_{i=1}^{N} \|D_i u_n(t,\cdot)\|_{L^\infty(B(m))} \right),$$

for some positive constant C_1, independent of n. The estimate (5.1.16) and the interior estimates in Theorem C.1.4 (see (C.1.16)) yield

$$\begin{aligned} \|\sqrt{t} D_i u_n(t,\cdot)\|_{L^\infty(B(m))} &\leq C\|u_n\|_{L^\infty((0,T)\times B(m+1))} \\ &\leq C_2 \left(\|\varphi\|_{L^\infty(B(m+1))} + K \right), \end{aligned}$$

for any $t \in (0,1)$, any $x \in B(m)$, any $i = 1,\ldots,N$ and some positive constant C_2, independent of n. Now, repeating step by step the last part of the proof of Theorem 2.2.1, we deduce that $u(t,\cdot)$ tends to φ uniformly in $B(m-1)$, as t tends to 0. Due to the arbitrariness of m, it follows that $u \in C([0,+\infty)\times\mathbb{R}^N)$.
∎

Thanks to Lemma 5.1.4, we can now prove the following theorem which gives a sufficient condition ensuring the compactness of $T(t)$ for any $t > 0$.

Theorem 5.1.5 *Assume that the semigroup $\{T(t)\}$ is conservative and suppose that there exist a strictly positive function $\varphi \in C^2(\mathbb{R}^N)$ and a convex function $g \in C^1([0,+\infty))$ such that*

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty, \quad \frac{1}{g} \text{ is integrable at } +\infty, \quad \mathcal{A}\varphi(x) \leq -g(\varphi(x)), \;\; x \in \mathbb{R}^N. \tag{5.1.19}$$

Then, $T(t)$ is compact in $C_b(\mathbb{R}^N)$ for any $t > 0$.

Proof. Without loss of generality, we can suppose that $g(0) < 0$. Indeed, if this is not the case we replace g with $g - C$ for some large enough positive constant C. Let us set

$$u(t,x) = (T(t)\varphi)(x) = \int_{\mathbb{R}^N} \varphi(y)p(t,x;dy), \quad t > 0, \; x \in \mathbb{R}^N.$$

Observe that, since g is convex and $1/g$ is integrable in a neighborhood of $+\infty$, then $g(x)$ tends to $+\infty$ as x tends to $+\infty$. In particular, g is bounded from below in $(0,+\infty)$. Hence, by (5.1.19), $\mathcal{A}\varphi$ is bounded from above. Consequently, by Lemma 5.1.4, it follows that $u \in C^{1,2}((0,+\infty)\times\mathbb{R}^N) \cap C([0,+\infty)\times\mathbb{R}^N)$. Moreover,

$$\frac{\partial}{\partial t}u(t,x) \leq \int_{\mathbb{R}^N} \mathcal{A}\varphi(y)p(t,x;dy) \leq -\int_{\mathbb{R}^N} g(\varphi(y))p(t,x;dy) \leq -g(u(t,x)),$$

for any $t > 0$ and any $x \in \mathbb{R}^N$, where the last inequality follows from Jensen's inequality.

For any fixed $x \in \mathbb{R}^N$ let $z(\cdot, x)$ be the solution of the Cauchy problem

$$\begin{cases} z'(t,x) = -g(z(t,x)), & t > 0,\, x \in \mathbb{R}^N, \\ z(0,x) = \varphi(x), & x \in \mathbb{R}^N. \end{cases} \tag{5.1.20}$$

Note that $z(\cdot, x)$ is defined in $(0, +\infty)$. Indeed, if $\varphi(x) = x_g$, where x_g denotes the zero of g, then $z(\cdot, x) \equiv x_g$, whereas $z(\cdot, x)$ is increasing and greater than x_g if $\varphi(x) > x_g$. Similarly, $z(\cdot, x)$ is decreasing and less than x_g if $\varphi(x) < x_g$. In any case, $z(\cdot, x)$ exists in $[0, +\infty)$.

Let us now prove that $z(\cdot, x)$ is bounded from above in $[\tau, +\infty)$ for any $\tau > 0$, uniformly with respect to $x \in \mathbb{R}^N$. First, we assume that $\varphi(x) > x_g$. Integrating the differential equation (5.1.20) gives

$$-\int_{\varphi(x)}^{z(t,x)} \frac{dz}{g(z)} = t.$$

Therefore,

$$\int_{z(t,x)}^{+\infty} \frac{dz}{g(z)} = \int_{\varphi(x)}^{+\infty} \frac{dz}{g(z)} - \int_{\varphi(x)}^{z(t,x)} \frac{dz}{g(z)} \geq -\int_{\varphi(x)}^{z(t,x)} \frac{dz}{g(z)} = t > \tau. \tag{5.1.21}$$

Note that all the integral terms are finite since the function $1/g$ is integrable in a neighborhood of $+\infty$. Since $1/g$ is not integrable in a neighborhood of x_g, there exists a unique value $\bar{z} = \bar{z}(\tau) > x_g$ such that

$$\int_{\bar{z}}^{+\infty} \frac{dz}{g(z)} = \tau. \tag{5.1.22}$$

The positivity of g in $(x_g, +\infty)$, (5.1.21) and (5.1.22) imply that

$$z(t,x) < \bar{z}, \qquad t > \tau. \tag{5.1.23}$$

Now, suppose that $\varphi(x) \leq x_g$. In such a case, $z(t,x) \leq x_g$ for any $t \in (\tau, +\infty)$. Summing up, we have proved that $z(t,x) \leq \max(\bar{z}, x_g) = \bar{z}$ for any $t > \tau$ and any $x \in \mathbb{R}^N$.

Now, we fix $\varepsilon > 0$ and let $\rho > 0$ be such that $\varphi(x) \geq \bar{z}/\varepsilon$ for any $x \in \mathbb{R}^N \setminus B(\rho)$. Then, we have

$$p(t,x;\mathbb{R}^N \setminus B(\rho)) \leq \varepsilon \bar{z}^{-1} \int_{\mathbb{R}^N \setminus B(\rho)} \varphi(y) p(t,x;dy) \leq \varepsilon \bar{z}^{-1} u(t,x) \leq \varepsilon,$$

for any $x \in \mathbb{R}^N$ and any $t > \tau$. The compactness of $T(t)$, for any $t > 0$, now follows from Proposition 5.1.3 and the arbitrariness of $\tau > 0$. Note that the proof of the quoted proposition shows that $T(t_0)$ is compact whenever the family $\{p(t_0, x; dy) : x \in \mathbb{R}^N\}$ is tight. ∎

Example 5.1.6 Consider the operator $\mathcal{A}$ defined by

$$\mathcal{A}f(x) = \Delta f(x) + \langle b(x), Df(x)\rangle, \quad x \in \mathbb{R},$$

on smooth functions, with the drift b satisfying

$$\langle b(x), x\rangle \le C - M|x|^{2+\varepsilon}, \qquad x \in \mathbb{R}^N,$$

for some $C \in \mathbb{R}$, $M, \varepsilon > 0$. Then, the associated semigroup $\{T(t)\}$ is compact. To see it, it suffices to apply Theorem 5.1.5 with

$$\varphi(x) = |x|^2, \qquad x \in \mathbb{R}^N, \qquad g(s) = -(2N + C) + Ms^{1+\frac{\varepsilon}{2}}, \qquad s > 0. \tag{5.1.24}$$

Finally, we give another condition implying that $T(t)$ is a compact operator for any $t > 0$. For this purpose, we introduce the one-dimensional differential operator $\mathcal{C}$, defined on smooth functions by

$$\mathcal{C}u(r) = q(r)\left[u''(r) + r^{-1}b(r)u'(r)\right], \qquad r > 0,$$

where

$$q(r) = \min_{x\in\partial B(r)}\left(\frac{1}{|x|^2}\sum_{i,j=1}^N q_{ij}(x)x_ix_j\right) := \min_{x\in\partial B(r)} \mathcal{Q}(x), \tag{5.1.25}$$

$$b(r) = \max_{x\in\partial B(r)}\left(\frac{1}{\mathcal{Q}(x)}\sum_{i=1}^N (q_{ii}(x) + b_i(x)x_i) - 1\right) := \max_{x\in\partial B(r)} \mathcal{B}(x). \tag{5.1.26}$$

Proposition 5.1.7 *If $+\infty$ is an entrance point for the operator $\mathcal{C}$ (see Definition 3.1.6), then $T(t)$ is a compact operator for any $t > 0$.*

To prove the proposition we need two preliminary lemmata.

Lemma 5.1.8 *Suppose that $+\infty$ is an entrance point for the operator $\mathcal{C}$. Then, for any $R > 0$, there exists a positive function $\varphi \in C^2(\mathbb{R}^N \setminus B(R))$ diverging to $+\infty$ as $|x|$ tends to $+\infty$ and such that $\varphi - \mathcal{A}\varphi \ge 0$ in $\mathbb{R}^N \setminus B(R)$. As a consequence, if $z \in C_b([0,T] \times \mathbb{R}^N \setminus B(R)) \cap C^{1,2}((0,T) \times \mathbb{R}^N \setminus \overline{B}(R))$ satisfies*

$$\begin{cases} D_tz(t,x) - \mathcal{A}z(t,x) \ge 0, & t \in (0,T),\, x \in \mathbb{R}^N \setminus \overline{B}(R),\\ z(t,x) \ge 0, & t \in (0,T),\, x \in \partial B(R),\\ z(0,x) \ge 0, & x \in \mathbb{R}^N \setminus B(R), \end{cases}$$

for some $R > 0$, then $z \ge 0$.

Proof. We limit ourselves to considering the case when $R = 1$, the general case being completely similar.

Let us determine a function φ satisfying the assertion. For this purpose, we observe that since $+\infty$ is an entrance, according to Proposition 3.1.7(iii), we can determine an increasing positive solution $\psi : [1, +\infty) \to \mathbb{R}$ of the equation $\psi - \mathcal{C}\psi = 0$ such that $\lim_{r\to+\infty} \psi(r) = +\infty$. This can be done adapting to this situation the proof of Proposition 3.1.4. The function φ can now be defined by setting $\varphi(x) = \psi(|x|)$ for any $x \in \mathbb{R}^N \setminus B(1)$. Indeed, a straightforward computation shows that

$$\varphi(x) - \mathcal{A}\varphi(x) = \psi(|x|) - \mathcal{Q}(x)\left(\psi''(|x|) + \frac{\mathcal{B}(x)}{|x|}\psi'(|x|)\right), \qquad x \in \mathbb{R}^N \setminus B(1),$$

where $\mathcal{Q}$ and $\mathcal{B}$ are given by (5.1.25) and (5.1.26). Since ψ and ψ' are both positive, it is now easy to check that

$$\varphi(x) - \mathcal{A}\varphi(x) \ge \mathcal{Q}(x)\left(\frac{1}{q(|x|)}\psi(|x|) - \psi''(|x|) - \frac{b(|x|)}{|x|}\psi'(|x|)\right) = 0,$$

for any $x \in \mathbb{R}^N \setminus B(1)$.

To prove the last part of the lemma, for any $\varepsilon > 0$, we introduce the function z_ε defined by $z_\varepsilon(t, x) = z(t, x) + \varepsilon e^t \varphi(x) + \varepsilon t$ for any $t \in [0, T]$ and any $x \in \mathbb{R}^N \setminus B(1)$. As it is immediately seen, $D_t z_\varepsilon - \mathcal{A}z_\varepsilon \ge \varepsilon$ in $(0, T) \times (\mathbb{R}^N \setminus \overline{B}(1))$ and z_ε tends to $+\infty$ as $|x|$ tends to $+\infty$, uniformly with respect to $t \in [0, T]$. This implies that z_ε has an absolute minimum at some point $(t_0, x_0) \in [0, T] \times \mathbb{R}^N \setminus B(1)$. We cannot have $t_0 > 0$ and $|x_0| > 1$, otherwise we should have $D_t z_\varepsilon(t_0, x_0) - \mathcal{A}z(t_0, x_0) \le 0$ which is a contradiction. Therefore, $t_0 = 0$ or $|x_0| = 1$. In any case we have $z_\varepsilon(t_0, x_0) \ge 0$. Letting ε go to 0 the assertion follows. ∎

Lemma 5.1.9 *Fix $\delta > 0$. Then, for any $f \in C_b([\delta, +\infty))$ the Cauchy-Dirichlet problem*

$$\begin{cases} D_t u(t, x) = \mathcal{C}u(t, x), & t > 0,\ x > \delta, \\ u(t, \delta) = 0, & t > 0, \\ u(0, x) = f(x), & x > \delta, \end{cases} \tag{5.1.27}$$

admits a solution $u \in C_b(([0, +\infty) \times [\delta, +\infty)) \setminus \{(0, \delta)\}) \cap C^{1,2}((0, +\infty) \times (\delta, +\infty))$. Moreover,

$$\|u\|_\infty \le \|f\|_\infty. \tag{5.1.28}$$

Further, the family of linear operators $\{S(t)\}$ defined by $S(t)f = u(t, \cdot)$ for any $t > 0$, where u is as above, is a positive semigroup of contractions. Finally, the function $v = S(\cdot)\mathbb{1}$ satisfies $D_t v \le 0$ and $D_r v \ge 0$ in $(0, +\infty) \times (\delta, +\infty)$.

Proof. Let us observe that a straightforward computation shows that the coefficients of the operator $\mathcal{C}$ belong to $C^{\alpha}_{\rm loc}([\delta,+\infty))$. Moreover, the proof of Lemma 5.1.8 shows that there exists a positive function $\psi \in C^2([\delta,+\infty))$, blowing up as x tends to $+\infty$, such that $\psi - \mathcal{C}\psi = 0$ in $(\delta,+\infty)$. Therefore, to prove the assertion, it suffices to apply the same arguments as in the proof of the forthcoming Theorem 11.2.1.

Let us give just the proof of the last part of the lemma. We set $u = S(\cdot)\mathbb{1}$ and we prove that $D_t u$ and $-D_r u$ are nonpositive in $(0,+\infty)\times(\delta,+\infty)$. As a first step we observe that, from the estimate (5.1.28), it follows that $u \le \mathbb{1}$. Now, using the positivity of the semigroup, we get

$$u(t+s,\cdot) = S(t+s)\mathbb{1} = S(s)S(t)\mathbb{1} \le S(t)\mathbb{1} = u(t,\cdot), \qquad s,t>0,$$

so that the function $u(\cdot,r)$ is nonincreasing in $(0,+\infty)$ for any $r>\delta$ and, consequently, $D_t u \le 0$ in $(0,+\infty)\times(\delta,+\infty)$.

Now we show that, for any $t>0$, the function $u(t,\cdot)$ is nondecreasing in $(0,+\infty)$. Since

$$\mathcal{C}u(t,r) = q(r)W(r)D_r\left(\frac{D_r u(t,\cdot)}{W}\right)(r), \qquad t>0,\ \ r>\delta,$$

where W is given by (3.1.3), we deduce that, for any $0<r<s$ and any $t>0$,

$$\frac{D_r u(t,s)}{W(s)} - \frac{D_r u(t,r)}{W(r)} = \int_r^s \frac{D_t u(t,\xi)}{q(\xi)W(\xi)}d\xi.$$

Since $D_t u$ is nonpositive in $(0,+\infty)\times(\delta,+\infty)$, then the function $w(t,\cdot) := D_r u(t,\cdot)/W$ is nonincreasing in $(\delta,+\infty)$, for any $t>0$. Set $l_t = \lim_{r\to+\infty} w(t,r)$. We claim that $l_t \ge 0$ for any $t>0$. Of course, once the claim is proved, we will immediately obtain that $w\ge 0$ and, consequently, $D_r u \ge 0$ in $(0,+\infty)\times(\delta,+\infty)$ since W is a strictly positive function. Suppose by contradiction that $l_t<0$ for some $t>0$. Then, there exist $r^*>0$ and a negative constant c such that

$$D_t u(t,r) \le cW(r), \qquad r\ge r^*.$$

Integrating in $[r^*,r]$ the previous inequality, we get

$$v(t,r) = v(t,r^*) + \int_{r^*}^r D_t u(t,s)ds \le u(t,\delta) + c\int_{r^*}^r W(s)ds.$$

The boundedness of v in $(0,+\infty)\times(\delta,+\infty)$ implies that W is integrable in $(r^*,+\infty)$ and, consequently, in $(\delta,+\infty)$. But this is a contradiction. Indeed, since Q is integrable in $(\delta,+\infty)$, then the function $1/(qW)$ is integrable in $(\delta,+\infty)$ as well. The integrability of W in $(\delta,+\infty)$ should imply the integrability of R in $(\delta,+\infty)$, which contradicts our assumptions. ∎

Now, we can prove Proposition 5.1.7.

Proof of Proposition 5.1.7. According to Proposition 5.1.3, to prove that $T(t)$ is compact for any $t > 0$, we show that the family of measures $\{p(t, x; dy) : x \in \mathbb{R}^N\}$ is tight for any $t > 0$., i.e. we fix $t_0 > 0$ and we prove that for any $\varepsilon > 0$ there exists $\rho > 0$ such that

$$p(t_0, x; B(\rho)) \geq 1 - \varepsilon, \qquad x \in \mathbb{R}^N. \tag{5.1.29}$$

Let $\{f_n\} \subset C_b(\mathbb{R}^N)$ be a sequence of continuous functions in $\mathbb{R}^N$ such that $\chi_{B(n-1)} \leq f_n \leq \chi_{B(n)}$ for any $n \in \mathbb{N}$. Observe that

$$p(t_0, x; B(n)) = \int_{B(n)} p(t_0, x; dy) \geq \int_{\mathbb{R}^N} f_n(y) p(t_0, x; dy) = (T(t_0) f_n)(x),$$

for any $x \in \mathbb{R}^N$ (see Theorem 2.2.5). Since f_n converges to the function $f = \mathbb{1}$ as n tends to $+\infty$, locally uniformly in $\mathbb{R}^N$, by Proposition 2.2.9 it follows that, the function $T(t_0) f_n$ converges locally uniformly in $\mathbb{R}^N$ to the function $\mathbb{1}$. Hence, it suffices to show that there exist $\rho > 0$ and $n_1 \in \mathbb{N}$ such that

$$(T(t_0) f_n)(x) \geq 1 - \varepsilon, \qquad x \in \mathbb{R}^N \setminus \overline{B}(\rho), \tag{5.1.30}$$

for any $n \geq n_1$. To prove (5.1.30) we use a comparison argument with radial functions. For this purpose we split the proof into two steps. First, in Step 1, we study the main properties of the function $v_\delta = S(\cdot)\mathbb{1}$ that we need in the sequel. Here $\{S(t)\}$ is the semigroup in $C_b([\delta, +\infty))$ defined in Lemma 5.1.9. Then, in Step 2, by means of the function v_δ, we construct a radial function w such that $w(t_0, \cdot) \geq 1 - \varepsilon$ and such that, for $|x|$ and n large enough, it satisfies $(T(t_0) f_n)(x) \geq w(t_0, x)$.

Step 1. We claim that, for any δ sufficiently large,

$$0 \leq v_\delta(t_0, r) \leq \eta := 1 - (1 - \varepsilon)^{1/2}, \qquad r \geq \delta. \tag{5.1.31}$$

For this purpose, for any $\delta > 0$, we set

$$\psi_\delta(r) = 1 - \int_0^{+\infty} e^{-t} v(t, r) dt, \qquad r \geq \delta.$$

The function ψ_δ is well defined since v is bounded in $(0, +\infty) \times (\delta, +\infty)$. Moreover, ψ_δ satisfies

$$\begin{cases} \psi_\delta(r) - \mathcal{C}\psi_\delta(r) = 0, & r > \delta, \\ \psi_\delta(\delta) = 1. \end{cases} \tag{5.1.32}$$

Of course, $\psi_\delta(\delta) = 1$ since $v(t, 0) = 0$ for any $t > 0$. To show that ψ_δ satisfies the differential equation in (5.1.32), we introduce, for any $n \in \mathbb{N}$, the function $\psi_{\delta,n} : [\delta, +\infty) \to \mathbb{R}$ defined by

$$\psi_{\delta,n}(r) = \int_0^{n} e^{-t} v(t, r) dt, \qquad r \geq \delta.$$

It is immediate to check that $\psi_{\delta,n} \in C^2((\delta,+\infty))$ and it solves the differential equation

$$\psi_{\delta,n}(r) - \mathcal{C}\psi_{\delta,n}(r) = 0, \qquad r > \delta. \tag{5.1.33}$$

Moreover, $\psi_{\delta,n}$ converges uniformly to ψ_δ in $[\delta,+\infty)$. Let us now show that $\psi_\delta \in C^2((\delta,+\infty))$ and it solves the differential equation in (5.1.32) pointwise. For this purpose, we observe that, since the sequence $\{\psi_{\delta,n}\}$ is bounded in $C_b([\delta,+\infty))$, using the same compactness arguments as in the proof of Theorem 2.1.1, one can easily show that, up to a subsequence, $\psi_{\delta,n}$ converges to ψ_δ in $W^{2,p}(r_1,r_2)$ for any $\delta < r_1 < r_2$ and any $p \in (1,+\infty)$. According to the Sobolev embedding theorems (see [4, Theorem 5.4]), we deduce that $\psi_\delta \in C^1((\delta,+\infty))$ and $\psi'_{\delta,n}$ converges to ψ'_δ locally uniformly in $(\delta,+\infty)$. Now, since $q(r)\psi''_{\delta,n}(r) = \psi_{\delta,n}(r) - r^{-1}q(r)b(r)\psi'_{\delta,n}(r)$ for any $r \in (\delta,+\infty)$ and any $n \in \mathbb{N}$, it follows that $q\psi''_{\delta,n}$ (and consequently $\psi''_{\delta,n}$ since q does not vanish in $[\delta,+\infty)$) converges locally uniformly in $(\delta,+\infty)$. Therefore, $\psi_\delta \in C^2((\delta,+\infty))$. Now, letting n go to $+\infty$ in (5.1.33), we easily see that ψ_δ solves the differential equation in (5.1.32).

Let us now prove that δ can be chosen sufficiently large so that

$$\psi_\delta(r) \geq 1 - \eta t_0 e^{-t_0}, \qquad r \geq \delta. \tag{5.1.34}$$

For this purpose, denote by φ the unique positive decreasing solution of the equation $\varphi - \mathcal{C}\varphi = 0$ in $(1,+\infty)$, such that $\lim_{r\to+\infty}\varphi(r) = 1$ (the existence of such a solution can be established arguing as in the proof of the properties (ii) and (iv) in Proposition 3.1.4). Then, $\psi_\delta = \varphi/\varphi(\delta)$. Indeed, the same arguments as in the proof of Proposition 3.1.4 show that, up to a multiplicative constant, ψ_δ is the unique bounded solution to the problem (5.1.32). It is now clear that we can fix $\hat{\delta}$ such that $\psi_{\hat{\delta}}$ satisfies (5.1.34).

Now, let $\lambda_m = e^{-t_0}\eta + m^{-1}$ $(m \in \mathbb{N})$ and introduce the set

$$E_m = \{t \geq 0 : \; e^{-t} \lim_{r\to+\infty} v_{\hat{\delta}}(t,r) \geq \lambda_m\}.$$

Since v is bounded, the set E_m is bounded for any $m \in \mathbb{N}$ and it is not empty since it contains $t = 0$. Moreover, using the Chebyshev inequality, we get

$$\begin{aligned}
\int_{E_m} dr = \frac{1}{\lambda_m}\int_{E_m} \lambda_m dr &\leq \frac{1}{\lambda_m}\int_{E_m} e^{-t}\lim_{r\to+\infty} v_{\hat{\delta}}(t,r)dt \\
&= \frac{1}{\lambda_m}\lim_{r\to+\infty}\int_{E_m} e^{-t} v_{\hat{\delta}}(t,r)dt \\
&= \frac{1}{\lambda_m}\lim_{r\to+\infty}\int_{E_m} e^{-t} v_{\hat{\delta}}(t,r)dt \\
&\leq \frac{1}{\lambda_m}\limsup_{r\to+\infty}(1 - \psi_{\hat{\delta}}(r)) \\
&\leq \frac{\eta}{\lambda_m} t_0 e^{-t_0}.
\end{aligned}$$

Therefore, due to the choice of λ_m, the Lebesgue measure of E_m is strictly less than t_0. From this fact, and recalling that $v_{\hat{\delta}}(t,\cdot)$ is increasing in $[\hat{\delta},+\infty)$ for any $t>\hat{\delta}$, it follows that we can determine $t_{1,m}<t_0$ such that

$$e^{-t_{1,m}}v_{\hat{\delta}}(t_{1,m},r)<\lambda_m,\qquad r>\hat{\delta}.$$

Since the sequence $\{t_{1,m}\}$ is bounded, up to a subsequence, we can assume that $t_{1,m}$ converges to a point $t_1\in(0,t_0]$ as m tends to $+\infty$. Moreover, by continuity

$$e^{-t_1}v_{\hat{\delta}}(t_1,r)\le e^{-t_0}\eta,\qquad r\ge\hat{\delta}$$

and, consequently, $v_{\hat{\delta}}(t_1,\cdot)\le\eta$ in $[\hat{\delta},+\infty)$. Since $v_{\hat{\delta}}(\cdot,r)$ is nonincreasing in $(0,+\infty)$ for any $r>\hat{\delta}$, the estimate (5.1.31) follows.

Step 2. Now, we prove (5.1.30) with $\rho=\hat{\delta}+1$. Since, as it has already been pointed out, $T(t)f_n$ converges locally uniformly in $[0,+\infty)\times\mathbb{R}^N$ to $\mathbb{1}$, we can fix $n_1\in\mathbb{N}$ such that

$$(T(t)f_n)(x)\ge 1-\eta,\qquad t\in(0,t_0),\;\; x\in\overline{B}(\hat{\delta}+1),\qquad n\ge n_1.$$

Let us now introduce the function $w:[0,+\infty)\times(\mathbb{R}^N\setminus B(\hat{\delta}+1))$ defined by

$$w(t,x)=(1-\eta)(1-v_{\hat{\delta}}(t,|x|)),\qquad t\ge 0,\;\; x\in\mathbb{R}^N\setminus B(\hat{\delta}+1).$$

Of course, $w\in C_b([0,+\infty)\times(\mathbb{R}^N\setminus B(\hat{\delta}+1)))\cap C^{1,2}((0,+\infty)\times(\mathbb{R}^N\setminus\overline{B}(\hat{\delta}+1)))$, it satisfies $w\le 1-\eta$ in $[0,+\infty)\times(\mathbb{R}^N\setminus B(\hat{\delta}+1))$, $w(0,\cdot)=0$ and, by (5.1.31), $w(t_0,\cdot)\ge(1-\eta)^2=1-\varepsilon$ in $\mathbb{R}^N\setminus B(\hat{\delta}+1)$. Moreover, it satisfies the differential equation $D_tw-\mathcal{A}w=g$, where

$$g(t,x)=(\eta-1)\mathcal{Q}(x)\left[\frac{D_tv(t,|x|)}{\mathcal{Q}(x)}-D_{rr}v(t,|x|)-\frac{\mathcal{B}(x)}{|x|}D_rv(t,|x|)\right],$$

for any $t>0$ and any $x\in\mathbb{R}^N\setminus B(\hat{\delta}+1)$. Here, $\mathcal{Q}$ and $\mathcal{B}$ are defined by (5.1.25) and (5.1.26). Since D_tv and $-D_rv$ are nonpositive in $(0,+\infty)\times(\hat{\delta},+\infty)$, we deduce that

$$g(t,x)\le(\eta-1)\frac{\mathcal{Q}(x)}{q(|x|)}\left(D_tv(t,|x|)-\mathcal{C}v(t,|x|)\right)=0.$$

From all the results above, it follows that the function $z=T(\cdot)f_n-w$ satisfies

$$\begin{cases}D_tz(t,x)-\mathcal{A}z(t,x)\ge 0, & t\in(0,t_0),\, x\in\mathbb{R}^N\setminus\overline{B}(\hat{\delta}+1),\\ z(t,x)\ge 0, & t\in(0,t_0),\, x\in\mathbb{R}^N\setminus\partial B(\hat{\delta}+1),\\ z(0,x)=f_n(x), & x\in\mathbb{R}^N\setminus B(\hat{\delta}+1),\end{cases}$$

for any $n\ge n_1$. The maximum principle in Lemma 5.1.8 implies that $z\ge 0$ in $(0,t_0)\times(\mathbb{R}^N\setminus B(\hat{\delta}+1))$ and (5.1.30) follows. ∎

5.1.2 The nonconservative case

We now consider the nonconservative case. In particular, here we do not assume that the coefficient c identically vanishes in $\mathbb{R}^N$.

The following theorem shows that $T(t)$ is compact and maps $C_0(\mathbb{R}^N)$ into itself, for some $t > 0$, if and only if it maps the function $\mathbb{1}$ into a function vanishing at infinity. To prove such a result, we need the following lemma.

Lemma 5.1.10 *Suppose that $\mathcal{F}$ is a bounded subset of $C_0(\mathbb{R}^N)$ such that for any $\varepsilon > 0$ there exist $\delta > 0$ with the property*

$$\sup_{f\in\mathcal{F}}\ \sup_{x\in\mathbb{R}^N\setminus B(\delta)} |f(x)| < \varepsilon.$$

Further, suppose that, for any compact set $H \subset \mathbb{R}^N$ and any $\varepsilon > 0$, there exists $\sigma > 0$ such that

$$\sup_{f\in\mathcal{F}}\ \sup_{\substack{|x-y|\le\sigma\\ x,y\in H}} |f(x) - f(y)| \le \varepsilon.$$

Then, $\mathcal{F}$ is totally bounded in $C_0(\mathbb{R}^N)$.

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ be such that $|f(x)| \le \varepsilon$ for any $|x| \ge \delta$ and any $f \in \mathcal{F}$. According to our assumptions and the Ascoli-Arzelà Theorem, the set $\mathcal{F}_{|B(2\delta)}$, of all the restrictions to $B(2\delta)$ of the functions in $\mathcal{F}$, is totally bounded. Hence, we can find out $k \in \mathbb{N}$ and $g_1, \ldots, g_k \in C(\overline{B}(2\delta))$ such that

$$\mathcal{F}_{|\overline{B}(2\delta)} \subset \bigcup_{j=1}^{k} (g_j + B(\varepsilon)),$$

where $g_j + B(\varepsilon)$ $(j = 1, \ldots, k)$ denotes the closed ball in $C(\overline{B}(2\delta))$ with centre at g_j and radius ε. Now let φ be any smooth function such that $\chi_{B(\delta)} \le \varphi \le \chi_{B(2\delta)}$. Then,

$$\mathcal{F} \subset \bigcup_{j=1}^{k} (\varphi g_j + B(3\varepsilon)), \tag{5.1.35}$$

where the closed balls $\varphi g_j + B(3\varepsilon)$ are now meant in $C_b(\mathbb{R}^N)$. To check (5.1.35), fix $f \in \mathcal{F}$ and let $j \in \{1, \ldots, k\}$ be such that $\|f - g_j\|_{C(\overline{B}(2\delta))} \le \varepsilon$. Then, in particular,

$$\|f - \varphi g_j\|_{C(\overline{B}(\delta))} \le \varepsilon.$$

Moreover,

$$\begin{aligned}
\|f - \varphi g_j\|_{C_b(\mathbb{R}^N\setminus B(\delta))} &\le \|f\|_{C_b(\mathbb{R}^N\setminus B(\delta))} + \|g_j\|_{C(\overline{B}(2\delta)\setminus B(\delta))}\\
&\le 2\|f\|_{C_b(\mathbb{R}^N\setminus B(\delta))} + \|f - g_j\|_{C(\overline{B}(2\delta))}\\
&\le 3\varepsilon.
\end{aligned}$$

Summing up, we have

$$\|f-\varphi g_j\|_{C_b(\mathbb{R}^N)} \le 3\varepsilon,$$

and (5.1.35) follows. ■

We can now prove the following result.

Theorem 5.1.11 *Fix $t>0$. Then $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$ if and only if $T(t)$ is compact and $C_0(\mathbb{R}^N)$ is invariant for $T(t)$.*

Proof. Suppose that $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$. Then, using (2.2.8), we get

$$|(T(t)f)(x)| = \left|\int_{\mathbb{R}^N} f(y)p(t,x;dy)\right| \le \|f\|_\infty \int_{\mathbb{R}^N} p(t,x;dy) = \|f\|_\infty (T(t)\mathbb{1})(x), \tag{5.1.36}$$

for any $f \in C_b(\mathbb{R}^N)$ and any $x \in \mathbb{R}^N$. Therefore, $T(t)$ maps $C_b(\mathbb{R}^N)$ in $C_0(\mathbb{R}^N)$. In particular, $C_0(\mathbb{R}^N)$ is invariant.

Let us now prove that $T(t)$ is a compact operator. For this purpose, let

$$B_\infty(1) = \{f \in C_b(\mathbb{R}^N) : \|f\|_\infty \le 1\}$$

be the unit ball in $C_b(\mathbb{R}^N)$. According to (5.1.36), $T(t)(B_\infty(1))$ is a set of functions vanishing uniformly at infinity. This implies that, for any $\varepsilon>0$, there exists $\delta>0$ such that

$$|(T(t)f)(x)| \le \varepsilon, \qquad |x| \ge \delta, \tag{5.1.37}$$

for any $f \in B_\infty(1)$. Moreover, from the interior Schauder estimates in Theorem C.1.4, it follows that, for any $\rho>0$, there exists a positive constant $C = C(\rho,t)$ such that

$$\|T(t)f\|_{C^{2+\alpha}(\overline{B}(\rho))} \le C\|f\|_{C(\overline{B}(2\rho))} \le C, \tag{5.1.38}$$

for any $f \in B_\infty(1)$, where α is as in Hypothesis 5.1.1(ii). Therefore, the set of all the restrictions of the functions in $T(t)(B_\infty(1))$ to $\overline{B}(\rho)$ is a bounded set of equicontinuous functions. From (5.1.37), (5.1.38) and Lemma 5.1.10 it follows that $T(t)(B_\infty(1))$ is compact in $C_b(\mathbb{R}^N)$.

Conversely, fix $t>0$ and suppose that $T(t)$ is compact and $C_0(\mathbb{R}^N)$ is invariant for $T(t)$. Let $\{f_n\}$ be a sequence of continuous functions such that $\chi_{B(n-1)} \le f_n \le \chi_{B(n)}$ for any $n \in \mathbb{N}$. By Proposition 2.2.9, $(T(t)f_n)(x)$ tends to $(T(t)\mathbb{1})(x)$ for any $x \in \mathbb{R}^N$. Since $T(t)$ is compact, then $T(t)f_n$ tends to $T(t)\mathbb{1}$ uniformly in $\mathbb{R}^N$. Moreover, since $C_0(\mathbb{R}^N)$ is invariant, then $T(t)f_n \in C_0(\mathbb{R}^N)$ for any $n \in \mathbb{N}$, and, therefore, $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$ as well. ■

In Section 5.2 we will determine some sufficient conditions implying that $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$ for any $t>0$. By virtue of Theorem 5.1.11, such conditions turn out to be sufficient conditions for the compactness of $T(t)$ for any $t>0$.

We conclude this section with the following proposition.

Proposition 5.1.12 *Suppose that, for any $t > 0$, $T(t)$ is a compact operator and $C_0(\mathbb{R}^N)$ is invariant under $T(t)$. Then, the semigroup is norm-continuous in $(0, +\infty)$ and the resolvent $R(\lambda, \widehat{A})$ is compact for any $\lambda > c_0$.*

Proof. We limit ourselves to proving that the map $t \mapsto T(t)f$ is continuous from the right in $(0, +\infty)$ for any $f \in B_\infty(1) = \{f \in C_b(\mathbb{R}^N) : \|f\|_\infty \le 1\}$. Indeed, once this property is proved, repeating the proof of Proposition 5.1.3, we can easily show that the map $t \mapsto T(t)$ is norm-continuous in $(0, +\infty)$ and $R(\lambda, \widehat{A})$ is compact for any $\lambda > c_0$. Observe that, by Theorem 5.1.11, $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$ for any $t > 0$. Hence, applying (5.1.36), written for $f \in C_b(\mathbb{R}^N)$, we easily see that $T(t)$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ for any $t > 0$. By Proposition 2.2.7, $T(t)g$ tends to g uniformly in $\mathbb{R}^N$, as t tends to 0, for any $g \in C_0(\mathbb{R}^N)$. Hence,

$$\lim_{h \to 0^+} \|(T(h) - I)T(t)f\|_\infty = 0,$$

for any $t > 0$ and any $f \in C_b(\mathbb{R}^N)$, and this implies that $t \mapsto T(t)f$ is right-continuous in $(0, +\infty)$. ∎

5.2 On the inclusion $T(t)(C_b(\mathbb{R}^N)) \subset C_0(\mathbb{R}^N)$

In this section we determine sufficient conditions implying that $T(t)$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$. If not explicitly specified, we do not assume that $c \equiv 0$. Let us observe that, without loss of generality, we can limit ourselves to determining sufficient conditions which ensure that $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$. Indeed, as it has been already pointed out in the proof of Proposition 5.1.12, applying (5.1.36) written for $f \in C_b(\mathbb{R}^N)$, we easily deduce that

$$|(T(t)f)(x)| \le \|f\|_\infty (T(t)\mathbb{1})(x), \qquad t > 0, \quad x \in \mathbb{R}^N. \tag{5.2.1}$$

Hence, if $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$, then $T(t)f \in C_0(\mathbb{R}^N)$ as well, for any $f \in C_b(\mathbb{R}^N)$.

Using (5.2.1), we can easily show that

$$|(R(\lambda, \widehat{A})f)(x)| \le \|f\|_\infty |(R(\lambda, \widehat{A})\mathbb{1})(x)|, \qquad x \in \mathbb{R}^N, \quad \lambda > c_0, \tag{5.2.2}$$

which implies that $R(\lambda, \widehat{A})$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ if and only if $R(\lambda, \widehat{A})\mathbb{1} \in C_0(\mathbb{R}^N)$. We recall that, for any $f \in C_b(\mathbb{R}^N)$,

$$(R(\lambda, \widehat{A})f)(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)dt, \qquad x \in \mathbb{R}^N, \tag{5.2.3}$$

where $\widehat{A}$ is the weak generator of the semigroup. See Sections 2.1 and 2.3.

The following proposition shows that $T(t)$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ if and only if $R(\lambda, \widehat{A})$ does for some (and hence all) $\lambda > 0$. Before stating and proving it, we consider the following lemma.

Lemma 5.2.1 *Let, as usual, $c_0 = \sup_{x\in\mathbb{R}^N} c(x)$. Then, for any $x \in \mathbb{R}^N$, the function $t \mapsto (S(t)\mathbb{1})(x) := e^{-c_0 t}(T(t)\mathbb{1})(x)$ is not increasing in $[0, +\infty)$.*

Proof. It can be obtained applying the proof of Lemma 4.1.8. See Remark 4.1.9. ∎

We can now prove the following proposition.

Proposition 5.2.2 *The following properties are equivalent:*

(*i*) $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$ *for any* $t > 0$;

(*ii*) $R(\lambda, \widehat{A})\mathbb{1} \in C_0(\mathbb{R}^N)$ *for any* $\lambda > c_0$;

(*iii*) $R(\lambda, \widehat{A})\mathbb{1} \in C_0(\mathbb{R}^N)$ *for some* $\lambda > c_0$;

(*iv*) $D(\widehat{A}) = D_{\max}(\mathcal{A}) \cap C_0(\mathbb{R}^N)$, *where* $D(\widehat{A})$ *is defined in Section* 2.3.

Proof. We can limit ourselves to proving that "(i) ⇒ (ii)" and "(iii) ⇒ (i)", since the implication "(ii) ⇒ (iii)" is trivial and "(ii) ⇔ (iv)" follows immediately, from Proposition 2.3.6, observing that (ii) and (5.2.2) imply that $R(\lambda, \widehat{A})f \in C_0(\mathbb{R}^N)$ for any $f \in C_b(\mathbb{R}^N)$.

"(i) ⇒ (ii)". Fix $\lambda > 0$. Taking (5.2.3) and Lemma 5.2.1 into account, we deduce that

$$
\begin{aligned}
(R(\lambda, \widehat{A})\mathbb{1})(x) &= \int_0^{+\infty} e^{-\lambda t}(T(t)\mathbb{1})(x)dt \\
&= \int_0^{+\infty} e^{-(\lambda-c_0)t}(S(t)\mathbb{1})(x)dt \\
&= \int_0^{\varepsilon} e^{-(\lambda-c_0)t}(S(t)\mathbb{1})(x)dt + \int_\varepsilon^{+\infty} e^{-(\lambda-c_0)t}(S(t)\mathbb{1})(x)dt \\
&\le \varepsilon + (S(\varepsilon)\mathbb{1})(x)\int_\varepsilon^{+\infty} e^{-(\lambda-c_0)t}dt \\
&\le \varepsilon + (S(\varepsilon)\mathbb{1})(x)\int_0^{+\infty} e^{-(\lambda-c_0)t}dt,
\end{aligned}
$$

for any $\varepsilon > 0$ and any $x \in \mathbb{R}^N$, where we use the fact that $\{S(t)\}$ is a semigroup of contractions. Since $T(\varepsilon)\mathbb{1} \in C_0(\mathbb{R}^N)$, then $S(\varepsilon)\mathbb{1} \in C_0(\mathbb{R}^N)$ as well. This implies that

$$\limsup_{|x|\to+\infty}(R(\lambda, \widehat{A})\mathbb{1})(x) \le \varepsilon.$$

From the arbitrariness of $\varepsilon > 0$ and the nonnegativity of the function $R(\lambda, \widehat{A})\mathbb{1}$, we deduce that $R(\lambda, \widehat{A})\mathbb{1} \in C_0(\mathbb{R}^N)$.

"(iii) ⇒ (i)". Fix $t > 0$. Again by the formula (5.2.3) and Lemma 5.2.1 we get

$$
\begin{aligned}
(R(\lambda, \widehat{A})\mathbb{1})(x) &= \int_0^{+\infty} e^{-(\lambda - c_0)s}(S(s)\mathbb{1})(x)ds \\
&\geq \int_0^t e^{-(\lambda - c_0)s}(S(s)\mathbb{1})(x)ds \\
&\geq (S(t)\mathbb{1})(x)\int_0^t e^{-(\lambda - c_0)s}ds, \qquad x \in \mathbb{R}^N,
\end{aligned}
$$

that is

$$0 \leq S(t)\mathbb{1} \leq \left(\frac{\lambda - c_0}{1 - e^{-(\lambda - c_0)t}}\right) R(\lambda, \widehat{A})\mathbb{1}.$$

Thus, we conclude that $S(t)\mathbb{1}$ (and hence $T(t)\mathbb{1}$) belongs to $C_0(\mathbb{R}^N)$. ■

An interesting consequence of the fact that the semigroup maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ is given by the next proposition.

Proposition 5.2.3 *Suppose that $c \equiv 0$ and $T(t_0)$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ for some $t_0 > 0$. Then, $\|T(t)\|_{L(C_b(\mathbb{R}^N))}$ decreases exponentially to 0 as t tends to $+\infty$.*

Proof. We begin the proof observing that, since $T(t)\mathbb{1} = T(t_0)T(t - t_0)\mathbb{1}$ for any $t \geq t_0$, then $T(t)$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ for any $t \geq t_0$. Now, we prove that there exists a positive constant $k_0 \in (0, 1)$ such that $(T(t_0)\mathbb{1})(x) \leq k_0$ for any $x \in \mathbb{R}^N$. For this purpose, we notice that, since $T(t_0)\mathbb{1} \in C_0(\mathbb{R}^N)$, there exists $\delta > 0$ such that $(T(t_0)\mathbb{1})(x) \leq 1/2$ for any $|x| \geq \delta$. Moreover, since $T(t)\mathbb{1} \leq \mathbb{1}$ for any $t > 0$, $(T(t_0)\mathbb{1})(x) < 1$ for any $x \in B(\delta)$ otherwise by the classical maximum principle (see Proposition C.2.3(iii)), $T(t_0)\mathbb{1}$ should be constant in $B(\delta)$ and equal to $\mathbb{1}$, which, of course, cannot be the case.

Now, the semigroup property allows us to show that $\|T(t)\|_{L(C_b(\mathbb{R}^N))}$ decreases to 0 exponentially as t tends to $+\infty$. For this purpose, we split any $t > t_0$ as $t = nt_0 + r$ with $n \in \mathbb{N}$ and $r \in [0, t_0)$. Recalling that $\|T(r)\|_{L(C_b(\mathbb{R}^N))} = \|T(r)\mathbb{1}\|_\infty \leq 1$ for any $r > 0$, we get

$$
\begin{aligned}
\|T(t)\|_{L(C_b(\mathbb{R}^N))} &= \|(T(t_0))^n T(r)\mathbb{1}\|_{L(C_b(\mathbb{R}^N))} \\
&\leq \|T(t_0)\|^n_{L(C_b(\mathbb{R}^N))}\|T(r)\mathbb{1}\|_\infty \\
&\leq \exp(n\log(k_0)),
\end{aligned}
$$

for any $t \geq t_0$. Now observing that $n \geq t/t_0 - 1$, we easily deduce that $\exp(n\log(k_0)) \leq k_0^{-1}e^{\omega t}$, where $\omega = t_0^{-1}\log(k_0) < 0$. This finishes the proof. ■

The following result gives a sufficient condition which guarantees that $T(t)\mathbb{1}$ belongs to $C_0(\mathbb{R}^N)$, in terms of a suitable Lyapunov function φ.

Proposition 5.2.4 *Suppose that there exist $\lambda_0 > c_0$, a compact set K and a function $\varphi \in C^2(\mathbb{R}^N \setminus K) \cap C_0(\mathbb{R}^N \setminus K)$ such that*

$$\varphi(x) > 0, \qquad x \in \mathbb{R}^N \setminus K, \qquad \inf_{x\in\mathbb{R}^N\setminus K} \big(\lambda_0\varphi(x) - \mathcal{A}\varphi(x)\big) := a > 0, \quad (5.2.4)$$

Then, $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$ for any $t > 0$.

Proof. By virtue of Proposition 5.2.2, we can limit ourselves to proving that $R(\lambda_0, \widehat{A})\mathbb{1}$ belongs to $C_0(\mathbb{R}^N)$.

Let now n_0 be the smallest integer such that $K \subset B(n_0)$. Moreover, let $\gamma = \inf_{x\in\partial B(n_0)} \varphi(x) > 0$ and set $\delta = \min\{a, (\lambda_0 - c_0)\gamma\}$. We will prove that $R(\lambda_0, \widehat{A})\delta \in C_0(\mathbb{R}^N)$. Of course, this will imply that $R(\lambda_0, \widehat{A})\mathbb{1} = \delta^{-1}R(\lambda_0, \widehat{A})\delta$ belongs to $C_0(\mathbb{R}^N)$. For this purpose, we recall that $R(\lambda_0, \widehat{A})\delta$ is the pointwise limit of the sequence $\{u_n\}$, where, for any $n \in \mathbb{N}$, the function u_n is the unique solution in $\bigcap_{1\le p<+\infty} W^{2,p}(B(n))$ to the problem

$$\begin{cases} \lambda_0 u_n(x) - \mathcal{A}u_n(x) = \delta, & x \in B(n), \\ u_n(x) = 0, & x \in \partial B(n). \end{cases}$$

Moreover, $0 \le u_n \le (\lambda_0 - c_0)^{-1}\delta$; see the proof of Theorem 2.1.1.

Now, we observe that, for any $n > n_0$, the function φ satisfies

$$\begin{cases} \lambda_0\varphi(x) - \mathcal{A}\varphi(x) \ge a, & x \in B(n) \setminus \overline{B}(n_0), \\ \varphi(x) \ge \gamma, & x \in \partial B(n_0), \\ \varphi(x) > 0, & x \in \partial B(n). \end{cases}$$

Therefore, the classical maximum principle (see Theorem C.2.2(i)) implies that

$$\varphi(x) \ge u_n(x), \qquad x \in B(n) \setminus \overline{B}(n_0).$$

Letting n go to $+\infty$, we get

$$0 \le (R(\lambda_0, \widehat{A})\delta)(x) \le \varphi(x), \qquad x \in \mathbb{R}^N \setminus \overline{B}(n_0).$$

Hence, $R(\lambda_0, \widehat{A})\delta \in C_0(\mathbb{R}^N)$. ∎

Example 5.2.5 Consider the operator $\mathcal{A}$ defined on smooth functions by

$$\mathcal{A}f(x) = \Delta f(x) + \langle b(x), Df(x)\rangle, \qquad x \in \mathbb{R}^N,$$

where $b \in C^\alpha_{\rm loc}(\mathbb{R}^N)$ ($\alpha \in (0,1)$). Moreover, let φ be the function defined by $\varphi(x) = (1+|x|^2)^{-1}$ for any $x \in \mathbb{R}^N$. We have $D\varphi(x) = -2(\varphi(x))^2 x$ and $D^2\varphi(x) = -2(\varphi(x))^2 I_{N\times N} + 8(\varphi(x))^3 x \otimes x$ for any $x \in \mathbb{R}^N$. Therefore,

$$\mathcal{A}\varphi(x) = 8|x|^2(\varphi(x))^3 - 2N(\varphi(x))^2 - 2(\varphi(x))^2\langle b(x), x\rangle, \qquad x \in \mathbb{R}^N. \quad (5.2.5)$$

The condition (5.2.4) reads as

$$\langle b(x), x\rangle \geq \frac{a}{2}(1+|x|^2)^2 - \frac{\lambda_0}{2}(1+|x|^2) - N + 4\frac{|x|^2}{1+|x|^2}, \qquad x \in \mathbb{R}^N \setminus K,$$

for some compact set $K \subset \mathbb{R}^N$, some $\lambda_0 > 0$ and some positive constant a. Such a condition is satisfied, for instance, if

$$\langle b(x), x\rangle \geq C + M|x|^4, \quad (5.2.6)$$

for some $C \in \mathbb{R}$ and $M > a/2$. In such a case, we can take any $\lambda_0 > 0$. According to Proposition 5.2.4, the semigroup $\{T(t)\}$, associated with the operator $\mathcal{A}$, is such that $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$ for any $t > 0$.

The condition (5.2.6) is satisfied, for instance, when the operator $\mathcal{A}$ is given by

$$\mathcal{A}_+ f(x) = \Delta f(x) + |x|^2\langle x, Df(x)\rangle, \qquad x \in \mathbb{R}^N.$$

This example should be compared with Example 4.1.13, where we considered the operator

$$\mathcal{A}_- f(x) = \Delta f(x) - |x|^2\langle x, Df(x)\rangle, \qquad x \in \mathbb{R}^N,$$

and showed that the associated semigroup is conservative. The operators $\mathcal{A}_+$ and $\mathcal{A}_-$ differ only in the sign of the drift term. The reason for the difference in the behaviour of the two semigroups can be understood looking at the stochastic equations associated with these two operators. The operator $\mathcal{A}_-$ satisfies the Hypotheses 2.5.3 and 2.5.4. Therefore, all the results in Section 2.5 hold. The stochastic equation associated with it is

$$d\xi^x_t = -|\xi^x_t|^2\xi^x_t dt + dW_t, \qquad t > 0,$$

with the initial condition $\xi^x_0 = x \in \mathbb{R}^N$. This equation has a solution defined for any $t > 0$. Heuristically, one can think of this equation as a perturbation of the ordinary differential equation

$$\frac{d}{dt}\xi^x_t = -|\xi^x_t|^2\xi^x_t, \qquad t > 0,$$

whose solutions exist for any $t > 0$.

Repeating the same argument in the case of the operator $\mathcal{A}_+$ leads to the differential equation

$$\frac{d}{dt}\xi^x_t = |\xi^x_t|^2\xi^x_t, \qquad t > 0,$$

whose not vanishing solutions explode in a finite time. And also the solutions of the stochastic equation

$$d\xi_t^x = |\xi_t^x|^2 \xi_t^x dt + dW_t, \qquad t > 0,$$

with the initial condition $\xi_0^x = x \in \mathbb{R}^N$, do not exist for all $t > 0$. Indeed, the life time of the Markov process associated with $\{T(t)\}$ is finite since

$$\mathbb{P}_x\{t < \tau\} = (T(t)\mathbb{1})(x) < 1.$$

Example 5.2.6 Let $\mathcal{A}$ be the one-dimensional operator defined by

$$\mathcal{A}\varphi(x) = \varphi''(x) + |x|^3\varphi'(x), \qquad x \in \mathbb{R}.$$

The associated semigroup $\{T(t)\}$ is not conservative. Indeed, using the notation of Chapter 3 (see the formulas (3.1.3), (3.1.4) and (3.1.5)), it is easy to see that $Q \in L^1(-\infty, 0)$, $Q \notin L^1(0, +\infty)$, $R \notin L^1(-\infty, 0)$ and $R \in L^1(0, +\infty)$. Therefore, the point $-\infty$ is entrance and, hence, unaccessible while the point $+\infty$ is exit and, hence, accessible. According to Theorem 3.2.2, we conclude that the semigroup $\{T(t)\}$ is not conservative.

We now claim that $R(\lambda, \widehat{A})\mathbb{1}$ and $T(t)\mathbb{1}$ do not belong to $C_0(\mathbb{R}^N)$, for any $\lambda > 0$ and any $t > 0$, respectively. To show that $u = R(\lambda, \widehat{A})\mathbb{1}$ is not in $C_0(\mathbb{R})$, we observe that u is a solution of the elliptic equation

$$\lambda u(x) - u''(x) - |x|^3 u'(x) = 1, \qquad x \in \mathbb{R}, \tag{5.2.7}$$

and it satisfies $0 \le u \le 1/\lambda$. Moreover, since the coefficients of the operator $\mathcal{A}$ belong to $C^{2+\theta}_{\mathrm{loc}}(\mathbb{R})$ for any $\theta \in (0,1)$, it is readily seen that $u \in C^{4+\theta}_{\mathrm{loc}}(\mathbb{R})$.

We claim that $u'(x) \le 0$ for any $x \in \mathbb{R}$. This, of course, will imply that $u \notin C_0(\mathbb{R})$. By contradiction, suppose that $u'(x) > 0$ for any $x \in (a, b)$ for some $a, b \in \mathbb{R}$. Let

$$a_0 = \inf\{\alpha : \ u'(x) > 0 \ \ \forall x \in (\alpha, b)\}.$$

If $a_0 > -\infty$, we have $u'(a_0) = 0$ and $u'(x) > 0$ for any $x \in (a_0, b)$. Therefore, for such values of x,

$$u''(x) = \lambda u(x) - 1 - |x|^3 u'(x) \le 0.$$

But this is a contradiction. Similarly, if $a_0 = -\infty$ we have $u'(x) > 0$ and $u''(x) \le 0$ for any $x < b$, so that

$$\lim_{x \to -\infty} u'(x) > 0,$$

which is a contradiction since u is bounded. We conclude that $u' \le 0$. Since u does not identically vanish in $\mathbb{R}$, then $u \notin C_0(\mathbb{R})$.

To prove that, for any $t > 0$, $T(t)\mathbb{1}$ does not belong to $C_0(\mathbb{R}^N)$, we show that $D_x T(t)\mathbb{1} \le 0$. For this purpose, let $u = T(\cdot)\mathbb{1}$. According to Lemma 4.1.8 we know that $D_t u(t,x) \le 0$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Suppose that $D_x u(t,x) > 0$ at some point $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Then, we have

$$D_{xx}u(t,x) = D_t u(t,x) - |x|^3 D_x u(t,x) \le 0.$$

Repeating the same arguments as above, we see that the assumption on the positiveness of $D_x u(t,x)$ leads us to a contradiction. Therefore, $D_x u(t,x) \le 0$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Since $T(t)\mathbb{1}$ does not identically vanish in $\mathbb{R}^N$, we conclude that $T(t)\mathbb{1} \notin C_0(\mathbb{R}^N)$.

5.3 Invariance of $C_0(\mathbb{R}^N)$

In this section we determine sufficient conditions implying that $C_0(\mathbb{R}^N)$ is invariant under $T(t)$ for some $t > 0$. By the results in Section 5.2 we know that this is the case if $T(t)\mathbb{1}$ vanishes at infinity. But this is the case also under weaker assumptions as the next propositions show.

In Proposition 5.2.2 we have seen that $T(t)$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ for any $t > 0$ if and only if $R(\lambda, \widehat{A})$ does, for some (and hence all) $\lambda > 0$. As we are going to show, the same property holds true as far as the invariance of $C_0(\mathbb{R}^N)$ is concerned. Moreover, in the case when $C_0(\mathbb{R}^N)$ is invariant under the semigroup, then the restriction of the semigroup to $C_0(\mathbb{R}^N)$ defines a strongly continuous semigroup.

Proposition 5.3.1 *$C_0(\mathbb{R}^N)$ is invariant under the semigroup $\{T(t)\}$ if and only if it is invariant under $R(\lambda, \widehat{A})$ for any (equivalently, some) $\lambda > 0$. In such a case, the restriction of $\{T(t)\}$ to $C_0(\mathbb{R}^N)$ defines a strongly continuous semigroup. Its infinitesimal generator A_0 is the part of $\widehat{A}$ in $C_0(\mathbb{R}^N)$, i.e.,*

$$\begin{cases} D(A_0) = \{u \in D_{\max}(\mathcal{A}) \cap C_0(\mathbb{R}^N) : \mathcal{A}u \in C_0(\mathbb{R}^N)\}, \\ A_0 u = \mathcal{A}u, \qquad \text{for any } u \in D(A_0). \end{cases} \tag{5.3.1}$$

Proof. Suppose that $T(t)$ maps $C_0(\mathbb{R}^N)$ into itself for any $t > 0$. Then, by Proposition 2.2.7 we deduce that the restriction of $\{T(t)\}$ to $C_0(\mathbb{R}^N)$ is a strongly continuous semigroup in $C_0(\mathbb{R}^N)$. Since the resolvent operator associated with the restriction to $C_0(\mathbb{R}^N)$ of $\{T(t)\}$ is the restriction of $R(\lambda, \widehat{A})$ to $C_0(\mathbb{R}^N)$, we immediately deduce that $R(\lambda, \widehat{A})$ maps $C_0(\mathbb{R}^N)$ into itself for any $\lambda > c_0$.

Conversely, let us assume that $R(\lambda, \widehat{A})$ maps $C_0(\mathbb{R}^N)$ into itself for any $\lambda > c_0$. It is easy to show that, for any $\lambda > c_0$, the operator $\lambda I - A_0$ with domain (5.3.1) is bijective from $D(A_0)$ into $C_0(\mathbb{R}^N)$. Moreover,

$$R(\lambda, A_0) = R(\lambda, \widehat{A})_{|C_0(\mathbb{R}^N)}, \qquad \lambda > c_0 \tag{5.3.2}$$

and, consequently,

$$\|R(\lambda, A_0)\|_{L(C_0(\mathbb{R}^N))} \leq \frac{1}{\lambda - c_0}, \qquad \lambda > c_0. \tag{5.3.3}$$

Indeed, if $u \in D(A_0)$ is such that $\lambda u - A_0 u = 0$, then $u \in D(\widehat{A})$ (see (2.3.13)) and consequently $\lambda u - \widehat{A}u = 0$, implying that $u = 0$. Now take $f \in C_0(\mathbb{R}^N)$. The function $u = R(\lambda, \widehat{A})f$ is the unique solution in $D(\widehat{A})$ of the equation $\lambda u - \mathcal{A}u = f$. Since $R(\lambda, \widehat{A})$ maps $C_0(\mathbb{R}^N)$ into itself, then $u \in D(\widehat{A}) \cap C_0(\mathbb{R}^N)$ and $\mathcal{A}u \in C_0(\mathbb{R}^N)$. Still from (2.3.13) we deduce that $u \in D(A_0)$ and it solves the equation $\lambda u - A_0 u = f$. The formula (5.3.2) now easily follows. Similarly, (5.3.3) follows recalling that

$$\|R(\lambda, \widehat{A})\|_{L(C_b(\mathbb{R}^N))} \leq \frac{1}{\lambda - c_0}, \qquad \lambda > c_0;$$

see (2.1.4) and Section 2.3.

Now, the Hille-Yosida theorem (see Theorem B.1.5) implies that the operator A_0 with domain (5.3.1) is the generator of a strongly continuous semigroup $\{T_0(t)\}$ in $C_0(\mathbb{R}^N)$. By general results for strongly continuous semigroups we know that

$$R(\lambda, A_0)f = \int_0^{+\infty} e^{-\lambda t} T_0(t) f dt,$$

for any $\lambda > c_0$, where the integral converges in $C_0(\mathbb{R}^N)$. Since $R(\lambda, A_0)$ is the restriction of $R(\lambda, \widehat{A})$ to $C_0(\mathbb{R}^N)$ and

$$(R(\lambda, \widehat{A})g)(x) = \int_0^{+\infty} e^{-\lambda t} (T(t)g)(x) dt, \qquad x \in \mathbb{R}^N,$$

for any $g \in C_b(\mathbb{R}^N)$ (see (2.2.14), (2.3.1)), we deduce that

$$\int_0^{+\infty} e^{-\lambda t} \left((T(t) - T_0(t))f\right)(x) dt = 0,$$

for any $f \in C_0(\mathbb{R}^N)$, any $x \in \mathbb{R}^N$ and any $\lambda > c_0$. The uniqueness of the Laplace transform implies that $T(t)f = T_0(t)f$ for any $t > 0$. Hence, $T(t)$ maps $C_0(\mathbb{R}^N)$ into itself for any $t > 0$.

Finally, we show that, if $C_0(\mathbb{R}^N)$ is invariant under $R(\lambda_0, \widehat{A})$ for some $\lambda_0 > c_0$, then it is invariant under $R(\lambda, \widehat{A})$ for any $\lambda > c_0$. For this purpose, we set

$$\mathcal{F} = \{\lambda \in (c_0, +\infty) : C_0(\mathbb{R}^N) \text{ is invariant under } R(\lambda, \widehat{A})\}$$

and prove that $\mathcal{F}$ is open and closed in $(c_0, +\infty)$. This will easily imply that $\mathcal{F} = (c_0, +\infty)$, since, by assumptions, $\mathcal{F}$ is not empty. We begin by proving that $\mathcal{F}$ is an open set. We fix $\lambda_1 \in \mathcal{F}$, $f \in C_0(\mathbb{R}^N)$ and prove that, if λ is sufficiently close to λ_1, the equation $\lambda u - \mathcal{A}u = f$ admits a solution $u \in D(A_0)$. As it is immediately seen, $u \in D(A_0)$ is a solution to the previous equation if and only if

$$(\lambda - \lambda_1)R(\lambda_1, \widehat{A})u + u = R(\lambda_1, \widehat{A})f. \tag{5.3.4}$$

Since $C_0(\mathbb{R}^N)$ is invariant under $R(\lambda_1, \widehat{A})$, the operator $G = (\lambda - \lambda_1)R(\lambda_1, \widehat{A}) + I$ is bounded from $C_0(\mathbb{R}^N)$ into itself. Moreover, if $|\lambda - \lambda_1| < |\lambda_1 - c_0|$, then $\|(\lambda - \lambda_1)R(\lambda_1, \widehat{A})\|_{C_0(\mathbb{R}^N)} < 1$ (see again Sections 2.1 and 2.3). Therefore, G is bijective in $C_0(\mathbb{R}^N)$ and the equation (5.3.4) is uniquely solvable in $C_0(\mathbb{R}^N)$. This implies that $R(\lambda, \widehat{A})f \in C_0(\mathbb{R}^N)$.
Let us now suppose that $\lambda \in (c_0, +\infty)$ is the limit of a sequence $\{\lambda_n\} \subset \mathcal{F}$. Then, for any $n \in \mathbb{N}$, $R(\lambda_n, \widehat{A})$ is a bounded operator in $C_0(\mathbb{R}^N)$. Since $R(\lambda_n, \widehat{A})$ converges to $R(\lambda, \widehat{A})$ in $L(C_b(\mathbb{R}^N))$, it follows that $R(\lambda, \widehat{A})$ belongs to $L(C_0(\mathbb{R}^N))$ as well, and $\mathcal{F}$ is closed. ∎

The next theorem gives a sufficient condition for the invariance of $C_0(\mathbb{R}^N)$. It should be compared with Proposition 5.2.4.

Theorem 5.3.2 *Suppose that there exist $\lambda_0 > 0$, a compact set F and a strictly positive function $\varphi \in C^2(\mathbb{R}^N \setminus F) \cap C_0(\mathbb{R}^N \setminus F)$ such that*

$$\lambda_0\varphi(x) - \mathcal{A}\varphi(x) \geq 0, \qquad x \in \mathbb{R}^N \setminus F. \tag{5.3.5}$$

Then $C_0(\mathbb{R}^N)$ is invariant under $T(t)$ for any $t > 0$.

Proof. We prove that $C_0(\mathbb{R}^N)$ is invariant under $R(\lambda_0 + 1, \widehat{A})$. Then, the conclusion will follow from Proposition 5.3.1.

We take a compactly supported function $f \in C_0(\mathbb{R}^N)$ and prove that $u = R(\lambda_0, \widehat{A})f$ belongs to $C_0(\mathbb{R}^N)$. The general case will follow by density since any function $f \in C_0(\mathbb{R}^N)$ can be approximated uniformly in $\mathbb{R}^N$ by a sequence of compactly supported functions.

We fix $n_0 \in \mathbb{N}$ such that $B(n_0)$ contains both F and the support of f. Moreover, replacing φ with $\delta\varphi$ for a sufficiently large positive constant δ, we can assume that

$$\inf_{x \in \partial B(n_0)} \varphi \geq (\lambda_0 - c_0)^{-1}\|f\|_\infty.$$

Finally, for any $n \in \mathbb{N}$, let u_n be the (unique) solution in $\bigcap_{1 \leq p < +\infty} W^{2,p}(B(n))$ of the equation

$$\begin{cases} \lambda_0 u_n(x) - \mathcal{A}u_n(x) = f(x), & x \in B(n), \\ u_n(x) = 0, & x \in \partial B(n). \end{cases}$$

According to the proof of Theorem 2.1.1, u is the pointwise limit of the sequence $\{u_n\}$ and

$$\|u_n\|_\infty \leq \frac{1}{\lambda_0 - c_0}\|f\|_\infty,$$

for any $n \in \mathbb{N}$. Since f vanishes outside $B(n_0)$, the function $w_n = \varphi - u_n$ satisfies $\lambda_0 w - \mathcal{A}w \geq 0$ in $B(n) \setminus \overline{B}(n_0)$ $(n > n_0)$, as well as the boundary conditions $w \geq 0$ on $\partial B(n_0) \cup \partial B(n)$. The classical maximum principle (see Theorem C.2.2(i)) now implies that $w \geq 0$ in $B(n) \setminus \overline{B}(n_0)$. Thus, letting n go to $+\infty$, we obtain that

$$0 \leq (R(\lambda_0, \widehat{A})f)(x) \leq \varphi(x), \qquad x \in \mathbb{R}^N \setminus \overline{B}(n_0).$$

This finishes the proof. ■

Example 5.3.3 Consider again the operator $\mathcal{A}$ and the function φ defined in Example 5.2.5. Taking (5.2.5) into account, it is immediate to see that the condition (5.3.5) reads as

$$\langle b(x), x\rangle \geq -\frac{\lambda_0}{2}(1+|x|^2) - N + 4\frac{|x|^2}{1+|x|^2}, \qquad x \in \mathbb{R}^N,$$

for some $\lambda_0 > 0$. If it is satisfied then $C_0(\mathbb{R}^N)$ is invariant. This holds, for instance, if we have

$$\langle b(x), x\rangle \geq C - M|x|^2, \qquad x \in \mathbb{R}^N, \tag{5.3.6}$$

for some $C \in \mathbb{R}$ and $M > 0$. Indeed, it suffices to take $\lambda_0 > 2M$.

The condition (5.3.6) should be compared with the condition (5.2.6), which guarantees that the semigroup associated with the operator $\mathcal{A}$ maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ and hence, in particular, leaves $C_0(\mathbb{R}^N)$ invariant.

We conclude this section with the following proposition which gives a sufficient condition ensuring that $C_0(\mathbb{R}^N)$ is not preserved by the action of the semigroup. It is a immediate corollary of Theorem 5.1.11.

Proposition 5.3.4 *Fix $t > 0$. If $T(t)\mathbb{1} = \mathbb{1}$ and $T(t)$ is compact, then $C_0(\mathbb{R}^N)$ is not invariant for $T(t)$.*

Chapter 6

Uniform estimates for the derivatives of $T(t)f$

6.0 Introduction

In this chapter we prove some uniform estimates for the derivatives of the function $T(t)f$, when $f \in C_b(\mathbb{R}^N)$ and $\{T(t)\}$ is the semigroup associated with the uniformly elliptic operator

$$\mathcal{A} = \sum_{i,j=1}^{N} q_{ij}(x)D_{ij} + \sum_{j=1}^{N} b_j(x)D_j,$$

with unbounded coefficients in $\mathbb{R}^N$.

The problem of estimating the derivatives of $T(t)f$ has been studied in literature with both analytic ([13, 15, 108]) and probabilistic methods ([29, 30, 143]).

Here, we present the results of [18]. More precisely, we prove uniform estimates for the first-, second- and third-order derivatives of $T(t)f$. First, we show that, for any $\omega > 0$ and any $k, l \in \mathbb{N}$, with $0 \le k \le l \le 3$, there exists a positive constant $C_{k,l} = C_{k,l}(\omega)$ such that

$$\|T(t)f\|_{C_b^l(\mathbb{R}^N)} \le C_{k,l} t^{-\frac{l-k}{2}} e^{\omega t} \|f\|_{C_b^k(\mathbb{R}^N)}, \qquad f \in C_b^k(\mathbb{R}^N),\ t > 0. \quad (6.0.1)$$

Although we limit ourselves to the case when $l \le 3$, the techniques that we present work as well for $l > 3$ under suitable additional assumptions on the coefficients.

To prove (6.0.1) we use the Bernstein method (see [14]) and approximate $T(t)f$ by solutions of Cauchy problems in bounded domains. We assume weak dissipativity-type and growth conditions on the coefficients of $\mathcal{A}$. We notice that some dissipativity condition is necessary, because in general the estimate (6.0.1) fails; see Example 6.1.11.

By interpolation, we can then extend the estimate (6.0.1) to the case when $k, l \in \mathbb{R}_+$, $0 \le k \le l \le 3$. This allows us to prove optimal Schauder estimates for the solution of the nonhomogeneous Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + g(t,x), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases}$$

as well as for the elliptic equation $\lambda u - \mathcal{A}u = f$, $(\lambda > 0)$.

6.1 Uniform estimates

Throughout this section we assume the following hypotheses:

Hypotheses 6.1.1 (i) $q_{ij}(x) = q_{ji}(x)$ for any $i,j = 1,\dots,N$ and any $x \in \mathbb{R}^N$, and

$$\sum_{i,j=1}^{N} q_{ij}(x)\xi_i\xi_j \geq \kappa(x)|\xi|^2, \qquad \xi, x \in \mathbb{R}^N,$$

for some function $\kappa : \mathbb{R}^N \to \mathbb{R}$ such that

$$\inf_{x\in\mathbb{R}^N} \kappa(x) = \kappa_0 > 0;$$

(ii) there exist a function $\varphi \in C^2(\mathbb{R}^N)$ and two positive constants λ_0 and C such that

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty, \qquad \mathcal{A}\varphi(x) - \lambda_0\varphi(x) \leq C, \qquad x \in \mathbb{R}^N;$$

(iii) there exists a constant $C > 0$ such that

$$\left|\sum_{i,j=1}^{N} q_{ij}(x)x_j\right| \leq C(1+|x|^2)\kappa(x), \qquad \mathrm{Tr}(Q(x)) \leq C(1+|x|^2)\kappa(x), \tag{6.1.1}$$

$$\sum_{i=1}^{N} b_i(x)x_i \leq C(1+|x|^2)\kappa(x), \tag{6.1.2}$$

for any $x \in \mathbb{R}^N$, $i = 1,\dots,N$.

Moreover, in the next theorems we always assume that one of the following hypotheses holds true:

(iv-1) $q_{ij}, b_j \in C^{1+\delta}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\delta \in (0,1)$ and any $i,j = 1,\dots,N$, and there exist a positive constant C and a function $d : \mathbb{R}^N \to \mathbb{R}$ with $L_1 := \sup_{x\in\mathbb{R}^N}\{d(x)/\kappa(x)\} < +\infty$, such that $|D_k q_{ij}(x)| \leq C\kappa(x)$ for any $i,j,k = 1,\dots,N$ and

$$\sum_{i,j=1}^{N} D_i b_j(x)\xi_i\xi_j \leq d(x)|\xi|^2, \qquad x, \xi \in \mathbb{R}^N; \tag{6.1.3}$$

(iv-2) $q_{ij}, b_j \in C^{2+\delta}_{\rm loc}(\mathbb{R}^N)$ for some $\delta \in (0,1)$ and any $i,j = 1,\ldots,N$; Hypothesis 6.1.1(iv-1) holds true and there exist a positive function $r : \mathbb{R}^N \to \mathbb{R}$ and three constants $K_1 \in \mathbb{R}$, $L_2, L_3 > 0$ such that

$$|D^\beta b_j(x)| \le r(x), \qquad x \in \mathbb{R}^N,\ \ j = 1,\ldots,N,\ \ |\beta| = 2,$$

$$d(x) + L_2 r(x) \le L_3 \kappa(x), \qquad x \in \mathbb{R}^N$$

and

$$\sum_{i,j,h,k=1}^N D_{hk} q_{ij}(x) m_{ij} m_{hk} \le K_1 \kappa(x) \sum_{h,k=1}^N m_{hk}^2, \qquad x \in \mathbb{R}^N,$$

for any symmetric matrix $M = (m_{hk})$;

(iv-3) $q_{ij}, b_j \in C^{3+\delta}_{\rm loc}(\mathbb{R}^N)$ for some $\delta \in (0,1)$ and any $i,j = 1,\ldots,N$; Hypothesis 6.1.1(iv-2) holds true and there exists a constant $C > 0$ such that $|D^\beta b_j(x)| \le r(x)$ and $|D^\beta q_{ij}(x)| \le C\kappa(x)$ for any $i,j = 1,\ldots,N$, any $|\beta| = 3$ and any $x \in \mathbb{R}^N$.

Remark 6.1.2 Differently from what we did in the previous chapters, here we assume that the function κ is far from zero.

Remark 6.1.3 We remark that in 6.1.1(iv-l) ($l = 2,3$) we can take

$$r(x) = L_4(1 + |d(x)|),$$

for any $L_4 > 0$. It is sufficient to fix L_2 and L_3 satisfying

$$L_2 < L_4^{-1}, \qquad L_3 = (1 + L_2 L_4)\left(\sup_{x\in\mathbb{R}^N} \frac{d(x)}{\kappa(x)}\right)^+ + L_2 L_4 \kappa_0^{-1}$$

in (iv-2).

Remark 6.1.4 In some situation, Hypothesis 6.1.1(iii) is easily implied by Hypothesis 6.1.1(iv-1). This is the case, for instance, when there exists a positive constant K such that

$$\int_0^1 \kappa(tx)dt \le K\kappa(x), \qquad x \in \mathbb{R}^N. \tag{6.1.4}$$

We limit ourselves to showing that in such a situation, the condition (6.1.2) follows from Hypothesis 6.1.1(iv-1). The same argument can be used to prove that also (6.1.1) is a consequence of Hypothesis 6.1.1(iv-1).

To show (6.1.2) we observe that taking $(x,\xi)=(tz,z)$ in (6.1.3) gives

$$\frac{d}{dt}\sum_{j=1}^N b_j(tz)z_j \le L_1\kappa(tz)|z|^2, \qquad z\in\mathbb{R}^N. \tag{6.1.5}$$

Integrating (6.1.5) with respect to $t\in[0,1]$ and taking (6.1.4) into account, we get

$$\begin{aligned}\sum_{j=1}^N b_j(z)z_j &\le \sum_{j=1}^N b_j(0)z_j + L_1|z|^2\int_0^1 \kappa(tz)dt\\ &\le |z||b(0)| + KL_1|z|^2\kappa(z)\\ &\le C'|z|(1+|z|)\kappa(z),\end{aligned} \tag{6.1.6}$$

for any $z\in\mathbb{R}^N$, where $b(0)=(b_1(0),\ldots,b_N(0))$, $C'=\max\{|b(0)|\kappa_0^{-1},KL_1\}$.

Let us observe that (6.1.4) is satisfied, for instance, in the case when $\kappa(x)=\overline{\kappa}(|x|)$ for any $x\in\mathbb{R}^N$ and some nondecreasing function $\overline{\kappa}:\mathbb{R}_+\to\mathbb{R}_+$, and in the case when $\kappa(x)=c+\widetilde{\kappa}(x)$, c and $\widetilde{\kappa}$ being, respectively, a positive constant and a nonnegative homogeneous function of degree $\alpha>0$.

In the sequel, for any $n\in\mathbb{N}$, we denote by u_n the (unique) classical solution of the Cauchy-Dirichlet problem

$$\begin{cases} D_t u_n(t,x) = \mathcal{A}u_n(t,x), & t>0,\ x\in B(n),\\ u_n(t,x)=0, & t>0,\ x\in\partial B(n),\\ u_n(0,x)=f(x), & x\in B(n),\end{cases} \tag{6.1.7}$$

corresponding to $f\in C_b(\mathbb{R}^N)$ (i.e., the unique solution in $C_b([0,+\infty)\times\overline{B}(n))\cap C^{1,2}((0,+\infty)\times\overline{B}(n))$). Here, $\vartheta_n:\mathbb{R}^N\to\mathbb{R}$ is defined by

$$\vartheta_n(x)=\varphi(|x|/n), \qquad x\in\mathbb{R}^N, \qquad n\in\mathbb{N}, \tag{6.1.8}$$

$\varphi\in C_c^\infty(\mathbb{R})$ being any nonincreasing function such that $\chi_{(-1/2,1/2)}\le\varphi\le\chi_{(-1,1)}$.

Remark 6.1.5 Repeating the same arguments as in Theorems 2.2.1 and in Remark 2.2.3 and taking Remark 4.1.4 and Theorem 4.1.3 into account, it is easy to check that u_n converges to $u:=T(\cdot)f$ locally uniformly in $(0,+\infty)\times\mathbb{R}^N$ as n tends to $+\infty$. From the interior Schauder estimate in Theorem C.1.4(ii), we easily deduce that

$$\lim_{n\to+\infty}\sup_{\varepsilon\le t\le T}\|u_n(t,\cdot)-u(t,\cdot)\|_{C^3(B(k))}=0, \tag{6.1.9}$$

for any $0<\varepsilon<T$ and any $k\in\mathbb{N}$.

To prove the main result of this section we need the following lemma.

Lemma 6.1.6 *Let $\Omega \subset \mathbb{R}^N$ be any smooth bounded open set. Assume that Hypothesis* 6.1.1(*i*) *is satisfied in $\overline{\Omega}$ and that the coefficients belong to $C^{k+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $k \in \{1,2,3\}$. Then, for any $f \in C_c^j(\overline{\Omega})$ $(j = 0,\ldots,k)$, the function $(t,x) \mapsto t^{(k-j)/2}(D^k T_n(t)f)(x)$ is continuous in $[0,T]\times\overline{\Omega}_\delta$ for any $T>0$ and any $\delta>0$, where $\Omega_\delta = \{x\in\Omega : \delta \le \inf_{y\in\partial\Omega}|x-y|\}$.*

Proof. The proof follows from a density argument. We limit ourselves to proving it in the case when $j = k = 3$, the other cases being completely similar, and even simpler. It is well known that for any $f \in C_c^\infty(\Omega)$, the function $D^kT_n(\cdot)f$ $(k = 1,2,3)$ is continuous in $[0,T]\times\overline{\Omega}$ (see Proposition C.3.2). Using a method similar to that used in the proof of the following Theorem 6.1.7, we can easily show that there exists a positive constant $C = C_{T,\delta}$, independent of f, such that

$$\|T(t)f\|_{C^3(\overline{\Omega}_\delta)} \le C\|f\|_{C^3(\overline{\Omega})}, \qquad t\in[0,T]. \tag{6.1.10}$$

Now with any $f\in C_c^3(\overline{\Omega})$ we associate a sequence of smooth functions $\{f_n\}\subset C_c^\infty(\Omega)$ converging to f in $C^k(\overline{\Omega})$. Since, for any $n\in\mathbb{N}$ and $l\in\{1,2,3\}$, the function $D^lT(\cdot)f_n$ is continuous in $[0,T]\times\overline{\Omega}$ for any $T>0$, taking (6.1.10) into account, we easily get the assertion. ■

Now, we prove the main result of this section.

Theorem 6.1.7 *Let Hypotheses* 6.1.1(*i*)–6.1.1(*iii*), 6.1.1(*iv-l*) *be satisfied for some $l\in\{1,2,3\}$. Then, for any $\omega>0$ and any $k=0,\ldots,l$, there exist constants $C_{k,l}=C_{k,l}(\omega)>0$ such that*

$$\|T(t)f\|_{C_b^l(\mathbb{R}^N)} \le C_{k,l}t^{-\frac{l-k}{2}}e^{\omega t}\|f\|_{C_b^k(\mathbb{R}^N)}, \qquad f\in C_b^k(\mathbb{R}^N),\ t>0 \tag{6.1.11}$$

holds true. In particular, if $k=l$ we can take $\omega=0$ in (6.1.11).

Proof. We begin the proof considering the case when $k=0$ and $l=3$. For any $n\in\mathbb{N}$, let ϑ_n be the smooth function defined in (6.1.8). We fix $\omega>0$ and $t_0>0$ such that

$$e^{\omega t_0}t^{-\frac{3}{2}} \ge 1, \qquad t\ge t_0, \tag{6.1.12}$$

and define the function

$$\begin{aligned} v_{0,3,n}(t,x) = {} & |u_n(t,x)|^2 + at\vartheta_n^2|Du_n(t,x)|^2 + a^2t^2\vartheta_n^4|D^2u_n(t,x)|^2 \\ & + a^3t^3\vartheta_n^6|D^3u_n(t,x)|^2, \end{aligned}$$

for any $t\in[0,t_0]$ and any $x\in B(n)$, where $u_n = T_n(\cdot)(\vartheta_n f)$ is the solution of the Cauchy-Dirichlet problem (6.1.7). For notational convenience, throughout the rest of the proof, we drop out the dependence on n, when there is no damage of confusion. Observing that u_n is in $C([0,+\infty);C(\overline{B}(n)))$ (see Proposition B.2.5(i) and Theorem C.3.6(iv)) and taking Lemma 6.1.6 into account, it follows that $v_{0,3}$ is continuous in $[0,t_0]\times\overline{B}(n)$ and, with some computations, one can see that $v_{0,3}$ solves the Cauchy problem

$$\begin{cases} D_t v_{0,3}(t,x)=\mathcal{A}v_{0,3}(t,x)+g(t,x), & t\in(0,t_0],\ x\in B(n),\\ v_{0,3}(t,x)=0, & t\in[0,t_0],\ x\in\partial B(n),\\ v_{0,3}(0,x)=(\vartheta f)^2(x), & x\in\overline{B}(n), \end{cases} \tag{6.1.13}$$

where $g=\sum_{j=1}^8 g_j$ with

$$\begin{aligned} g_1 &= -2\sum_{i,j=1}^N q_{ij}D_iuD_ju-2at\vartheta^2\sum_{i,j,h=1}^N q_{ij}D_{ih}uD_{jh}u\\ &\quad -2a^2t^2\vartheta^4\sum_{i,j,h,k=1}^N q_{ij}D_{ihk}uD_{jhk}u-2a^3t^3\vartheta^6\sum_{i,j,h,k,l=1}^N q_{ij}D_{ihkl}uD_{jhkl}u, \end{aligned} \tag{6.1.14}$$

$$\begin{aligned} g_2 &= -2at|Du|^2\sum_{i,j=1}^N q_{ij}D_i\vartheta D_j\vartheta-12a^2t^2\vartheta^2|D^2u|^2\sum_{i,j=1}^N q_{ij}D_i\vartheta D_j\vartheta\\ &\quad -30a^3t^3\vartheta^4|D^3u|^2\sum_{i,j=1}^N q_{ij}D_i\vartheta D_j\vartheta, \end{aligned} \tag{6.1.15}$$

$$\begin{aligned} g_3 &= -2at\vartheta\mathcal{A}(\vartheta)|Du|^2-4a^2t^2\vartheta^3\mathcal{A}(\vartheta)|D^2u|^2-6a^3t^3\vartheta^5\mathcal{A}(\vartheta)|D^3u|^2\\ &\quad -8at\vartheta\sum_{i,j,h=1}^N q_{ij}D_j\vartheta D_huD_{ih}u-16a^2t^2\vartheta^3\sum_{i,j,h,k=1}^N q_{ij}D_j\vartheta D_{hk}uD_{ihk}u\\ &\quad -24a^3t^3\vartheta^5\sum_{i,j,h,k,l=1}^N q_{ij}D_j\vartheta D_{hkl}uD_{ihkl}u, \end{aligned} \tag{6.1.16}$$

$$\begin{aligned} g_4 &= 2at\vartheta^2\sum_{j,h=1}^N D_hb_jD_juD_hu+4a^2t^2\vartheta^4\sum_{j,h,k=1}^N D_hb_jD_{jk}uD_{hk}u\\ &\quad +6a^3t^3\vartheta^6\sum_{j,h,k,l=1}^N D_hb_jD_{jkl}uD_{hkl}u, \end{aligned} \tag{6.1.17}$$

$$
\begin{aligned}
g_5 = {} & 2at\vartheta^2 \sum_{i,j,h=1}^{N} D_h q_{ij} D_h u D_{ij} u + 4a^2t^2\vartheta^4 \sum_{i,j,h,k=1}^{N} D_h q_{ij} D_{hk} u D_{ijk} u \\
& + 6a^3t^3\vartheta^6 \sum_{i,j,h,k,l=1}^{N} D_h q_{ij} D_{hkl} u D_{ijkl} u,
\end{aligned}
\tag{6.1.18}
$$

$$
\begin{aligned}
g_6 = {} & 2a^2t^2\vartheta^4 \bigg(\sum_{i,j,h,k=1}^{N} D_{hk} q_{ij} D_{ij} u D_{hk} u + \sum_{j,h,k=1}^{N} D_{hk} b_j D_j u D_{hk} u \bigg) \\
& + 6a^3t^3\vartheta^6 \bigg(\sum_{i,j,h,k,l=1}^{N} D_{hk} q_{ij} D_{ijl} u D_{hkl} u + \sum_{j,h,k,l=1}^{N} D_{hk} b_j D_{jl} u D_{hkl} u \bigg),
\end{aligned}
\tag{6.1.19}
$$

$$
g_7 = 2a^3t^3\vartheta^6 \bigg(\sum_{i,j,h,k,l=1}^{N} D_{hkl} q_{ij} D_{ij} u D_{hkl} u + \sum_{j,h,k,l=1}^{N} D_{hkl} b_j D_j u D_{hkl} u \bigg),
\tag{6.1.20}
$$

$$
g_8 = a\vartheta^2|Du|^2 + 2a^2t\vartheta^4|D^2u|^2 + 3a^3t^2\vartheta^6|D^3u|^2. \tag{6.1.21}
$$

Taking the ellipticity condition 6.1.1(i) into account, we easily deduce that

$$
g_1 \le -2\kappa|Du|^2 - 2at\vartheta^2\kappa|D^2u|^2 - 2a^2t^2\vartheta^4\kappa|D^3u|^2 - 2a^3t^3\vartheta^6\kappa|D^4u|^2 \tag{6.1.22}
$$

and that $g_2 \le 0$.

To estimate the function g_3 we observe that, by virtue of (6.1.1) and recalling that $\varphi' = 0$ in $(-1/2, 1/2)$, it can be easily shown that

$$
\begin{aligned}
& |\mathrm{Tr}(Q(x)D^2\vartheta(x))| \\
& \le |\varphi'(|x|/n)| \bigg(\frac{1}{|x|n}\mathrm{Tr}(Q(x)) + \frac{1}{|x|^3n}\langle Q(x)x, x\rangle \bigg) \\
& \quad + |\varphi''(|x|/n)| \frac{1}{|x|^2n^2}\langle Q(x)x, x\rangle \\
& \le C\kappa(x) \sup_{\frac{n}{2}\le|x|\le n} \bigg[|\varphi'(|x|/n)| \bigg(\frac{1+|x|^2}{|x|n} + \frac{1+|x|^2}{|x|^3n}\sum_{i=1}^{N}|x_i| \bigg) \\
& \qquad + C|\varphi''(|x|/n)| \frac{1+|x|^2}{|x|^2n^2}\sum_{i=1}^{N}|x_i| \bigg] \\
& \le 4C\left(1+2\sqrt{N}\right)\|\varphi'\|_\infty\kappa(x) + 4C\sqrt{N}\|\varphi''\|_\infty\kappa(x) =: C'\kappa(x),
\end{aligned}
\tag{6.1.23}
$$

for any $x \in B(n)$ and any $n \geq 1$. Similarly,

$$\left| \sum_{i,j=1}^N q_{ij}(x) D_j \vartheta(x) \right| \leq C\kappa(x) \sup_{\frac{n}{2} \leq |x| \leq n} \left(|\varphi'(|x|/n)| \frac{1+|x|^2}{|x|n} \right)$$
$$\leq 4C\|\varphi'\|_\infty \kappa(x) =: C''\kappa(x), \tag{6.1.24}$$

for any $i = 1, \ldots, N$. Using (6.1.2) and recalling that φ' is nonincreasing in $\mathbb{R}$, we obtain that

$$\begin{aligned} -\sum_{i=1}^N b_i(x) D_i \vartheta(x) &= -\varphi'(|x|/n) \frac{1}{|x|n} \sum_{j=1}^N b_j(x) x_j \\ &\leq C\kappa(x) \sup_{\frac{n}{2} \leq |x| \leq n} \left(-\varphi'(|x|/n) \frac{1+|x|^2}{|x|n} \right) \\ &\leq C''\kappa(x), \end{aligned} \tag{6.1.25}$$

for any $x \in B(n)$.

Taking (6.1.23), (6.1.24) and (6.1.25) into account and recalling that for any $a, b, \varepsilon > 0$ it holds that $ab \leq (4\varepsilon)^{-1} a^2 + \varepsilon^2 b^2$, we can now show that

$$\begin{aligned} g_3 \leq\; & 2aC't\kappa\vartheta|Du|^2 + 4a^2C't^2\kappa\vartheta^3|D^2u|^2 + 6a^3C't^3\kappa\vartheta^5|D^3u|^2 \\ & + 2aC''t\kappa\vartheta|Du|^2 + 4a^2C''t^2\kappa\vartheta^3|D^2u|^2 + 6a^3C''t^3\kappa\vartheta^5|D^3u|^2 \\ & + 8aC''t\kappa \left(\frac{N}{4\varepsilon}|Du|^2 + \varepsilon\vartheta^2|D^2u|^2 \right) \\ & + 16a^2C''t^2\kappa \left(\frac{N}{4\varepsilon}\vartheta^2|D^2u|^2 + \varepsilon\vartheta^4|D^3u|^2 \right) \\ & + 24a^3C''t^3\kappa \left(\frac{N}{4\varepsilon}\vartheta^4|D^3u|^2 + \varepsilon\vartheta^6|D^4u|^2 \right) \\ \leq\; & 2a\left(C' + C'' + C''\frac{N}{\varepsilon} \right) t\kappa|Du|^2 \\ & + 4a\left(aC't + aC''t + 2C''\varepsilon + aC''t\frac{N}{\varepsilon} \right) t\kappa\vartheta^2|D^2u|^2 \\ & + 2a^2\left(3aC't + 3aC''t + 8C''\varepsilon + 3aC''t\frac{N}{\varepsilon} \right) t^2\kappa\vartheta^4|D^3u|^2 \\ & + 24a^3C''t^3\varepsilon\kappa\vartheta^6|D^4u|^2. \end{aligned} \tag{6.1.26}$$

Taking advantage of Hypothesis 6.1.1(iv-1), we deduce that

$$g_4 \leq 2atd\,\vartheta^2|Du|^2 + 4a^2t^2d\,\vartheta^4|D^2u|^2 + 6a^3t^3d\,\vartheta^6|D^3u|^2. \tag{6.1.27}$$

The terms g_5, g_6 and g_7 can be estimated in a similar way taking Hypothesis 6.1.1(iv-3) into account, and they yield

$$
\begin{aligned}
g_5 \le\ & atC\frac{N^2}{2\varepsilon}\kappa|Du|^2 + aCN\left(2\varepsilon + at\frac{N}{\varepsilon}\right)t\kappa\,\vartheta^2|D^2u|^2 \\
& + a^2CN\left(4\varepsilon + 3at\frac{N}{2\varepsilon}\right)t^2\kappa\,\vartheta^4|D^3u|^2 + 6a^3t^3\varepsilon CN\kappa\,\vartheta^4|D^4u|^2,
\end{aligned}
\tag{6.1.28}
$$

$$
\begin{aligned}
g_6 \le\ & a^2t^2\vartheta^2 r\frac{N^2}{2\varepsilon}|Du|^2 + a^2t^2\vartheta^4\left[2K_1\kappa + N\left(2\varepsilon + 3at\frac{N}{2\varepsilon}\right)r\right]|D^2u|^2 \\
& + 6a^3t^3\vartheta^6\left(K_1\kappa + \varepsilon Nr\right)|D^3u|^2,
\end{aligned}
\tag{6.1.29}
$$

$$
\begin{aligned}
g_7 \le\ & a^3t^3\vartheta^2 r\frac{N^3}{2\varepsilon}|Du|^2 + a^3t^3\vartheta^4 C\kappa\frac{N^3}{2\varepsilon}|D^2u|^2 \\
& + 2a^3t^3\vartheta^6\varepsilon N\left(C\kappa N + r\right)|D^3u|^2.
\end{aligned}
\tag{6.1.30}
$$

From (6.1.22), (6.1.26)-(6.1.30) we immediately deduce that for any $t \in (0, t_0]$ we have

$$
\begin{aligned}
g \le\ & \Bigg\{-\kappa_0 + a + \kappa\left[-1 + 2at\left(C' + C'' + C''\frac{N}{\varepsilon}\right) + atC\frac{N^2}{2\varepsilon} + a^2t^2C\frac{N^2}{2\varepsilon}\right] \\
& \qquad + at\left[2d + at(1 + atN)\frac{N^2}{2\varepsilon}r\right]\Bigg\}|Du|^2 \\
& + a\Bigg\{-\kappa_0 + 2a + \kappa\Bigg[-1 + 4\left(aC't + aC''t + 2C''\varepsilon + aC''t\frac{N}{\varepsilon}\right) \\
& \qquad + CN\left(2\varepsilon + at\frac{N}{\varepsilon}\right) + 2K_1^+at + a^2t^2C\frac{N^3}{2\varepsilon}\Bigg] \\
& \qquad + at\left[4d + \left(2\varepsilon N + 3at\frac{N^2}{2\varepsilon}\right)r\right]\Bigg\}t\vartheta^2|D^2u|^2 \\
& + a^2\Bigg\{-\kappa_0 + 3a + \kappa\Bigg[-1 + 2\left(3aC't + 3aC''t + 8C''\varepsilon + 3aC''t\frac{N}{\varepsilon}\right) \\
& \qquad + CN\left(4\varepsilon + 3at\frac{N}{2\varepsilon}\right) + 6atK_1^+ + 2at\varepsilon CN^2\Bigg] \\
& \qquad + 2at\big(3d + 4\varepsilon Nr\big)\Bigg\}t^2\vartheta^4|D^3u|^2 \\
& + 2a^3\left(-1 + 12C''\varepsilon + 3\varepsilon CN\right)t^3\kappa\,\vartheta^6|D^4u|^2.
\end{aligned}
\tag{6.1.31}
$$

We now choose (a,ε), sufficiently small, satisfying the following set of inequalities:

$$\begin{cases} 3a-\kappa_0\le 0, \qquad 4\varepsilon N\le 3L_2, \qquad at_0(1+at_0N)\dfrac{N^2}{2\varepsilon}\le 2L_2,\\[2mm] 2\varepsilon N+3at_0\dfrac{N^2}{2\varepsilon}\le 4L_2,\\[2mm] -1+at_0\left[4\left(C'+C''+C''\dfrac{N}{\varepsilon}+L_3\right)+C\dfrac{N^2}{\varepsilon}+2K_1^+\right]\\ \qquad +2\varepsilon\left(CN+4C''\right)+a^2t_0^2C\dfrac{N^3}{2\varepsilon}\le 0,\\[2mm] -1+at_0\left[6\left(C'+C''+C''\dfrac{N}{\varepsilon}+K_1^++L_3\right)+3C\dfrac{N^2}{2\varepsilon}+2\varepsilon CN^2\right]\\ \qquad +4\varepsilon\left(CN+4C''\right)\le 0. \end{cases}$$

With such a choice of (a,ε) we get $g(t,x)\le 0$ for any $t\in(0,t_0]$ and any $x\in B(n)$. From the classical maximum principle we now deduce that

$$|v_{0,3,n}(t,x)|\le \|\vartheta_n f\|_\infty^2\le \|f\|_\infty^2, \qquad t\in[0,t_0],\ x\in B(n).$$

Taking the limit as n tends to $+\infty$, by (6.1.9) we deduce that (6.0.1) holds for any $t\in[0,t_0]$, some constant $C_{0,3}=C_{0,3}(t_0)>0$ and with $\omega=0$. Using the semigroup property we can then extend the estimate to all the positive t. Indeed, taking (6.1.12) into account, we get, for any $t>t_0$,

$$\begin{aligned}\|T(t)f\|_{C_b^3(\mathbb{R}^N)}&=\|T(t_0)T(t-t_0)f\|_{C_b^3(\mathbb{R}^N)}\le C_{0,3}t_0^{-\frac{3}{2}}\|T(t-t_0)f\|_\infty\\ &\le C_{0,3}t_0^{-\frac{3}{2}}\|f\|_\infty\le C_{0,3}t_0^{-\frac{3}{2}}e^{\omega t}t^{-\frac{3}{2}}\|f\|_\infty,\end{aligned}$$

and, so, (6.0.1) follows with $C_{0,3}(\omega)=\max\{C_{0,3}(t_0),C_{0,3}(t_0)t_0^{-3/2}\}$.

In the other cases the proof is very similar. It suffices to apply the quoted arguments to the function

$$v_{k,l,n}(t,x)=\sum_{j=0}^{l}a^jt^{(j-k)^+}(\vartheta_n(x))^{2j}|D^ju_n(t,x)|^2, \qquad t\in(0,t_0],\ x\in B(n).$$

Let us just show that, if $k=l$, we can take $\omega=0$ in (6.0.1). We only consider the case when $l=3$. A straightforward computation shows that $v_{3,3,n}$ is a classical solution to the Cauchy-Dirichlet problem (6.1.13) with $v_{3,3,n}(0,\cdot)=\sum_{j=0}^3|D^j(\vartheta_n f)|^2$ and g_n being replaced with $\widetilde{g}_n=\sum_{j=1}^7\widetilde{g}_{j,n}$, where $\widetilde{g}_{j,n}$ ($j=1,\ldots,7$) are defined by the right-hand sides of (6.1.14)-(6.1.20) after replacing each t, therein appearing as coefficients of u_n or of its derivatives, with $t=1$. Arguing as above we can easily show that $\widetilde{g}_n$ can be estimated, for any $t > 0$, by the last side of (6.1.31), where we set $t = 1$ and replace the terms $-\kappa_0 + a$, $-\kappa_0 + 2a$ and $-\kappa_0 + 3a$ simply with $-\kappa$. It is now clear that we can take (a, ε) such that $\widetilde{g}_n(t, x) \leq 0$ for any $t > 0$, any $x \in \mathbb{R}^N$ and any $n \in \mathbb{N}$, and, consequently,

$$|v_{3,3,n}(t,x)| \leq \|\vartheta f\|_\infty^2 + \|D(\vartheta f)\|_\infty^2 + \|D^2(\vartheta f)\|_\infty^2 + \|D^3(\vartheta f)\|_\infty^2, \qquad (6.1.32)$$

for any $t \in [0, t_0]$, any $x \in \overline{B}(n)$ and any $n \in \mathbb{N}$. The estimate (6.1.32) yields (6.0.1) taking the limit as n tends to $+\infty$. ∎

Now, by interpolation, we can extend (6.0.1) to the case when at least one between k and l is not integer.

Theorem 6.1.8 *Let Hypotheses* 6.1.1(*i*)–6.1.1(*iii*), 6.1.1(*iv-l*) *be satisfied for some* $l \in \{1, 2, 3\}$. *Then, for any* $\omega > 0$, *any* $k, m = 0, \ldots, l-1$ *and any* $\alpha, \theta \in [0, 1]$ *such that* $k + \alpha \leq m + \theta$, *there exist constants* $C_{k+\alpha,m+\theta} = C_{k+\alpha,m+\theta}(\omega) > 0$ *such that*

$$\|T(t)f\|_{C_b^{m+\theta}(\mathbb{R}^N)} \leq C_{k+\alpha,m+\theta}\, t^{-\frac{m+\theta-k-\alpha}{2}} e^{\omega t} \|f\|_{C_b^{k+\alpha}(\mathbb{R}^N)}, \qquad (6.1.33)$$

for any $f \in C_b^{k+\alpha}(\mathbb{R}^N)$ *and any* $k = 0, \ldots, m$. *In particular, if* $k + \alpha = m + \theta$ *we can take* $\omega = 0$ *in* (6.1.33).

Proof. The proof follows from an interpolation argument. We limit ourselves to sketching it in a particular case, since the same techniques can also be applied to all the other cases. So, let us assume that $k = m = 2$ and $0 < \alpha \leq \theta < 1$. Moreover, fix $\omega > 0$ and $t > 0$. Setting $\omega' = \omega/(\theta - \alpha)$, from (6.1.11) with $(k, l) = (2, 2)$ and $(k, l) = (2, 3)$, we deduce that

$$\|T(t)\|_{L(C_b^2(\mathbb{R}^N), C_b^2(\mathbb{R}^N))} \leq C_{2,2}, \qquad \|T(t)\|_{L(C_b^2(\mathbb{R}^N), C_b^3(\mathbb{R}^N))} \leq C_{2,3}\, t^{-\frac{1}{2}} e^{\omega' t}, \qquad (6.1.34)$$

for any $t > 0$. Recalling that $(C_b^2(\mathbb{R}^N); C_b^3(\mathbb{R}^N))_{\beta,\infty} = C_b^{2+\beta}(\mathbb{R}^N)$, for any $\beta \in (0, 1)$ (see Theorem A.4.8), and applying Proposition A.4.2, we easily see from (6.1.34) that $T(t)$ is bounded from $C_b^2(\mathbb{R}^N)$ into $C_b^{2+\beta}(\mathbb{R}^N)$ for any $\beta \in (0, 1)$ and

$$\|T(t)\|_{L(C_b^2(\mathbb{R}^N), C_b^{2+\beta}(\mathbb{R}^N))} \leq C_{2,2+\beta}\, t^{-\frac{\beta}{2}} e^{\beta\omega' t}, \qquad t > 0, \qquad (6.1.35)$$

where $C_{2,2+\beta} = C_{2,2}^{1-\beta} C_{2,3}^{\beta}$.

Applying the same argument to (6.1.11), with $(k, l) = (3, 3)$ and to (6.1.35), we deduce that $T(t)$ is bounded from $C_b^{2+\alpha}(\mathbb{R}^N)$ into $C_b^{2+\beta+(1-\beta)\alpha}(\mathbb{R}^N)$ and

$$\|T(t)\|_{L(C_b^{2+\alpha}(\mathbb{R}^N), C_b^{2+\beta+(1-\beta)\alpha}(\mathbb{R}^N))} \leq C_{2+\alpha,2+\beta+(1-\beta)\alpha}\, t^{-(1-\alpha)\frac{\beta}{2}} e^{(1-\alpha)\beta\omega' t},$$

where $C_{2+\alpha,2+\beta+(1-\beta)\alpha} = C_{2,2+\beta}^{1-\alpha} C_{3,3}^{\alpha}$. Now the assertion follows, taking $\beta = (\theta-\alpha)/(1-\alpha)$. ∎

In some cases, we can extend Theorems 6.1.7 and 6.1.8 to the case when in 6.1.1(iv-j), the j-th-order derivatives of the coefficients are merely continuous in $\mathbb{R}^N$. As the following theorem shows, this is the case when the condition (6.1.4) is satisfied and there exist $m, M > 0$ such that

$$\int_0^1 d(tx+y)dt \le M\kappa(x+y), \qquad x \in \mathbb{R}^N,\ y \in B(m). \tag{6.1.36}$$

As it is immediately seen, the previous condition is satisfied, for instance, when d is bounded from above.

Theorem 6.1.9 *Suppose that Hypotheses* 6.1.1(*i*)–6.1.1(*iii*), 6.1.1(*iv-j*) ($j = 1, 2, 3$) (*with the j-th-order derivatives of the coefficients merely continuous in* $\mathbb{R}^N$) *and the conditions* (6.1.4) *and* (6.1.36) *are satisfied. Then,* (6.1.11) *and* (6.1.33) *hold true for any $k, l \in \mathbb{N}$, $k \le l \le j$.*

Proof. The proof follows from a density argument. Of course, we can limit ourselves to dealing with (6.1.11), since, as Theorem 6.1.8 shows, (6.1.33) follows easily from (6.0.1).

For any $\varepsilon > 0$ let $\varphi^\varepsilon(x) = \varepsilon^{-N}\varphi(x/\varepsilon)$, where $\varphi \in C_c^\infty(\mathbb{R}^N)$ is any non-negative even function compactly supported in $B(1)$ and with integral 1. We denote by f^ε the convolution between f and φ^ε.

Let $\mathcal{A}^\varepsilon$ be defined as $\mathcal{A}$ with q_{ij} and b_j being replaced, respectively, with q_{ij}^ε and b_j^ε for any $i, j = 1, \ldots, N$. As it is immediately seen, q_{ij}^ε and b_j^ε ($i, j = 1, \ldots, N$) satisfy Hypotheses 6.1.1(i) and 6.1.1(iv-j), with κ, d, r being replaced with $\kappa^\varepsilon, d^\varepsilon$ and r^ε, and with $C^\varepsilon = C$, $L_i^\varepsilon = L_i$ for $i = 1, 2, 3$, $K_1^\varepsilon = K_1$ (if $j > 1$) and $\kappa_0^\varepsilon \ge \kappa_0$.

Let us check that q_{ij}^ε and b_j^ε ($i, j = 1, \ldots, N$) satisfy Hypothesis 6.1.1(iii) for some positive constant, independent of ε. For this purpose, we observe that (6.1.4) implies that

$$|q_{ij}(x)| \le |q_{ij}(0)| + CK\sqrt{N}|x|\kappa(x), \qquad x \in \mathbb{R}^N,\ i, j = 1, \ldots, N,$$

which yields (6.1.1).

A straightforward computation now shows that

$$|q_{ij}^\varepsilon(x)| \le |q_{ij}(0)| + CK\sqrt{N}(|x|+\varepsilon)\kappa^\varepsilon(x), \qquad x \in \mathbb{R}^N,\ i, j = 1, \ldots, N,$$

so that the q_{ij}^ε's satisfy the condition (6.1.1) with a constant being independent of $\varepsilon \in (0, 1]$.

Similarly, we can show that the b_j^ε's satisfy (6.1.2) with a positive constant being independent of $\varepsilon \le m$.

Indeed, combining (6.1.6) and (6.1.36), we deduce that

$$\begin{aligned}\int_0^1 d^\varepsilon(tx)dt &= \int_{B(\varepsilon)} dy \int_0^1 d(tx-y)\varphi^\varepsilon(y)dt\\ &\le M\int_{B(\varepsilon)} dy\int_0^1 \kappa(x-y)\varphi^\varepsilon(y)dt\\ &= M\kappa^\varepsilon(x),\end{aligned}$$

for any $x \in \mathbb{R}^N$ and any $\varepsilon \le m$. Now, arguing as in the proof of (6.1.6), we get

$$\begin{aligned}\sum_{j=1}^N b_j^\varepsilon(x)x_j &\le \sum_{j=1}^N b_j^\varepsilon(0)x_j + |x|^2\int_0^1 d^\varepsilon(tx)dt\\ &\le \sup_{y\in B(\varepsilon)} |b_j(y)||x| + M|x|^2\kappa^\varepsilon(x),\end{aligned}$$

for any $x \in \mathbb{R}^N$, which yields (6.1.2) with a constant being independent of $\varepsilon \le m$.

Now, applying the same arguments as in the proof of Theorem 6.1.7, we can show that, for any $\omega > 0$ and any $k, l \in \mathbb{N}$, $0 \le k \le l \le j$, there exists a positive constant $C_{k,l} = C_{k,l}(\omega)$, independent of ε and n, such that

$$\|T_n^\varepsilon(t)f\|_{C^l(\overline{B}(n))} \le C_{k,l}t^{-\frac{l-k}{2}}e^{\omega t}\|f\|_{C^k(\overline{B}(n))}, \qquad t>0,\ \ f\in C_b^k(\overline{B}(n)),$$

where $T^\varepsilon(t)$ is defined as $T(t)$ with $\mathcal{A}$ being replaced with $\mathcal{A}^\varepsilon$. As n tends to $+\infty$, $T_n^\varepsilon(t)f$ tends to a solution $u^\varepsilon =: T^\varepsilon(t)f$ of the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}^\varepsilon u(t,x), & t>0,\, x\in\mathbb{R}^N,\\ u(0,x) = f(x), & x\in\mathbb{R}^N,\end{cases}$$

which satisfies

$$\|T^\varepsilon(t)f\|_{C^l(\mathbb{R}^N)} \le C_{k,l}t^{-\frac{l-k}{2}}e^{\omega t}\|f\|_{C^k(\mathbb{R}^N)}, \qquad t>0. \tag{6.1.37}$$

Theorem C.1.4 and the estimate (6.1.37) imply that there exists an infinitesimal sequence $\{\varepsilon_n\}$ such that $T^{\varepsilon_n}(t)f$ and its space derivatives up to the $(j-1)$-th-order converge in $C^{1+\delta/2,2+\delta}_{\rm loc}((0,+\infty)\times\mathbb{R}^N)$ to a function $u =: S(t)f$ satisfying

$$\|S(t)f\|_{C^l(\mathbb{R}^N)} \le C_{k,l}t^{-\frac{l-k}{2}}e^{\omega t}\|f\|_{C^k(\mathbb{R}^N)}, \qquad t>0. \tag{6.1.38}$$

Since the coefficients q_{ij} and b_j $(i,j=1,\ldots,N)$ are locally Lipschitz continuous, $q_{ij}^{\varepsilon_n}$ and $b_j^{\varepsilon_n}$ converge locally uniformly in $\mathbb{R}^N$ as n tends to $+\infty$, respectively to q_{ij} and b_j, so that $S(t)f$ satisfies the differential equation

$D_tw-\mathcal{A}w=0$ in $(0,+\infty)\times\mathbb{R}^N$. Moreover, for any $f\in C^2_b(\mathbb{R}^N)$, $S(t)f$ converges to f as t tends to 0, locally uniformly in x. This can be seen by a localization argument similar to the one used in the proof of Theorem 2.2.1. For this purpose, we fix $k\in\mathbb{N}$ and let $\vartheta=\vartheta_k$ be as in the proof of Theorem 6.1.7. Moreover, for any $n\in\mathbb{N}$, we set $v_n=\vartheta T^{\varepsilon_n}(\cdot)f$ and observe that v_n is a solution to the Cauchy problem

$$\begin{cases} D_tv_n(t,x)=\mathcal{A}^{\varepsilon_n}v_n(t,x)+\psi_n(t,x), & t\in[0,T],\, x\in B(k),\\ v_n(t,x)=0, & t\in[0,T],\, x\in\partial B(k),\\ v_n(0,x)=\vartheta(x)f(x), & x\in\overline{B}(k),\end{cases}$$

where

$$\psi_n(t,\cdot)=-2T^{\varepsilon_n}(t)f\cdot\mathcal{A}^{\varepsilon_n}(\vartheta)-2\sum_{i,j=1}^N q_{ij}^{\varepsilon_n}D_iT^{\varepsilon_n}(t)f\cdot D_j\vartheta.$$

Since the coefficients q_{ij}^{ε} and b_j^{ε} $(i,j=1,\ldots,N)$ converge locally uniformly in $\mathbb{R}^N$, they are equibounded (with respect to $n\in\mathbb{N}$) in $B(k)$ and there exists a positive constant $\widetilde{C}$ such that $\|\mathcal{A}^{\varepsilon_n}g\|_{C(\overline{B}(k))}\le\widetilde{C}\|g\|_{C^2(\overline{B}(k))}$ for any $g\in C^2(\overline{B}(k))$. Therefore, from (6.1.37) we deduce that there exists a constant $\overline{C}$, independent of n, such that

$$|\psi_n(t,x)|\le\overline{C}t^{-\frac{1}{2}}\|f\|_\infty,\qquad t\in(0,T],\quad x\in B(k),\quad n\in\mathbb{N}. \tag{6.1.39}$$

The estimate (6.1.39) implies that v_n can be written by the usual variation-of-constants formula as

$$v_n(t,\cdot)=T_{k,n}(t)(\vartheta f)+\int_0^t T_{k,n}(t-s)\psi_n(s,\cdot)ds,\qquad t\in(0,T],$$

where $\{T_{k,n}(t)\}$ is the semigroup associated with the realization $A_{k,n}$ of $\mathcal{A}^{\varepsilon_n}$ in $C(\overline{B}(k))$ with homogeneous Dirichlet boundary conditions (see Theorem C.3.6(iv)).

Since $\vartheta f\in D(A_{k,n})=\{u\in\bigcap_{1\le p<+\infty}W^{2,p}(B(k)):\mathcal{A}^{\varepsilon_n}u\in C(\overline{B}(k))\}$, the domain of the realization of $\mathcal{A}^{\varepsilon_n}$ in $C(\overline{B}(k))$ (see Theorem C.3.6) and $\{T_{k,n}\}$ is a semigroup of contractions in $C(\overline{B}(k))$ for any $n\in\mathbb{N}$, then

$$\begin{aligned}\|T_{k,n}(t)(\vartheta f)-\vartheta f\|_{\overline{C}(B(k))}&=\left\|\int_0^t T_{k,n}(s)\mathcal{A}^{\varepsilon_n}(\vartheta f)ds\right\|_{C(\overline{B}(k))}\\ &\le t\|\mathcal{A}^{\varepsilon_n}(\vartheta f)\|_{C(\overline{B}(k))}\le\overline{C}t\|\vartheta f\|_{C^2(\overline{B}(k))},\end{aligned}$$

for any $t\in(0,T)$, which readily yields

$$\lim_{t\to0^+}\sup_{n\in\mathbb{N}}\|T_{k,n}(t)(\vartheta f)-\vartheta f\|_{C(\overline{B}(k))}=0. \tag{6.1.40}$$

From (6.1.39) and (6.1.40) we can now easily show that

$$\lim_{t\to 0^+}\sup_{n\in\mathbb{N}}\|v_n(t,\cdot)-\vartheta f\|_{C(\overline{B}(k))}=0.$$

Recalling that $\vartheta\equiv 1$ in $B(k/2)$, we deduce that

$$\begin{aligned}
&|(S(t)f)(x)-f(x)|\\
&\le \sup_{n\in\mathbb{N}}|(T^{\varepsilon_n}(t)f)(x)-\vartheta(x)f(x)|+\limsup_{n\to+\infty}|(T^{\varepsilon_n}(t)f)(x)-(S(t)f)(x)|\\
&\le \sup_{n\in\mathbb{N}}\|v_k(t,\cdot)-\vartheta f\|_{C(\overline{B}(k))},
\end{aligned}\qquad(6.1.41)$$

for any $t\in(0,T)$ and any $x\in B(k/2)$. Taking the limit as t tends to 0 in both the first and the last side of (6.1.41) gives

$$\lim_{t\to 0^+}\sup_{x\in B(k/2)}|u(t,x)-f(x)|=0.$$

From the arbitrariness of $k\in\mathbb{N}$, we deduce that $S(t)f$ converges locally uniformly (with respect to x) to f as t tends to 0. Hence $S(t)f$ is a classical solution to the problem

$$\begin{cases}D_tw(t,x)=\mathcal{A}w(t,x), & t>0,\ x\in\mathbb{R}^N,\\ w(0,x)=f(x), & x\in\mathbb{R}^N,\end{cases}$$

and Hypothesis 6.1.1(ii) and the maximum principle in Theorem 4.1.3 imply that $T(t)f=S(t)f$. Since $C_b^2(\mathbb{R}^N)$ is dense in $C_b^1(\mathbb{R}^N)$ (which is endowed with the sup-norm) we can extend the previous equality to all the $f\in C_b^1(\mathbb{R}^N)$, obtaining (6.0.1) in the case when $k\ge 1$.

Finally, with any $f\in C(\mathbb{R}^N)$ and any $t>0$ we write $T(t)=T(t/2)T(t/2)f$ and observe that since $T(t/2)f\in C_b^1(\mathbb{R}^N)$, then $T(t)f=S(t/2)T(t/2)f$. Applying (6.1.38) with (t,f) being replaced with $(t/2,T(t/2)f)$, we easily get (6.0.1) also in the case when $k=0$. ■

Remark 6.1.10 In Theorem 6.1.7 we have shown that the estimate (6.1.11) holds true for some $\omega\ge 0$. In Chapter 7, we will show that, actually, we can take $\omega=0$ in (6.1.11) (and, consequently, in (6.1.33)) and that, under somewhat heavier assumptions on the coefficients, we can also take $\omega<0$.

To conclude this section we show that the estimates (6.0.1) may fail to hold without any dissipativity assumption.

Example 6.1.11 Consider the following one-dimensional operator

$$\mathcal{A}u(x)=u''(x)+p'(x)u'(x)=e^{-p(x)}\left(e^{p(x)}u'(x)\right)',\qquad x\in\mathbb{R},$$

where $p \in C^1(\mathbb{R})$ is a function that will be chosen later on. For any $f \in C_b(\mathbb{R})$, the solutions of the equation $\mathcal{A}u = f$ are given by

$$u(x) = C_1 + \int_0^x e^{-p(t)} \left(C_2 + \int_0^t f(s) e^{p(s)} ds \right) dt, \tag{6.1.42}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary real constants. Suppose now that $e^p \in L^1(\mathbb{R})$ and that

$$\int_{-\infty}^{+\infty} f(t) e^{p(t)} dt = 0. \tag{6.1.43}$$

Choose

$$C_2 = -\int_0^{+\infty} f(t) e^{p(t)} dt = \int_{-\infty}^0 f(t) e^{p(t)} dt.$$

If $x > 0$ then, from (6.1.42), we formally get

$$\begin{aligned} u(x) &= C_1 - \int_0^x e^{-p(t)} \int_t^{+\infty} f(s) e^{p(s)} ds\, dt \\ &= C_1 - \int_0^{+\infty} e^{p(s)} f(s) \int_0^{s\wedge x} e^{-p(t)} dt\, ds. \end{aligned}$$

Suppose now that the function

$$Q(s) = e^{p(s)} \int_0^s e^{-p(t)} dt, \quad s \in \mathbb{R}$$

belongs to $L^1(0, +\infty)$. Then, we get the estimate

$$|u(x)| \le |C_1| + \|f\|_\infty \int_0^{+\infty} Q(s) ds, \qquad x > 0,$$

which implies that $u \in C_b([0, +\infty))$. Similarly, assuming that $Q \in L^1(-\infty, 0)$ we get $u \in C_b((-\infty, 0])$.

Therefore, under the assumption $Q \in L^1(\mathbb{R})$ we conclude that $u \in D_{\max}(\mathcal{A})$. Note that the assumption $Q \in L^1(\mathbb{R})$ implies $e^p \in L^1(\mathbb{R})$. The derivative of u is given by

$$u'(x) = -e^{-p(x)} \int_x^{+\infty} f(s) e^{p(s)} ds, \qquad x \in \mathbb{R}.$$

We claim that we can choose the functions p and f so that $Q \in L^1(\mathbb{R})$ and (6.1.43) holds but u' is not bounded. Indeed take

$$p(x) = -x^4 + \log h(x),$$

where $h(x)$ is a regular function satisfying

$$\begin{aligned} & h(x) = \varepsilon_n, && \text{if } x = n - \frac{\delta_n}{2},\ n \in \mathbb{N}, \\ & \varepsilon_n \le h(x) \le 1, && \text{if } n - \delta_n < x < n,\ n \in \mathbb{N}, \\ & h(x) = 1, && \text{otherwise,} \end{aligned}$$

where

$$\varepsilon_n = \frac{1}{n}\, e^{\left(n-\frac{1}{2}\right)^4 - \left(n+\frac{1}{2}\right)^4}, \qquad \delta_n = \frac{e^{-n^4}}{n^2}\varepsilon_n\,.$$

Then, we have

$$Q(x) = e^{-x^4}\int_0^x e^{t^4}dt, \quad x<0, \qquad Q(x) = h(x)e^{-x^4}\int_0^x \frac{e^{t^4}}{h(t)}dt, \quad x>0.$$

Applying De L'Hôpital rule one sees that $\lim_{x\to-\infty} x^3 Q(x) = 1/4$ and conclude that $Q\in L^1(-\infty,0)$. For any $x>0$ we have

$$\begin{aligned}
Q(x) &\le e^{-x^4}\int_0^x \frac{e^{t^4}}{h(t)}dt\\
&\le e^{-x^4}\int_0^x e^{t^4}dt + e^{-x^4}\sum_{n=1}^{[x]+1}\int_{n-\delta_n}^n \frac{e^{n^4}}{\varepsilon_n}dt\\
&\le e^{-x^4}\int_0^x e^{t^4}dt + e^{-x^4}\sum_{n=1}^{\infty}\frac{\delta_n e^{n^4}}{\varepsilon_n}\\
&= e^{-x^4}\int_0^x e^{t^4}dt + e^{-x^4}\sum_{n=1}^{\infty}\frac{1}{n^2},
\end{aligned}$$

which shows that $Q\in L^1(0,+\infty)$. Now we choose a function $f\in C_b(\mathbb{R})$ satisfying (6.1.43) and such that $f(x)=1$ for any $x>0$. Then, for any $x>0$ we have

$$u'(x) = -\frac{e^{x^4}}{h(x)}\int_x^{+\infty} h(t)e^{-t^4}dt,$$

for any $x>0$, so that at $x=n-\delta_n/2$

$$\begin{aligned}
|u'(n-\delta_n/2)| &= \frac{e^{(n-\delta_n/2)^4}}{\varepsilon_n}\int_{n-\frac{\delta_n}{2}}^{+\infty} h(t)e^{-t^4}dt\\
&\ge \frac{e^{\left(n-\frac{1}{2}\right)^4}}{\varepsilon_n}\int_n^{n+\frac{1}{2}} e^{-t^4}dt\\
&\ge \frac{e^{\left(n-\frac{1}{2}\right)^4}}{\varepsilon_n}\frac{1}{2}e^{-\left(n+\frac{1}{2}\right)^4}\\
&= \frac{n}{2},
\end{aligned}$$

which shows that $u'(x)$ is unbounded.

Therefore, the function u belongs to $D_{\max}(\mathcal{A})$ but not to $C_b^1(\mathbb{R})$. This means that (6.0.1) with $k=0$, $l=1$ fails to hold.

We note that in this case the dissipativity assumption (6.1.3) fails since p'' is unbounded from above. Indeed, let $g:\mathbb{R}\to\mathbb{R}$ be the function defined by $g(x)=\log(h(x))$ for any $x\in\mathbb{R}$. Since $g(n-\delta_n)=g(n)=0$ and $g(n-\delta_n/2)=\log(\varepsilon_n)$ for any $n\in\mathbb{N}$, then, by the mean value theorem, there exist two points $y_n\in(n-\delta_n,n-\delta_n/2)$ and $z_n\in(n-\delta_n/2,n)$ such that

$$g'(y_n)=-g'(z_n)=\frac{2\log(\varepsilon_n)}{\delta_n},\qquad n\in\mathbb{N}.$$

Applying again the mean value theorem, it follows that there exists $x_n\in(y_n,z_n)$ such that

$$g''(x_n)=-\frac{4\log(\varepsilon_n)}{\delta_n(z_n-y_n)}\geq-\frac{4\log(\varepsilon_n)}{\delta_n^2}=-\frac{4\log(\varepsilon_n)}{\varepsilon_n^2}n^4e^{2n^4},\qquad n\in\mathbb{N}.$$

Since ε_n tends to 0 as n tends to $+\infty$, then, for n large enough it holds that

$$p''(x_n)=-12x_n^2+g''(x_n)\geq-12n^2+n^4e^{2n^4},$$

which shows that p'' is unbounded from above.

Remark 6.1.12 As already recalled in the introduction to this chapter, in the paper [29] and in the recent book [30], the author, starting from the stochastic equation

$$\begin{cases}d\xi_t^x=b(\xi_t^x)dt+\sigma(\xi_t^x)dW_t, & t>0,\\ \xi_0\equiv x,\end{cases}$$

and using probabilistic methods, proves uniform estimates, comparable with ours, for the space derivatives of the function $T(t)f$ when $f\in C_b(\mathbb{R}^N)$ and $\{T(t)\}$ is the semigroup associated with the stochastic differential equation.

More precisely, she assumes the following set of assumptions:

(i) $\sigma_{ij}\in C^3(\mathbb{R}^N)$ and there exists $k\geq0$ such that for any $j=0,\dots,3$ it holds

$$\sup_{x\in\mathbb{R}^N}\frac{\|D^\beta\sigma(x)\|}{1+|x|^{k-j}}<+\infty,\qquad|\beta|=j.$$

(ii) $b_i\in C^3(\mathbb{R}^N)$ and there exists $m\geq k$ such that for any $j=0,\dots,3$ it holds

$$\sup_{x\in\mathbb{R}^N}\frac{|D^\beta b(x)|}{1+|x|^{2m+1-j}}<+\infty,\qquad|\beta|=j.$$

There exist $a,\gamma>0$ and $c\in\mathbb{R}$ such that for any $x,h\in\mathbb{R}^N$ it holds

$$\langle b(x+h)-b(x),h\rangle\leq-a|h|^{2m+2}+c(1+|x|^\gamma).$$

(iii) For any $p > 0$ there exists $c_p \in \mathbb{R}$ such that

$$\langle b(x) - b(y), x - y\rangle + p\|\sigma(x) - \sigma(y)\|_2^2 \le c_p|x-y|^2.$$

Under the previous set of assumptions, she proves that $T(t)f \in C_b^3(\mathbb{R}^N)$ for any $t > 0$ and any $f \in B_b(\mathbb{R}^N)$ and that, for any $T > 0$, there exists a constant $C_T > 0$ such that

$$\|D^kT(t)f\|_\infty \le C_T t^{-\frac{k}{2}}\|f\|_\infty, \qquad t \in (0,T],\ \ k = 1,2,3.$$

Note that also our estimates can be extended to any $f \in B_b(\mathbb{R}^N)$. Indeed, since $\{T(t)\}$ is a semigroup of contractions in $B_b(\mathbb{R}^N)$ and it is also strong Feller (see Remark 2.2.10 and Proposition 2.2.12), then we can split $T(t)f = T(t/2)T(t/2)f$ for any $f \in B_b(\mathbb{R}^N)$ and write, for any $\omega > 0$ and some positive $C = C(\omega)$

$$\begin{aligned}\|D^kT(t)f\|_\infty &\le \|D^kT(t/2)\|_{L(C_b(\mathbb{R}^N);C_b^k(\mathbb{R}^N))}\|T(t/2)f\|_{C_b(\mathbb{R}^N)}\\ &\le Ce^{\frac{\omega}{2}t}t^{-\frac{k}{2}}\|f\|_\infty, \end{aligned} \tag{6.1.44}$$

for any $t > 0$ and any $k = 1,2,3$. Actually, in view of the results in Section 7.2 (see Remark 7.2.4), we can get rid of the exponential term in (6.1.44).

Remark 6.1.13 As it has been remarked in the introduction to this chapter, uniform estimates similar to those in Theorem 6.1.7 have been obtained in [108] under assumptions on the coefficients of the operator $\mathcal{A}$ which are comparable with ours. The main differences are in the method used in [108] to prove such uniform estimates. Indeed, A. Lunardi approximates the coefficients of the operator $\mathcal{A}$ rather than the whole space $\mathbb{R}^N$ by balls, centered at the origin, as we do here. Moreover, she does not assume that the potential c identically vanishes in $\mathbb{R}^N$. Here, we assume such an assumption in view of the pointwise estimates of Chapter 7.

6.2 Some consequences

The estimates (6.1.33) can be used to prove optimal Schauder estimates for the elliptic equation

$$\lambda u - \mathcal{A}u = f, \tag{6.2.1}$$

as well as for the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + g(t,x), & t > 0,\ x \in \mathbb{R}^N,\\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{6.2.2}$$

We begin with the elliptic equation and we prove the following lemma.

Lemma 6.2.1 *Let $\theta \in (0,3)$ be not an integer, let $I \subset \mathbb{R}$ be an interval, and let $\varphi : I \to C_b^\theta(\mathbb{R}^N)$ be such that, for any $x \in \mathbb{R}^N$, the real function $t \mapsto \varphi(t)(x)$ is continuous in I and*

$$\|\varphi(t)\|_{C_b^\theta(\mathbb{R}^N)} \le c(t), \qquad t \in I,$$

for some function $c \in L^1(I)$. Then, the function

$$f(x) = \int_I \varphi(t)(x)dt, \qquad x \in \mathbb{R}^N, \tag{6.2.3}$$

belongs to $C_b^\theta(\mathbb{R}^N)$ and

$$\|f\|_{C_b^\theta(\mathbb{R}^N)} \le K\|c\|_{L^1(I)},$$

for some positive constant K.

Proof. We begin the proof observing that, according to [141, Section 2.7.2], for any $\theta \in (0,3)$, $C_b^\theta(\mathbb{R}^N)$ can be characterized as the space of functions $g \in C_b(\mathbb{R}^N)$ such that

$$[[g]]_\theta = \sup_{\substack{x,h\in\mathbb{R}^N \\ h\neq 0}} |h|^{-\theta}\left|\sum_{l=0}^{3}(-1)^l f(x+lh)\right| < +\infty$$

and the norm

$$g \mapsto \|g\|_\infty + [[g]]_\theta$$

is equivalent to the classical norm of $C_b^\theta(\mathbb{R}^N)$. Hence, if φ is as in the statement of the lemma, then for any $x, h \in \mathbb{R}^N$, with $h \neq 0$, we have

$$\left|\sum_{l=0}^{3}(-1)^l \int_I \varphi(t)(x+lh)dt\right| \le \int_I \left|\sum_{l=0}^{3}(-1)^l \varphi(t)(x+lh)\right| dt \le K|h|^\theta \int_I c(t)dt, \tag{6.2.4}$$

for some positive constant K. The estimate (6.2.4) implies that the function f in (6.2.3) belongs to $C_b^\theta(\mathbb{R}^N)$, and the statement follows. ∎

Now, we observe that by virtue of Hypothesis 6.1.1(ii), Theorem 4.1.5 implies that, for any $f \in C_b(\mathbb{R}^N)$ and any $\lambda > c_0$, the function $R(\lambda)f$ defined by

$$(R(\lambda)f)(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)dt, \qquad x \in \mathbb{R}^N, \tag{6.2.5}$$

is the unique solution to the elliptic equation (6.2.1) in

$$D_{\max}(\mathcal{A}) = \left\{u \in C_b(\mathbb{R}^N) \cap \bigcap_{1\le p<+\infty} W^{2,p}_{\rm loc}(\mathbb{R}^N) : \ \mathcal{A}u \in C_b(\mathbb{R}^N)\right\}, \quad Au = \mathcal{A}u.$$

As a consequence, Proposition 2.3.6 implies that $(A, D_{\max}(\mathcal{A})) = (\widehat{A}, D(\widehat{A}))$ with equivalence of the corresponding norms.

With this remark in mind, we can now prove the following theorem, which provides us a partial characterization of $D_{\max}(\mathcal{A})$ and gives optimal Schauder estimates for the solution of the elliptic equation (6.2.1).

Theorem 6.2.2 *Suppose that Hypotheses* 6.1.1(*i*)–6.1.1(*iii*), 6.1.1(*iv-l*) *are satisfied (for some $l = 1, 2$). Then, $D_{\max}(\mathcal{A})$ is contained in $C^\theta_b(\mathbb{R}^N)$, for any $\theta \in (0, 1]$, if $l = 1$, and, for any $\theta \in (0, 2)$, if $l = 2$. Moreover, for any $\omega > 0$ and any $\theta \in (0, l)$, and also for $\theta = 1$ if $l = 1$, there exists a positive constant $C = C(\theta, \omega)$ such that*

$$\|u\|_{C^\theta_b(\mathbb{R}^N)} \le C\|u\|_\infty^{1-\frac{\theta}{2}}\|(\omega - \mathcal{A})u\|_\infty^{\frac{\theta}{2}}, \qquad u \in D_{\max}(\mathcal{A}). \tag{6.2.6}$$

Finally, let the previous assumptions be satisfied with $l = 3$. Then, for any $f \in C^\theta_b(\mathbb{R}^N)$ ($\theta \in (0, 1)$) and any $\lambda > 0$, there exist a unique solution $u \in C^{2+\theta}_b(\mathbb{R}^N)$ of the elliptic equation $\lambda u - \mathcal{A}u = f$ and a positive constant $C = C(\theta, \lambda)$ such that

$$\|u\|_{C^{2+\theta}_b(\mathbb{R}^N)} \le C\|f\|_{C^\theta_b(\mathbb{R}^N)}. \tag{6.2.7}$$

Proof. To prove the first part we assume that $l = 3$, the other case being completely similar and even simpler. Let $u \in D_{\max}(\mathcal{A})$ be a nonidentically vanishing function and fix $\omega > 0$. For any $\lambda > \omega$, set $\varphi = \lambda u - \mathcal{A}u$. Then,

$$u(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)\varphi)(x)dt, \qquad x \in \mathbb{R}^N,$$

so that, according to Lemma 6.2.1 and the estimate (6.1.33), $f \in C^\theta_b(\mathbb{R}^N)$ and, if $\theta \neq 1$, then

$$\begin{aligned}\|u\|_{C^\theta_b(\mathbb{R}^N)} &\le C\frac{\Gamma(1-\theta/2)}{(\lambda-\omega)^{1-\theta/2}}\|\varphi\|_\infty \\ &\le C\frac{\Gamma(1-\theta/2)}{(\lambda-\omega)^{1-\theta/2}}\left((\lambda-\omega)\|u\|_\infty + \|(\omega-\mathcal{A})u\|_\infty\right),\end{aligned} \tag{6.2.8}$$

for some positive constant C, independent of λ, where Γ denotes the Gamma function. Taking the minimum for $\lambda > \omega$ in (6.2.8), the estimate (6.2.6) follows.

Now, we prove that for any $f \in C^\theta_b(\mathbb{R}^N)$ ($\theta \in (0, 1)$) and any $\lambda > 0$, there exists a (unique) solution $u \in C^{2+\theta}_b(\mathbb{R}^N)$ of the equation (6.2.1). So we assume that Hypothesis 6.1.1(iv-3) is satisfied. By virtue of Theorem 4.1.5, to prove the assertion, it suffices to show that the function $u = R(\lambda)f$ has the claimed regularity properties.

We set

$$u(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)\varphi)(x)dt, \qquad x \in \mathbb{R}^N,$$

and prove that, for any $\alpha \in (\alpha, 1)$,

$$u \in (C_b^\alpha(\mathbb{R}^N), C_b^{2+\alpha}(\mathbb{R}^N))_{1-(\alpha-\theta)/2,+\infty}.$$

This is enough for our aims. Indeed, by a well known result in interpolation theory (see Theorem A.4.8) it holds that

$$(C_b^\alpha(\mathbb{R}^N), C_b^{2+\alpha}(\mathbb{R}^N))_{1-(\alpha-\theta)/2,+\infty} = C_b^{2+\theta}(\mathbb{R}^N).$$

In order to show that $u \in (C_b^\alpha(\mathbb{R}^N), C_b^{2+\alpha}(\mathbb{R}^N))_{1-(\alpha-\theta)/2,+\infty}$, we apply the same arguments as in [105]. For this purpose, we split, for any $\xi > 0$, $u = a_\xi + b_\xi$, where

$$a_\xi(x) = \int_0^\xi e^{-\lambda t}(T(t)f)(x)dt, \qquad b_\xi(x) = \int_\xi^{+\infty} e^{-\lambda t}(T(t)f)(x)dt,$$

for any $x \in \mathbb{R}^N$. From Lemma 6.2.1 and the estimate (6.1.33), where we take $\omega = \lambda/2$, we deduce that

$$\|a_\xi\|_{C_b^\alpha(\mathbb{R}^N)} \le C\|f\|_{C_b^\theta(\mathbb{R}^N)} \int_0^\xi t^{-\frac{\alpha-\theta}{2}} dt = C'\xi^{1-\frac{\alpha-\theta}{2}}\|f\|_{C_b^\theta(\mathbb{R}^N)},$$

$$\|b_\xi\|_{C_b^{2+\alpha}(\mathbb{R}^N)} \le C\|f\|_{C_b^\theta(\mathbb{R}^N)} \int_\xi^{+\infty} t^{-1-\frac{\alpha-\theta}{2}} dt = C''\xi^{-\frac{\alpha-\theta}{2}}\|f\|_{C_b^\theta(\mathbb{R}^N)},$$

for some positive constants C, C', C'', independent of ξ. Hence,

$$\xi^{-1+\frac{\alpha-\theta}{2}}\left(\|a_\xi\|_{C_b^\alpha(\mathbb{R}^N)} + \xi\|b_\xi\|_{C_b^{2+\alpha}(\mathbb{R}^N)}\right) \le C'\|f\|_{C_b^\theta(\mathbb{R}^N)} + C''\|f\|_{C_b^\theta(\mathbb{R}^N)},$$

for any $\xi > 0$. This implies that (see Definition A.4.1)

$$\sup_{\xi \in (0,1)} \xi^{-1+\frac{\alpha-\theta}{2}} K(\xi, u) \le (C' + C'')\|\varphi\|_{C_b^\theta(\mathbb{R}^N)},$$

implying that $u \in (C_b^\alpha(\mathbb{R}^N), C_b^{2+\alpha}(\mathbb{R}^N))_{1-(\alpha-\theta)/2,\infty}$ and, consequently, that (6.2.7) holds true. ∎

Remark 6.2.3 Actually, the results in Chapter 7 will show that, in some situations, (6.1.11) and, consequently, (6.1.33) hold with an exponential term of negative type (see Remark 7.2.4). In such a situation, the same arguments as in the proof of Theorem 6.2.2 show that we can take $\omega = 0$ in (6.2.6).

As far as the Cauchy problem (6.2.2) is concerned, we can prove two results which provide us a (unique) solution u to such a problem and give sharp estimates for its space derivatives. As in the classical case, we give the following definition.

Definition 6.2.4 *The function $u : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ given by*

$$u(t,x) = (T(t)u_0)(x) + \int_0^t (T(t-s)g(s,\cdot))(x)ds, \qquad t > 0,\ x \in \mathbb{R}^N, \quad (6.2.9)$$

is called mild solution to the problem (6.2.2).

Throughout the rest of this section we always assume that Hypotheses 6.1.1(i)–6.1.1(iii) and 6.1.1(iv-3) are satisfied.

Proposition 6.2.5 *Fix $\alpha, \theta \in (0,1)$, $\theta < \beta \le 2+\theta$. Moreover, let $u_0 \in C_b(\mathbb{R}^N)$ and let g be a continuous function in $(0,T] \times \mathbb{R}^N$ such that $g(t,\cdot) \in C_b^\beta(\mathbb{R}^N)$ for any $t \in (0,T]$ and*

$$\sup_{0<t\le T} t^\alpha \|g(t,\cdot)\|_{C_b^\beta(\mathbb{R}^N)} < +\infty.$$

Then, the function u defined in (6.2.9) *belongs to $C([0,T]\times\mathbb{R}^N) \cap C^{1,2}((0,T]\times\mathbb{R}^N)$. Moreover, u is the unique bounded classical solution to the problem* (6.2.2). *Finally, $u(t,\cdot) \in C_b^{2+\theta}(\mathbb{R}^N)$ for any $t \in (0,T]$ and there exists a positive constant $C > 0$, independent of u, such that*

$$\|u(t,\cdot)\|_\infty + t^{1+\frac{\theta}{2}} \|u(t,\cdot)\|_{C_b^{2+\theta}(\mathbb{R}^N)} \le C \left(\|u_0\|_\infty + \sup_{0<t\le T} t^\alpha \|g(t,\cdot)\|_{C_b^\beta(\mathbb{R}^N)} \right),$$

for any $t \in [0,T]$.

Proof. The uniqueness of the bounded classical solution to the problem (6.2.9) is a straightforward consequence of Hypothesis 6.1.1(ii) and Theorem 4.1.3. So, let us prove that the function u in (6.2.9) is actually a bounded classical solution to the problem (6.2.2).

Throughout the proof, we denote by C positive constants, independent of s and t, which may vary from line to line.

To begin with, we will deal with the function defined by the integral term in (6.2.9). We will denote it by v.

As a first step, we show that v is well defined and it is continuous in $[0,T]\times\mathbb{R}^N$. For this purpose, we observe that the function $(r,s,x) \mapsto (T(r)g(s,\cdot))(x)$ is continuous in $[0,T]\times(0,T]\times\mathbb{R}^N$. To see it, it suffices to observe that, for any $s \in [0,T]$, the function $(r,x) \mapsto (T(r)g(s,\cdot))(x)$ is continuous in $[0,T]\times\mathbb{R}^N$, and, by virtue of Proposition 2.2.9, for any $r \in [0,T]$ and any $x \in \mathbb{R}^N$, the function $s \mapsto (T(r)g(s,\cdot))(x)$ is continuous in $[0,T]$, uniformly with respect to r and x on compact subsets of $\mathbb{R}^N$. Moreover,

$$|T(r)(g(s,\cdot))(x)| \le \|g(s,\cdot)\|_\infty \le Cs^{-\alpha}, \qquad s \in (0,T], \quad x \in \mathbb{R}^N$$

for some $C > 0$. This is enough to conclude that v is continuous in $[0,T]\times\mathbb{R}^N$.

Let us now prove that $v(t,\cdot)\in C_b^{2+\theta}(\mathbb{R}^N)$ for any $t\in(0,T]$. By the estimate (6.1.33) we have

$$\|T(t-s)\|_{L(C_b^{\beta}(\mathbb{R}^N),C_b^{2+\theta}(\mathbb{R}^N))}\le C(t-s)^{1-\frac{\beta-\theta}{2}},$$

for any $0\le s<t\le T$. Therefore,

$$\|T(t-s)g(s)\|_{C_b^{2+\theta}(\mathbb{R}^N)}\le \frac{C}{s^{\alpha}(t-s)^{1-(\beta-\theta)/2}}\sup_{0<t\le T}t^{\alpha}\|g(t,\cdot)\|_{C_b^{\beta}(\mathbb{R}^N)}.$$

Since $s\mapsto(s^{\alpha}(t-s)^{1-(\beta-\alpha)/2})^{-1}$ is in $L^1(0,t)$, by Lemma 6.2.1, we have $v(t,\cdot)\in C_b^{2+\theta}(\mathbb{R}^N)$ for any $t\in(0,T]$, and

$$\begin{aligned}\|v(t,\cdot)\|_{C_b^{2+\theta}(\mathbb{R}^N)}&\le C\left(\int_0^t\frac{1}{s^{\alpha}(t-s)^{1-(\beta-\theta)/2}}ds\right)\sup_{0<t\le T}t^{\alpha}\|g(t,\cdot)\|_{C_b^{\beta}(\mathbb{R}^N)}\\&=C't^{-\alpha+\frac{\beta-\theta}{2}}\sup_{0<t\le T}t^{\alpha}\|g(t,\cdot)\|_{C_b^{\beta}(\mathbb{R}^N)}.\end{aligned}\tag{6.2.10}$$

Therefore, the function $t\mapsto v(t,\cdot)$ is bounded in $[\varepsilon,T]$ with values in $C^{2+\theta}(K)$ for any compact set $K\subset\mathbb{R}^N$ and any $\varepsilon\in(0,T)$. Since v is continuous, then it belongs to $C([0,T];C(K))$. By Propositions A.4.4 and A.4.6(i), the map v belongs to $C([\varepsilon,T]\times C^2(K))$. Therefore, the first- and second-order derivatives of v are continuous in $[\varepsilon,T]\times K$ and, hence, in $(0,T]\times\mathbb{R}^N$.

Let us now consider the regularity of v with respect to t. For this purpose we observe that, for any $t>0$, any $x\in\mathbb{R}^N$ and any $h>0$, sufficiently close to 0, one has

$$\begin{aligned}\frac{v(t+h,x)-v(t,x)}{h}&=\frac{1}{h}\int_t^{t+h}(T(t+h-s)g(s,\cdot))(x)ds\\&\quad+\frac{1}{h}\int_0^t((T(t+h-s)-T(t-s))g(s,\cdot))(x)ds.\end{aligned}\tag{6.2.11}$$

As it is easily seen, the first integral term in the right-hand side of (6.2.11) tends to $g(t,x)$ as h tends to 0^+. As far as the second term is concerned, we observe that, taking Theorem 2.2.1 into account, we can show that, for any $s\in(0,T]$ and any $x\in\mathbb{R}^N$, the function $t\mapsto(T(t-s)g(s,\cdot))(x)$ is continuously differentiable in $(s,T]$ and

$$D_t(T(t-s)g(s,\cdot))(x)=(\mathcal{A}T(t-s)g(s,\cdot))(x).$$

Therefore,

$$\begin{aligned}((T(t+h-s)-T(t-s))g(s,\cdot))(x)&=\int_0^h(D_tT(t-s+r)g(s,\cdot))(x)dr\\&=\int_0^h(\mathcal{A}T(t-s+r)g(s,\cdot))(x)dr.\end{aligned}$$

Now, using (6.1.33), we can show that

$$\begin{aligned}|(\mathcal{A}T(t+r-s)g(s,\cdot))(x)| &\le \frac{C}{s^\alpha}(t+r-s)^{\min\left(\frac{\beta}{2}-1,0\right)}\sup_{0<t\le T} t^\alpha\|g(t,\cdot)\|_{C_b^\beta(\mathbb{R}^N)},\\ &\le \frac{C}{s^\alpha}(t-s)^{\min\left(\frac{\beta}{2}-1,0\right)}\sup_{0<t\le T} t^\alpha\|g(t,\cdot)\|_{C_b^\beta(\mathbb{R}^N)},\end{aligned}$$

for any $t\in(s,T]$, any $x\in B(n)$, any $r\in(0,h)$ and some positive constant $C=C(n,T)$. Hence,

$$\begin{aligned}&|((T(t+h-s)-T(t-s))g(s,\cdot))(x)|\\ &\le \frac{C}{s^\alpha}(t-s)^{\min\left(\frac{\beta}{2}-1,0\right)}|h|\sup_{0<t\le T} t^\alpha\|g(t,\cdot)\|_{C_b^\beta(\mathbb{R}^N)}\end{aligned}\tag{6.2.12}$$

for any s,t,h as above. From (6.2.12) and the dominated convergence theorem, we obtain that the second term in the right-hand side of (6.2.11) tends to $\mathcal{A}v(t,x)$ as h tends to 0^+, for any $t>0$ and any $x\in\mathbb{R}^N$. Therefore, $v(\cdot,x)$ is differentiable from the right in $(0,T)$ for any $x\in\mathbb{R}^N$, and

$$D_t^+v(t,x)=\int_0^t(\mathcal{A}T(t-s)g(s,\cdot))(x)ds+g(t,x),\qquad t\in(0,T)\times\mathbb{R}^N.$$

Let us now prove that

$$\int_0^t(\mathcal{A}T(t-s)g(s,\cdot))(x)ds=\mathcal{A}\int_0^t(T(t-s)g(s,\cdot))(x)ds,\tag{6.2.13}$$

for any $t\in(0,T]$ and any $x\in\mathbb{R}^N$. This will imply that $D_tv^+=\mathcal{A}v+g$. But, since we have already shown that the functions $D^\gamma v$ ($|\gamma|=1,2$) are continuous in $(0,T]\times\mathbb{R}^N$, it follows that $\mathcal{A}v$ is continuous in $(0,T]\times\mathbb{R}^N$ as well. Therefore, D_tv exists in $(0,T]\times\mathbb{R}^N$ and it is therein continuous. Hence, v is differentiable with respect to the time variable in $(0,T]\times\mathbb{R}^N$ and $D_tv=\mathcal{A}v+g$.

We begin by proving (6.2.13). For this purpose, we observe that the function $s\mapsto D_x^\gamma(T(t-s)g(s,\cdot))_{|K}$ belongs to $C((0,T);C(K))\cap L^1((0,T);C(K))$ for any compact set $K\subset\mathbb{R}^N$, any multi-index γ with $|\gamma|=1,2$ and any $t\in(0,T]$. Since the realization of the derivative D_x^γ in $C(K)$ is a closed operator, it follows that

$$D_x^\gamma\int_0^t(T(t-s)g(s,\cdot))(x)ds=\int_0^t(D_x^\gamma T(t-s)g(s,\cdot))(x)ds,$$

for any $t\in(0,T]$ and any $x\in K$. Hence, the formula (6.2.13) immediately follows.

To conclude the proof, let us deal with the function $T(\cdot)u_0$. According to Theorem 2.2.1 we already know that such a function belongs to $C^{1,2}((0,+\infty)\times$

$\mathbb{R}^N) \cap C_b([0,+\infty) \times \mathbb{R}^N)$ and $D_tT(t)u_0 = \mathcal{A}T(t)u_0$ for any $t \in (0,+\infty)$. We now observe that, by virtue of (6.1.33), the function $t \mapsto t^{1+\theta/2}T(t)u_0$ is bounded with values in $C_b^{2+\theta}(\mathbb{R}^N)$. Hence, the function

$$u(t,x) = (T(t)u_0)(x) + v(t,x), \qquad t \in [0,T],\ x \in \mathbb{R}^N,$$

belongs to $C^{1,2}((0,T] \times \mathbb{R}^N)$, it satisfies (6.2.2), and

$$\sup_{0<t\le T} t^{1+\frac{\theta}{2}}\|u(t,\cdot)\|_{C_b^{2+\theta}(\mathbb{R}^N)} \le C\left(\|u_0\|_\infty + \sup_{0<t\le T} t^\alpha \|g(t,\cdot)\|_{C_b^\beta(\mathbb{R}^N)}\right).$$

If, in addition, g is bounded with values in $C_b^{2+\theta}(\mathbb{R}^N)$, then $v(t,\cdot)$ is bounded up to $t=0$ with values in $C_b^{2+\theta}(\mathbb{R}^N)$ since, by (6.1.33),

$$\|T(t)\|_{L(C_b^\beta(\mathbb{R}^N);C_b^{2+\theta}(\mathbb{R}^N))} \le C(t-s)^{-1+\frac{\beta-\theta}{2}},$$

and the right-hand side of the previous inequality defines a function in $L^1(s,t)$. Taking once more advantage of (6.1.33), we obtain that also the function $t \mapsto T(t)u_0$ is bounded with values in $C_b^{2+\theta}(\mathbb{R}^N)$. Then, arguing as above, we easily see that $u \in C^{1,2}([0,T] \times \mathbb{R}^N)$. ∎

Proposition 6.2.6 *Let $T > 0$, $\theta \in (0,1)$ and let $g : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a bounded and continuous function such that $g(t,\cdot) \in C_b^\theta(\mathbb{R}^N)$ for any t and*

$$\sup_{t\in[0,T]} \|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)} < +\infty.$$

Moreover, let $u_0 \in C_b^{2+\theta}(\mathbb{R}^N)$. Then, the function u in (6.2.9) is the unique strict solution to the problem (6.2.2) (i.e., $D_tu, D_xu, D_{xx}u \in C_b([0,T] \times \mathbb{R}^N)$ and u solves (6.2.2)). Moreover, there exists a positive constant $C = C_T$ such that

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{C_b^{2+\theta}(\mathbb{R}^N)} \le C\left(\|u_0\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \sup_{t\in[0,T]} \|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)}\right). \quad (6.2.14)$$

Proof. From Proposition 6.2.5, we know that, under our assumptions, the function u defined by (6.2.9) is the unique classical solution to the problem (6.2.2) belonging to $C^{1,2}([0,T] \times \mathbb{R}^N)$. For notational convenience, throughout the rest of the proof, we denote by v the function defined by the integral term in (6.2.9), so that we can write $u(t,\cdot) = T(t)u_0 + v(t,\cdot)$ for any $t \in [0,T]$.

Let us prove that u is bounded with values in $C_b^{2+\theta}(\mathbb{R}^N)$ and it satisfies (6.2.14). For this purpose, we begin by observing that, due to (6.1.33), we already know that $T(t)u_0 \in C_b^{2+\theta}(\mathbb{R}^N)$ for any $t \ge 0$ and

$$\|T(t)\|_{C_b^{2+\theta}(\mathbb{R}^N)} \le C_1\|u_0\|_{C_b^{2+\theta}(\mathbb{R}^N)}, \quad (6.2.15)$$

for some positive constant C_1 and any $t \in [0,T]$. To show that v is bounded with values in $C_b^{2+\theta}(\mathbb{R}^N)$ we adapt the techniques used in the proof of Theorem 6.2.2, namely, we show that, for any $t \in [0,T]$ and any $\alpha \in (\theta,1)$,

$$v(t,\cdot) \in (C_b^\alpha(\mathbb{R}^N), C_b^{2+\alpha}(\mathbb{R}^N))_{1-(\alpha-\theta)/2,\infty},$$

recalling that $(C_b^\alpha(\mathbb{R}^N), C_b^{2+\alpha}(\mathbb{R}^N))_{1-(\alpha-\theta)/2,+\infty} = C_b^{2+\theta}(\mathbb{R}^N)$ with equivalence of the corresponding norms (see Theorem A.4.8). For this purpose, we split $v(t,\cdot) = a_\xi(t,\cdot) + b_\xi(t,\cdot)$, where, for any $\xi \in (0,1)$, we set

$$a_\xi(t,x) = \begin{cases} \displaystyle\int_{t-\xi}^t (T(t-s)g(s,\cdot))(x)ds, & \text{if } \xi \le t, \\ \displaystyle\int_0^t (T(t-s)g(s,\cdot))(x)ds, & \text{if } \xi > t, \end{cases}$$

$$b_{t,\xi}(x) = \begin{cases} \displaystyle\int_0^{t-\xi} (T(t-s)g(s,\cdot))(x)ds, & \text{if } \xi \le t, \\ 0, & \text{if } \xi > t \end{cases}$$

for any $x \in \mathbb{R}^N$. Taking Lemma 6.2.1 and the estimate (6.1.33) into account, we easily deduce that, for any $\alpha \in (\theta,1)$ and any $t \in [0,T]$, $a_\xi(t,\cdot) \in C_b^\alpha(\mathbb{R}^N)$ and $b_\xi(t,\cdot) \in C_b^{2+\alpha}(\mathbb{R}^N)$. Moreover,

$$\begin{aligned}\|a_\xi(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^N)} &\le C_2 \sup_{0\le s\le t} \int_{\max(t-\xi,0)}^t (t-s)^{-\frac{\alpha-\theta}{2}} ds, \\ &\le C_3 \xi^{1-\frac{\alpha-\theta}{2}} \sup_{0\le s\le t} \|g(s,\cdot)\|_{C_b^\theta(\mathbb{R}^N)}\end{aligned}$$

and

$$\begin{aligned}\|b_\xi(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^N)} &\le C_4 \sup_{0\le s\le t} \int_0^{\max(t-\xi,0)} (t-s)^{-1-\frac{\alpha-\theta}{2}} ds, \\ &\le C_5 \xi^{-\frac{\alpha-\theta}{2}} \sup_{0\le s\le t} \|g(s,\cdot)\|_{C_b^\theta(\mathbb{R}^N)},\end{aligned}$$

for some positive constants $C_2, \ldots, C_5$, independent of ε, so that

$$\begin{aligned}\xi^{-1+\frac{\alpha-\theta}{2}} K(\xi, v(t,\cdot)) &\le \xi^{-1+\frac{\alpha-\theta}{2}} \|a_\xi(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^N)} + \xi^{\frac{\alpha-\theta}{2}} \|b_\xi(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^N)} \\ &\le (C_4 + C_5) \sup_{0\le s\le T} \|g(s,\cdot)\|_{C_b^\theta(\mathbb{R}^N)},\end{aligned}$$

for any $\xi \in (0,1)$, implying that $v(t,\cdot) \in (C_b^\alpha(\mathbb{R}^N), C_b^{2+\alpha}(\mathbb{R}^N))_{1-(\alpha-\theta)/2,\infty}$ for any $t \in (0,T)$ and that

$$\sup_{0\le t\le T} \|v(t,\cdot)\|_{C_b^{2+\theta}(\mathbb{R}^N)} \le C_6 \sup_{0\le t\le T} \|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)}, \tag{6.2.16}$$

for some positive constant C_6. Summing up, we have proved that $u(t,\cdot) \in C_b^{2+\theta}(\mathbb{R}^N)$ for any $t \in [0,T]$. Moreover, combining (6.2.15) and (6.2.16), the estimate (6.2.14) now follows. ∎

Chapter 7

Pointwise estimates for the derivatives of $T(t)f$

7.0 Introduction

This chapter is devoted to prove some pointwise estimates for the first-, second- and third-order derivatives of $T(t)f$, when $f \in C_b^k(\mathbb{R}^N)$ $(k = 0, 1, 2, 3)$. Under the same assumptions on the coefficients as in Chapter 6, we prove that, for any $k = 1, 2, 3$ and any $p \in (1, +\infty)$, there exists a constant $M_{k,p} > 0$ such that

$$\left(\sum_{i=0}^{k} |D^i T(t)f)(x)|^2\right)^{\frac{p}{2}} \le M_{k,p}\left(T(t)\left(\sum_{i=0}^{k} |D^i f|^2\right)^{\frac{p}{2}}\right)(x), \tag{7.0.1}$$

for any $t > 0$, any $x \in \mathbb{R}^N$ and any $f \in C_b^k(\mathbb{R}^N)$. Under somewhat heavier assumptions, we show that

$$\left(\sum_{i=1}^{k} |D^i T(t)f)(x)|^2\right)^{\frac{p}{2}} \le \hat{M}_{k,p} e^{\sigma_{k,p} t}\left(T(t)\left(\sum_{i=1}^{k} |D^i f|^2\right)^{\frac{p}{2}}\right)(x), \tag{7.0.2}$$

for any $t > 0$ and any $x \in \mathbb{R}^N$, where $\hat{M}_{k,p}$ and $\sigma_{k,p}$ are, respectively, a positive and a negative constant.

The estimates (7.0.1) and (7.0.2) are then used to prove the sharper estimate

$$\begin{aligned}|(D^k T(t)f)(x)|^p \le{}& \left(\frac{\widetilde{\sigma}_{k,\min\{p,2\}}}{1 - e^{-\widetilde{\sigma}_{k,\min\{p,2\}} t}}\varphi_{k,\min\{p,2\}}(t)\right)^{\max\{1,\frac{p}{2}\}}\\ &\times \left(T(t)((|f|^2 + \ldots + |D^{k-1} f|^2)^{\frac{p}{2}})\right)(x),\end{aligned} \tag{7.0.3}$$

holding for any $(t, x) \in (0, +\infty) \times \mathbb{R}^N$, any $f \in C_b^{k-1}(\mathbb{R}^N)$ and any $p > 1$, where $\widetilde{\sigma}_{k,p}$ is a real constant and $\varphi_{k,r} \in C([0, +\infty))$ is a suitable function which behaves as $t^{1-r/2}$ near 0 and it is such that the term in the first brackets in the right-hand side of (7.0.3) stays bounded at infinity, or it decreases to 0 exponentially. Taking the semigroup property into account, from (7.0.3) we get

$$|(D^k T(t)f)(x)|^p \le C_{k,p}\frac{e^{\omega_{k,p} t}}{t^{pk/2}}(T(t)(|f|^p))(x), \qquad t > 0,\ x \in \mathbb{R}^N, \tag{7.0.4}$$

for any $f \in C_b(\mathbb{R}^N)$, any $p > 1$ and some constants $C_{k,p} > 0$, blowing up as p tends to 1, and $\omega_{k,p} \in \mathbb{R}$.

In the particular case when $q_{ij}(x) = \delta_{ij}$, i.e., when $\mathcal{A} = \Delta + \sum b_i(x)D_i$, we prove the estimate (7.0.1) also for $p = 1$. Such pointwise estimates are typical for transition semigroups of Markov processes, and they have been already studied for the first-order derivatives ($k = 1$); see [11, 13, 15]. Here, we present the results of [18].

On the contrary, the estimate (7.0.4) cannot be extended, in general, to the case when $p = 1$. Counterexamples are easily obtained in the simple case $\mathcal{A} = \Delta$ (see Example 7.3.3).

In the case when $\omega_{1,p} \le 0$, the estimate (7.0.4) with $k = 1$ allows us to obtain a Liouville type theorem, namely, in such a situation we can show that if $\mathcal{A}u = 0$, then u is constant. If $\omega_{1,p} > 0$, in general, such a result fails. Counterexamples are given in [128] also in the one-dimensional case.

Sometimes in what follows, when there is no damage of confusion, we write u instead of $T(\cdot)f$.

7.1 The first type of pointwise gradient estimates

We begin by proving the following lemma which will be essential to prove the first type of pointwise estimates.

Proposition 7.1.1 *Let $k \in \{1,2,3\}$ and let Hypotheses 6.1.1(i)–6.1.1(iii) and 6.1.1(iv-k) be satisfied. Then, for any $f \in C_b^k(\mathbb{R}^N)$, the function $(t,x) \mapsto (D^kT(t)f)(x)$ is continuous in $[0,+\infty) \times \mathbb{R}^N$.*

Proof. Fix $f \in C_b^k(\mathbb{R}^N)$ and set $u = T(\cdot)f$. The regularity of u for $t > 0$ is a classical result, recalled in Theorem C.1.4. Thus we have only to prove the regularity at $t = 0$. We will do it using a localization argument.

Fix $x_0 \in \mathbb{R}^N$ and let Ω be a smooth bounded neighborhood of x_0. Moreover, let $\vartheta \in C_c^\infty(\Omega)$ be such that $\vartheta \equiv 1$ in a smaller neighborhood $\Omega_0 \subset \Omega$ of x_0. Set $v(t,x) = \vartheta(x)u(t,x)$ for any $t > 0$ and any $x \in \Omega$. Then, the function v satisfies the equation $D_tv - \mathcal{A}v = \psi$ in $(0,+\infty) \times \Omega$, where

$$\psi(t,x) = -u(t,x)\mathcal{A}\vartheta(x) - 2\sum_{i,j=1}^N q_{ij}(x)D_iu(t,x)D_j\vartheta(x),$$

for any $t > 0$ and any $x \in \Omega$, and the boundary condition $v(t,x) = 0$ for any $t > 0$ and any $x \in \partial\Omega$. Moreover, it is readily seen that there exists a constant $C > 0$ such that

$$\|\psi(t,\cdot)\|_\infty \le C\|u(t,\cdot)\|_{C^1(\overline{\Omega})} \le C\frac{C_{0,1}e^T}{\sqrt{t}}\|f\|_\infty, \qquad t \in (0,T],$$

where we have used the estimate (6.0.1) with $(k,l)=(0,1)$. In particular the function $t\mapsto \|\psi(t,\cdot)\|_\infty$ belongs to $L^1(0,T)$ for any $T>0$, and, therefore, we can write

$$v(t,x)=(T_\Omega(t)(\vartheta f))(x)+\int_0^t (T_\Omega(t-s)\psi(s,\cdot))(x)ds, \qquad t>0,\ x\in\Omega,$$

where $\{T_\Omega(t)\}$ is the semigroup associated with the realization of the operator $\mathcal{A}$ with homogeneous Dirichlet boundary condition in $C(\overline{\Omega})$. Now, let $k=1,2,3$ and $f\in C^k_b(\mathbb{R}^N)$; using the classical gradient estimates for $\{T_\Omega(t)\}$ (see Proposition C.3.2) and the estimate (6.0.1), we get

$$\begin{aligned}\|D^kT_\Omega(t-s)\psi(s,\cdot)\|_\infty &\le \frac{C_T}{\sqrt{t-s}}\|\psi(s,\cdot)\|_{C^{k-1}(\overline{\Omega})}\\ &\le C\frac{C_T}{\sqrt{t-s}}\|u(s,\cdot)\|_{C^k(\overline{\Omega})}\\ &\le C_{k,k}C\frac{C_T}{\sqrt{t-s}}\|f\|_{C^k_b(\mathbb{R}^N)},\end{aligned}$$

for any $0<s<t\le T$, where $C,C_T>0$ are constants. This means that the function $s\mapsto \|D^kT_\Omega(t-s)\psi(s,\cdot)\|_\infty$ belongs to $L^1(0,t)$ for any $t\in(0,T)$ and, therefore, we can write

$$\begin{aligned}|D^kv(t,x)-(D^kT_\Omega(t)(\vartheta f))(x)| &= \left|\int_0^t (D^kT_\Omega(t-s)\psi(s,\cdot))(x)ds\right|\\ &\le CC_{kk}C_T\int_0^t\frac{1}{\sqrt{t-s}}\|f\|_{C^k_b(\mathbb{R}^N)}ds\\ &=2CC_{kk}C_Tt\|f\|_{C^k_b(\mathbb{R}^N)},\end{aligned}$$

for any $t\in(0,T]$ and any $x\in\Omega$. This implies that the function D^kv is continuous in $[0,T]\times\Omega_0$ since, by virtue of Lemma 6.1.6, $D^k(T_\Omega(t)(\vartheta f))$ tends to $D^k(\vartheta f)$ uniformly in $\Omega_\delta:=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)\ge\delta\}$, for any $\delta>0$, as t tends to 0. Since $v\equiv u$ in a neighborhood of x_0, it follows that D^ku is continuous at $(0,x_0)$. ∎

Now, we can prove the following theorem.

Theorem 7.1.2 *Fix $k\in\{1,2,3\}$ and let Hypotheses 6.1.1(i)–6.1.1(iii) and 6.1.1(iv-k) be satisfied. Then, for any $f\in C^k_b(\mathbb{R}^N)$ and any $p\in(1,+\infty)$, there exists a positive constant $M_{k,p}$ such that*

$$\left(\sum_{j=0}^k |(D^jT(t)f)(x)|^2\right)^{\frac{p}{2}}\le M_{k,p}\left(T(t)\left(\sum_{j=0}^k|D^jf|^2\right)^{\frac{p}{2}}\right)(x), \tag{7.1.1}$$

for any $t>0$ and any $x\in\mathbb{R}^N$.

Proof. We begin with the case when $k=3$ and $p\in(1,2]$. For any $\delta>0$ we introduce the function $w_\delta:[0,+\infty)\times\mathbb{R}^N\to\mathbb{R}$ defined by

$$w_\delta=\left(\alpha|u|^2+\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2+\delta\right)^{\frac{p}{2}}, \tag{7.1.2}$$

where α,β,γ are positive constants to be fixed later on, and $u=T(\cdot)f$. By Theorems 2.2.1, C.1.4(ii) and Proposition 7.1.1, $w_\delta\in C_b([0,+\infty)\times\mathbb{R}^N)\cap C^{1,2}((0,+\infty)\times\mathbb{R}^N)$ and a straightforward computation shows that w_δ solves the Cauchy problem

$$\begin{cases} D_tw_\delta(t,\cdot)=\mathcal{A}w_\delta(t,\cdot)+g_\delta(t,\cdot), & t>0,\\[2mm] w_\delta(0,\cdot)=\left(\alpha|f|^2+\beta|Df|^2+\gamma|D^2f|^2+|D^3f|^2+\delta\right)^{\frac{p}{2}}, \end{cases} \tag{7.1.3}$$

where

$$\begin{aligned} g_\delta=p&\left(\alpha|u|^2+\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2+\delta\right)^{\frac{p}{2}-1}\\ &\times\Bigg(-\alpha\sum_{i,j=1}^N q_{ij}D_iuD_ju-\beta\sum_{i,j,h=1}^N q_{ij}D_{ih}uD_{jh}u\\ &\qquad-\gamma\sum_{i,j,h,k=1}^N q_{ij}D_{ihk}uD_{jhk}u-\sum_{i,j,h,k,l=1}^N q_{ij}D_{ihkl}uD_{jhkl}u\\ &\qquad+\beta\sum_{i,j,h=1}^N D_hq_{ij}D_huD_{ij}u+2\gamma\sum_{i,j,h,k=1}^N D_hq_{ij}D_{hk}uD_{ijk}u\\ &\qquad+3\sum_{i,j,h,k,l=1}^N D_hq_{ij}D_{hkl}uD_{ijkl}u+\beta\sum_{j,h=1}^N D_hb_jD_juD_hu\\ &\qquad+2\gamma\sum_{j,h,k=1}^N D_hb_jD_{jk}uD_{hk}u+3\sum_{j,h,k,l=1}^N D_hb_jD_{jkl}uD_{hkl}u\\ &\qquad+\gamma\sum_{i,j,h,k=1}^N D_{hk}q_{ij}D_{ij}uD_{hk}u+3\sum_{i,j,h,k,l=1}^N D_{hk}q_{ij}D_{ijl}uD_{hkl}u\\ &\qquad+\gamma\sum_{j,h,k=1}^N D_{hk}b_jD_juD_{hk}u+3\sum_{j,h,k,l=1}^N D_{hk}b_jD_{jl}uD_{hkl}u\\ &\qquad+\sum_{i,j,h,k,l=1}^N D_{hkl}q_{ij}D_{ij}uD_{hkl}u+\sum_{j,h,k,l=1}^N D_{hkl}b_jD_juD_{hkl}u\Bigg) \end{aligned}$$

$$
\begin{aligned}
&+ p(2-p)\left(\alpha|u|^2+\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2+\delta\right)^{\frac{p}{2}-2}\\
&\quad\times\sum_{i,j=1}^N q_{ij}\Bigg(\alpha u D_i u+\beta\sum_{h=1}^N D_h u D_{ih}u+\gamma\sum_{h,k=1}^N D_{hk}uD_{ihk}u\\
&\qquad\qquad+\sum_{h,k,l=1}^N D_{hkl}uD_{ihkl}u\Bigg)\\
&\qquad\quad\times\Bigg(\alpha u D_j u+\beta\sum_{h=1}^N D_h u D_{jh}u+\gamma\sum_{h,k=1}^N D_{hk}uD_{jhk}u\\
&\qquad\qquad+\sum_{h,k,l=1}^N D_{hkl}uD_{jhkl}u\Bigg).
\end{aligned}
$$

Now, let $h,k\in\{0,1,2,3\}$ be fixed. Applying the Cauchy-Schwarz inequality twice (first to the inner product $(\xi,\eta)\mapsto\langle Q(x)\xi,\eta\rangle$ and then to the Euclidean one) we deduce that

$$
\begin{aligned}
&\sum_{i,j=1}^N q_{ij}\sum_{|\alpha|=h}D^\alpha u D_i D^\alpha u\sum_{|\beta|=k}D^\beta u D_j D^\beta u\\
&=\sum_{|\alpha|=h}\sum_{|\beta|=k}D^\alpha u D^\beta u\sum_{i,j=1}^N q_{ij}D_iD^\alpha u D_j D^\beta u\\
&\le\sum_{|\alpha|=h}|D^\alpha u|\left(\sum_{i,j=1}^N q_{ij}D_iD^\alpha uD_jD^\alpha u\right)^{\frac12}\\
&\qquad\times\sum_{|\beta|=k}|D^\beta u|\left(\sum_{i,j=1}^N q_{ij}D_iD^\beta uD_jD^\beta u\right)^{\frac12}\\
&\le|D^hu||D^ku|\left(\sum_{|\alpha|=h}\sum_{i,j=1}^N q_{ij}D_iD^\alpha uD_jD^\alpha u\right)^{\frac12}\\
&\qquad\times\left(\sum_{|\beta|=k}\sum_{i,j=1}^N q_{ij}D_iD^\beta uD_jD^\beta u\right)^{\frac12}.
\end{aligned}
\tag{7.1.4}
$$

This estimate can be used in order to get

$$\begin{aligned}
&\sum_{i,j=1}^N q_{ij}\bigg(\alpha u D_i u+\beta\sum_{h=1}^N D_h u D_{ih}u+\gamma\sum_{h,k=1}^N D_{hk}uD_{ihk}u\\
&\qquad\qquad+\sum_{h,k,l=1}^N D_{hkl}uD_{ihkl}u\bigg)\\
&\quad\times\bigg(\alpha u D_j u+\beta\sum_{h=1}^N D_h u D_{jh}u+\gamma\sum_{h,k=1}^N D_{hk}uD_{jhk}u\\
&\qquad\qquad+\sum_{h,k,l=1}^N D_{hkl}uD_{jhkl}u\bigg)\\
&\le\bigg[\alpha|u|\bigg(\sum_{i,j=1}^N q_{ij}D_i uD_j u\bigg)^{\frac12}+\beta|Du|\bigg(\sum_{i,j,h=1}^N q_{ij}D_{ih}uD_{jh}u\bigg)^{\frac12}\\
&\qquad+\gamma|D^2u|\bigg(\sum_{i,j,h,k=1}^N q_{ij}D_{ihk}uD_{jhk}u\bigg)^{\frac12}\\
&\qquad+|D^3u|\bigg(\sum_{i,j,h,k,l=1}^N q_{ij}D_{ihkl}uD_{jhkl}u\bigg)^{\frac12}\bigg]^2\\
&\le(\alpha|u|^2+\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2)\\
&\quad\times\bigg(\alpha\sum_{i,j=1}^N q_{ij}D_i uD_j u+\beta\sum_{i,j,h=1}^N q_{ij}D_{ih}uD_{jh}u\\
&\qquad+\gamma\sum_{i,j,h,k=1}^N q_{ij}D_{ihk}uD_{jhk}u+\sum_{i,j,h,k,l=1}^N q_{ij}D_{ihkl}uD_{jhkl}u\bigg).
\end{aligned}\tag{7.1.5}$$

Taking (7.1.5) into account, it is immediate to check that

$$\begin{aligned}
g_\delta\le p\bigg\{&(1-p)\bigg(\alpha\sum_{i,j=1}^N q_{ij}D_i uD_j u+\beta\sum_{i,j,h=1}^N q_{ij}D_{ih}uD_{jh}u\bigg)\\
&+(1-p)\bigg(\gamma\sum_{i,j,h,k=1}^N q_{ij}D_{ihk}uD_{jhk}u+\sum_{i,j,h,k,l=1}^N q_{ij}D_{ihkl}uD_{jhkl}u\bigg)\\
&+\beta\sum_{i,j,h=1}^N D_h q_{ij}D_h uD_{ij}u+2\gamma\sum_{i,j,h,k=1}^N D_h q_{ij}D_{hk}uD_{ijk}u\\
&+3\sum_{i,j,h,k,l=1}^N D_h q_{ij}D_{hkl}uD_{ijkl}u+\beta\sum_{j,h=1}^N D_h b_j D_j uD_h u
\end{aligned}$$

$$
\begin{aligned}
&+2\gamma\sum_{j,h,k=1}^N D_hb_jD_{jk}uD_{hk}u+3\sum_{j,h,k,l=1}^N D_hb_jD_{jkl}uD_{hkl}u\\
&+\gamma\sum_{i,j,h,k=1}^N D_{hk}q_{ij}D_{ij}uD_{hk}u+3\sum_{i,j,h,k,l=1}^N D_{hk}q_{ij}D_{ijl}uD_{hkl}u\\
&+\gamma\sum_{j,h,k=1}^N D_{hk}b_jD_juD_{hk}u+3\sum_{j,h,k,l=1}^N D_{hk}b_jD_{jl}uD_{hkl}u\\
&+\sum_{i,j,h,k,l=1}^N D_{hkl}q_{ij}D_{ij}uD_{hkl}u+\sum_{j,h,k,l=1}^N D_{hkl}b_jD_juD_{hkl}u\Bigg\}\\
&\qquad\times(\alpha|u|^2+\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2+\delta)^{\frac{p}{2}-1}.
\end{aligned}
$$

Hence, using Hypotheses 6.1.1(i) and 6.1.1(iv-3), we get, for any $\varepsilon,\varepsilon_1>0$,

$$
\begin{aligned}
g_\delta\le p\Bigg\{&\left((1-p)\alpha\kappa+C\beta\frac{N^2}{4\varepsilon_1}\kappa+\beta d+r\frac{N^3}{4\varepsilon_2}+r\gamma\frac{N^2}{4\varepsilon_3}\right)|Du|^2\\
&+\Bigg[(1-p+C\varepsilon_1N)\beta\kappa+\left(C\gamma\frac{N^2}{2\varepsilon_1}+C\frac{N^3}{4\varepsilon_1}+\gamma K_1\right)\kappa+2d\gamma\\
&\qquad+rN\left(\gamma\varepsilon_3+\frac{3N}{4\varepsilon_2}\right)\Bigg]|D^2u|^2\\
&+\Bigg[\left((1-p+2C\varepsilon_1N)\gamma+C\frac{3N^2}{4\varepsilon_1}+\varepsilon_1CN^2+3K_1\right)\kappa\\
&\qquad+3d+4r\varepsilon_2N\Bigg]|D^3u|^2+(1-p+3C\varepsilon_1N)\,\kappa|D^4u|^2\Bigg\}\\
&\qquad\times\left(\alpha|u|^2+\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2+\delta\right)^{\frac{p}{2}-1}.
\end{aligned}
\tag{7.1.6}
$$

Now, we choose $\varepsilon_1=(p-1)/(3CN)$ to make the coefficient of $|D^4u|^2$ vanish, $\varepsilon_2=3L_2/(4N)$ and $\varepsilon_3=\gamma^{-1}$, where L_2 is defined in Hypothesis 6.1.1(iv-3). Moreover, we choose β and γ satisfying

$$
\begin{cases}
\dfrac{N^4}{3L_2}+\dfrac{N^2\gamma^2}{4}\le L_2\beta,\\[2ex]
N\left(1+\dfrac{N^2}{L_2}\right)\le 2L_2\gamma.
\end{cases}
$$

With this choice of β and γ, from (7.1.6) we get

$$
\begin{aligned}
g_\delta\le p\Bigg\{&\left((1-p)\alpha\kappa+\frac{3C^2N^3}{4(p-1)}\beta\kappa+\beta(d+L_2r)\right)|Du|^2\\
&+\Bigg[\left(\frac{2-2p}{3}\beta+\frac{3C^2N^3}{4(p-1)}(2\gamma+N)+\gamma K_1\right)\kappa+2\gamma(d+L_2r)\Bigg]|D^2u|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\bigg[\left(\frac{1-p}{3}\gamma+\frac{9C^2N^3}{4(p-1)}+\frac{(p-1)N}{3}+3K_1\right)\kappa \\
&\qquad +3(d+L_2r)\bigg]|D^3u|^2\bigg\} \\
&\qquad\qquad \times\left(\alpha|u|^2+\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2+\delta\right)^{\frac{p}{2}-1} \\
\le\, & p\left\{\left((1-p)\alpha+\frac{3C^2N^3}{4(p-1)}\beta+\beta L_3\right)\kappa|Du|^2\right. \\
&+\left(\frac{2-2p}{3}\beta+\frac{3C^2N^3}{4(p-1)}(2\gamma+N)+\gamma K_1+2\gamma L_3\right)\kappa|D^2u|^2 \\
&\left.+\left(\frac{1-p}{3}\gamma+\frac{9C^2N^3}{4(p-1)}+\frac{(p-1)N}{3}+3(K_1+L_3)\right)\kappa|D^3u|^2\right\} \\
&\qquad\qquad \times\left(\alpha|u|^2+\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2+\delta\right)^{\frac{p}{2}-1}
\end{aligned}
\tag{7.1.7}
$$

Hence, up to taking larger β and γ and fixing α suitably large, we can make the right-hand side of (7.1.7) nonpositive in $(0,+\infty)\times\mathbb{R}^N$. The maximum principle in Theorem 4.1.3 now implies that

$$
w_\delta\le T(\cdot)\left(\alpha|f|^2+\beta|Df|^2+\gamma|D^2f|^2+|D^3f|^2+\delta\right),
$$

in $(0,+\infty)\times\mathbb{R}^N$, for any $\delta>0$. Taking the limit as δ tends to 0, from Proposition 2.2.9 we get

$$
\begin{aligned}
&\left(\alpha|u(t,x)|^2+\beta|Du(t,x)|^2+\gamma|D^2u(t,x)|^2+|D^3u(t,x)|^2\right)^{\frac{p}{2}} \\
&\le\left(T(t)\left(\alpha|f|^2+\beta|Df|^2+\gamma|D^2f|^2+|D^3f|^2\right)\right)(x),
\end{aligned}
$$

for any $t>0$ and any $x\in\mathbb{R}^N$. Then (7.1.1) follows.

Finally we consider the case when $p>2$. Since

$$
\begin{aligned}
&\left(\sum_{j=0}^{3}|(D^jT(t)f)(x)|^2\right)^{\frac{p}{2}} \\
&\le\left(M_{3,2}\left(T(t)\left(|f|^2+|Df|^2+|D^2f|^2+|D^3f|^2\right)\right)(x)\right)^{\frac{p}{2}},
\end{aligned}
$$

we get (7.1.1) observing that, since $T(t)\psi$ is given by (2.2.8) and the measures $p(t,x;dy)$ in (2.2.9) are probability measures for any $t>0$ and any $x\in\mathbb{R}^N$ (since $c\equiv 0$), then the Jensen inequality implies that

$$
(T(t)\psi)^{p/2}\le T(t)(\psi^{p/2}),
$$

for any $t>0$ and any nonnegative $\psi\in C_b(\mathbb{R}^N)$.

To get (7.1.1) in the case when $p \in (1,2)$ and $k = 1,2$, one can apply the previous arguments to the function

$$w_\delta = \left(\alpha|T(\cdot)f|^2 + |DT(\cdot)f|^2 + \delta\right)^{\frac{p}{2}}, \qquad \delta > 0,$$

if $k = 1$ and to the function

$$w_\delta = \left(\alpha|T(\cdot)f|^2 + \beta|DT(\cdot)f|^2 + |D^2T(\cdot)f|^2 + \delta\right)^{\frac{p}{2}},$$

for any $\delta > 0$, if $k = 2$, where α and β are positive constants to be properly fixed. Actually one gets

$$\left(\sum_{j=0}^{k} |(D^jT(t)f)(x)|^2 + \delta\right)^{\frac{p}{2}} \le M_{k,p}\left(T(t)\left(\sum_{j=0}^{k} |D^jf|^2 + \delta\right)^{\frac{p}{2}}\right)(x), \quad (7.1.8)$$

for any $t > 0$, any $x \in \mathbb{R}^N$ and $k = 1,2$.

In the case when $p > 2$ (7.1.1) then follows from the case $p = 2$, applying the Jensen inequality. This finishes the proof. ■

We now show that under more restrictive assumptions on the coefficients of the operator $\mathcal{A}$ we can improve the estimates in Theorem 7.1.2.

Hypotheses 7.1.3 (i) Hypotheses 6.1.1(i)–6.1.1(iii) are satisfied. Moreover, we always assume that one of the following hypotheses holds true:

(ii-1) $q_{ij}, b_j \in C^{1+\delta}_{\rm loc}(\mathbb{R}^N)$ for some $\delta \in (0,1)$ and any $i,j = 1,\dots,N$, and there exist $\sigma \in (0,1)$ and a constant $C > 0$ such that $|D^\beta q_{ij}(x)| \le C(\kappa(x))^\sigma$ for any $x \in \mathbb{R}^N$, any $|\beta| = 1$ and any $i,j = 1,\dots,N$. Moreover, there exist two constants $L > 0$, $p_0 \in (1,2]$ and a function $d : \mathbb{R}^N \to \mathbb{R}$ such that

$$\sum_{i,j=1}^{N} D_ib_j(x)\xi_i\xi_j \le d(x)|\xi|^2, \qquad x,\xi \in \mathbb{R}^N \quad (7.1.9)$$

and

$$0 \ge C_1(p_0) := \sup_{\mathbb{R}^N}\left(\frac{C^2N^3}{4(p_0-1)\kappa_0^{1-\sigma}}\kappa^\sigma + d\right); \quad (7.1.10)$$

(ii-2) $q_{ij}, b_j \in C^{2+\delta}_{\rm loc}(\mathbb{R}^N)$ for some $\delta \in (0,1)$ and any $i,j = 1,\dots,N$, and there exist $\sigma \in (0,1)$, a constant $C > 0$ and a positive function $r : \mathbb{R}^N \to \mathbb{R}$ such that $|D^2b_i(x)| \le r(x)$ and $|D^\beta q_{ij}(x)| \le C(\kappa(x))^\sigma$ for any $x \in \mathbb{R}^N$, any $|\beta| = 1,2$ and any $i,j = 1,\dots,N$. Moreover, the condition (7.1.9) is satisfied and there exist two constants $L > 0$ and $p_0 \in (1,2]$ such that

$$0 \ge C_2(p_0) := \sup_{\mathbb{R}^N}\left(d + Lr + \frac{C^2N^3}{4(p_0-1)}\kappa_0^{\sigma-1}\kappa^\sigma\right). \quad (7.1.11)$$

Finally, there exists a constant $K_1 \in \mathbb{R}$ such that

$$\sum_{i,j,h,k=1}^{N} D_{hk}q_{ij}(x)m_{ij}m_{hk} \le K_1(\kappa(x))^{\sigma} \sum_{h,k=1}^{N} m_{hk}^2,$$

for any symmetric matrix $M = (m_{hk})_{h,k=1}^{N}$ and any $x \in \mathbb{R}^N$;

(ii-3) $q_{ij}, b_j \in C^{3+\delta}_{\rm loc}(\mathbb{R}^N)$ for some $\delta \in (0,1)$ and any $i,j = 1,\ldots,N$; Hypothesis 7.1.3(ii-2) holds true. Moreover, $|D^{\beta}b_i(x)| \le r(x)$ and $|D^{\beta}q_{ij}(x)| \le C(\kappa(x))^{\sigma}$ for any $i,j = 1,\ldots,N$, any $|\beta| = 3$ and any $x \in \mathbb{R}^N$.

Remark 7.1.4 Hypotheses 7.1.3(ii-k) ($k = 1,2,3$) allow us to consider the case when the coefficients are of polynomial type.

We are now able to improve the result in Theorem 7.1.2.

Theorem 7.1.5 *Let $k = 1,2,3$ and let Hypotheses 7.1.3(i), 7.1.3(ii-k) be satisfied. Then, there exists a positive constant $\hat{M}_{k,p}$ such that*

$$\left(\sum_{j=1}^{k} |(D^jT(t)f)(x)|^2\right)^{\frac{p}{2}} \le \hat{M}_{k,p}e^{\omega_{k,p}t}\left(T(t)\left(\sum_{j=1}^{k}|D^jf|^2\right)^{\frac{p}{2}}\right)(x), \quad (7.1.12)$$

for any $t > 0$, any $x \in \mathbb{R}^N$, any $f \in C^k_b(\mathbb{R}^N)$ and any $p \in (p_0,+\infty)$, where $\omega_{1,p} = pC_1(p)$, with $C_1(p)$ being defined as in (7.1.10) *with p instead of p_0, and $\omega_{k,p}$ ($k = 2,3$) is any real number greater than $p\kappa_0^{\sigma}C_2(p)$, where $C_2(p)$ is defined according to* (7.1.11) *with p instead of p_0. In particular, for $k = 1,2,3$, we can take $\omega_{k,p} < 0$.*

Proof. The proof is close to that of Theorem 7.1.2. Hence, we just point out the main differences. To get (7.1.12), in the case when $p \in (p_0, 2]$ and $k = 3$, one has to deal with the function w_δ defined in (7.1.2) with $\alpha = 0$ and β,γ to be determined. As it is immediately seen, the function w_δ so defined solves the Cauchy problem (7.1.3) with g_δ satisfying (7.1.6) (with $\alpha = 0$ everywhere).

Taking Hypothesis 7.1.3(ii-3) into account and using the same techniques as in the proof of Theorem 7.1.2 to estimate the right-hand side of (7.1.6), we get

$$\begin{aligned} g_\delta \le p\Bigg\{ & \left(\left(C\frac{N^2}{4\varepsilon_2}\kappa^{\sigma} + d\right)\beta + r\frac{N^2}{4\varepsilon_1}(N+\gamma^2)\right)|Du|^2 \\ & + \Bigg[\left((1-p)\kappa_0^{1-\sigma} + C\varepsilon_2 N\right)\beta\kappa^{\sigma} + \left(C\gamma\frac{N^2}{2\varepsilon_3} + C\frac{N^3}{4\varepsilon_3} + K_1\gamma\right)\kappa^{\sigma} \\ & \quad + 2d\gamma + rN\left(\varepsilon_1 + \frac{3N}{4\varepsilon_1}\right)\Bigg]|D^2u|^2 \end{aligned}$$

$$+\Bigg[\left((1-p)\kappa_0^{1-\sigma}+2C\varepsilon_3 N\right)\gamma\kappa^\sigma+\left(C\frac{3N^2}{4\varepsilon_3}+\varepsilon_3 CN^2+3K_1\right)\kappa^\sigma$$
$$+3d+4r\varepsilon_1 N\Bigg]|D^3u|^2+\left((1-p)\kappa_0^{1-\sigma}+3C\varepsilon_3 N\right)\kappa^\sigma|D^4u|^2\Bigg\}$$
$$\times\left(\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2+\delta\right)^{\frac{p}{2}-1}. \tag{7.1.13}$$

Due to the estimate (7.1.11), we can find out $s_1\in(0,1)$ such that

$$0>C_2(p,s_1)=\sup_{x\in\mathbb{R}^N}\left(\frac{d(x)+Lr(x)}{(\kappa(x))^\sigma}+\frac{C^2N^3}{4(p-1)(1-s_1)}\kappa_0^{\sigma-1}\right). \tag{7.1.14}$$

We now choose

$$\varepsilon_2=(p-1)(1-s_1)\frac{\kappa_0^{1-\sigma}}{CN},\qquad \varepsilon_3=(p-1)\frac{\kappa_0^{1-\sigma}}{3CN}$$

and ε_1 in order to minimize the function $g:(0,+\infty)\to\mathbb{R}$ defined by $g(x)=N\max\{4x/3,(4x^2+3N)/(8\gamma x)\}$. We obtain $\varepsilon_1=3\sqrt{N}/(2\sqrt{8\gamma-3})$, if $\gamma\geq 3/8$. With these choices of ε_1, ε_2 and ε_3, we get

$$g_\delta\leq p\Bigg\{\beta\left(\frac{C^2N^3}{4(p-1)(1-s_1)}\kappa_0^{\sigma-1}\kappa^\sigma+d+r\frac{\sqrt{(8\gamma-3)N^3}}{6\beta}(N+\gamma^2)\right)|Du|^2$$
$$+\Bigg[s_1(1-p)\kappa_0^{1-\sigma}\beta\kappa^\sigma+\left((2\gamma+N)\frac{3C^2N^3}{4(p-1)\kappa_0^{1-\sigma}}+K_1\gamma\right)\kappa^\sigma$$
$$+2\gamma\left(d+\frac{2N^{\frac{3}{2}}}{\sqrt{8\gamma-3}}r\right)\Bigg]|D^2u|^2$$
$$+\Bigg[\frac{1}{3}(1-p)\kappa_0^{1-\sigma}\gamma\kappa^\sigma+\left(\frac{9C^2N^3}{4(p-1)\kappa_0^{1-\sigma}}+\frac{1}{3}(p-1)N\kappa_0^{1-\sigma}+3K_1\right)\kappa^\sigma$$
$$+3\left(d+\frac{2N^{\frac{3}{2}}}{\sqrt{8\gamma-3}}r\right)\Bigg]|D^3u|^2\Bigg\}$$
$$\times\left(\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2+\delta\right)^{\frac{p}{2}-1}.$$

Now we choose $\beta,\gamma>0$ such that

$$\begin{cases}A_1(\beta,\gamma):=\dfrac{\sqrt{(8\gamma-3)N^3}}{6\beta}(N+\gamma^2)-L\leq 0,\\[2ex] A_2(\gamma):=\dfrac{2N^{\frac{3}{2}}}{\sqrt{8\gamma-3}}-L\leq 0,\\[2ex] C_3(\beta,\gamma,p):=s_1(1-p)\kappa_0^{1-\sigma}\beta+(4\gamma+3N)\dfrac{C^2N^3}{4(p-1)\kappa_0^{1-\sigma}}+K_1\gamma<0,\\[2ex] C_4(\gamma,p):=\dfrac{1}{3}(1-p)\kappa_0^{1-\sigma}\gamma+\dfrac{3C^2N^3}{2(p-1)\kappa_0^{1-\sigma}}+\dfrac{1}{3}(p-1)N\kappa_0^{1-\sigma}+3K_1<0.\end{cases}$$

and we get

$$\begin{aligned} g_\delta(t,x) &\le \omega_{3,p}\left(\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2\right)\\ &\qquad\times\left(\beta|Du|^2+\gamma|D^2u|^2+|D^3u|^2+\delta\right)^{\frac{p}{2}-1}\\ &\le \omega_{3,p}w(t,x)-\omega_{3,p}\delta^{\frac{p}{2}}, \end{aligned}\tag{7.1.15}$$

for any $t>0$ and any $x\in\mathbb{R}^N$, where

$$\omega_{3,p}=p\kappa_0^\sigma \inf_{\substack{\beta>0,\gamma>3/8,\ A_1(\beta,\gamma),\ A_2(\gamma)\le 0\\ C_3(\beta,\gamma,p),\ C_4(\gamma,p)<0,\ j=2,3}} \max\{C_2(p),C_3(\beta,\gamma,p)/\gamma,C_4(\gamma,p)\}.$$

Note that the previous infimum is $C_2(p)$. Of course, $\omega_{3,p}\ge C_2(p)$. Hence, we just need to show that $\omega_{3,p}\le C_2(p)$. For this purpose, we observe that, since $\lim_{\gamma\to+\infty}C_4(\gamma,p)=-\infty$, we can determine γ_0 such that $C_3(\gamma_0,p)<C_2(p)$. Similarly, since $\lim_{\beta\to+\infty}C_3(\beta,\gamma_0,p)=-\infty$, we can determine $\beta_0>0$ such that $C_3(\beta_0,\gamma_0,p)/\gamma_0<C_2(p)$. Of course, without loss of generality we can assume that $\gamma_0>3/8$ and $\max\{A_1(\beta_0,\gamma_0),A_2(\gamma_0)\le 0\}$. Therefore, $\omega_{3,p}\le C_2(p)$. So, let us fix $\beta,\gamma\ge 1$ such that $\omega_{3,p}=p\kappa_0^\sigma C_2(p)$.

Now, from (7.1.15) we easily deduce that

$$D_tw_\delta(t,x)\le \mathcal{A}w_\delta(t,x)+\omega_{3,p}w_\delta(t,x)-\omega_{3,p}\delta^{\frac{p}{2}}.$$

Set now $z_\delta=e^{-\omega_{3,p}t}(w_\delta-\delta^{p/2})$ and observe that z_δ satisfies

$$\begin{cases} D_tz_\delta(t,\cdot)\le \mathcal{A}z_\delta(t,\cdot), & t>0,\\ z_\delta(0,\cdot)\le\left(\beta|Df|^2+\gamma|D^2f|^2+|D^3f|^2+\delta\right)^{\frac{p}{2}}. \end{cases}\tag{7.1.16}$$

The maximum principle in Theorem 4.1.3 implies that

$$z_\delta(t,x)\le\left(T(t)\left(\beta|Df|^2+\gamma|D^2f|^2+|D^3f|^2+\delta\right)^{\frac{p}{2}}\right)(x),\qquad t>0,\quad x\in\mathbb{R}^N,$$

for any $\delta>0$. Taking the limit as δ tends to 0, from Proposition 2.2.9 we get

$$\begin{aligned} &\left(\beta|Du(t,x)|^2+\gamma|D^2u(t,x)|^2+|D^3u(t,x)|^2\right)^{\frac{p}{2}}\\ &\le e^{\omega_{3,p}t}\left(T(t)\left(\beta|Df|^2+\gamma|D^2f|^2+|D^3f|^2\right)^{\frac{p}{2}}\right)(x),\qquad t>0,\ x\in\mathbb{R}^N, \end{aligned}$$

and (7.1.12) follows with $\hat{M}_{3,p}=\max\{\beta^{p/2},\gamma^{p/2}\}$.

Next, to get (7.1.12) in the case when $p>2$, it suffices to repeat the same arguments as in the proof of Theorem 7.1.2.

To get (7.1.1) in the case when $p\in(1,2)$ and $k=2$, one can apply the above arguments to the function

$$w_\delta=\left(\beta|DT(\cdot)f|^2+|D^2T(\cdot)f|^2+\delta\right)^{\frac{p}{2}},\qquad \delta>0,$$

if $k = 2$, where β is a positive constant to be determined. From (7.1.13), where we disregard the terms whose coefficients do not depend on β, γ, we take $\gamma = 1$, $\varepsilon_1 = 2L/N$, $\varepsilon_3 = (p-1)\kappa_0^{1-\sigma}/(2CN)$ and we make the same choices of ε_2 as above; we get

$$g_\delta \le p\Big\{\beta\left(\frac{C^2N^3}{4(p-1)(1-s_1)}\kappa_0^{\sigma-1}\kappa^\sigma + d + \frac{N^3}{8\beta L}r\right)|Du|^2 \\ + \left(s_1(1-p)\beta\kappa_0^{1-\sigma} + \frac{C^2N^3}{(p-1)\kappa_0^{1-\sigma}} + K_1\right)\kappa^\sigma|D^2u|^2\Big\} \\ \times\left(\beta|Du|^2 + |D^2u|^2 + \delta\right)^{\frac{p}{2}-1}.$$

Choosing β satisfying

$$\begin{cases} N^3 \le 8\beta L^2, \\ C_5(\beta,p) := s_1(1-p)\beta\kappa_0^{1-\sigma} + \dfrac{C^2N^3}{(p-1)\kappa_0^{1-\sigma}} + K_1 < 0, \end{cases}$$

we get

$$(\beta|Du|^2 + |D^2u|^2 + \delta)^{\frac{p}{2}} \le e^{\omega_{2,p}t}\left(T(\cdot)(\beta|Df|^2 + |D^2u|^2 + \delta)^{\frac{p}{2}}\right) + \delta^{\frac{p}{2}}, \quad (7.1.17)$$

with

$$\omega_{2,p} = p\kappa_0^\sigma \min_{\substack{\beta\ge N^4/(8L^2)\\ C_5(\beta,p)<0}} \max\{C_2(p,s_1), C_5(\beta,p)\} = C_2(p,s_1).$$

Finally, in the case when $k = 1$, taking $\beta = 1$ and $\varepsilon_2 = (p-1)\kappa_0^{1-\sigma}/(CN)$ in (7.1.13) and disregarding the terms whose coefficients are independent of β, we get

$$g_\delta \le p\left(\frac{C^2N^3}{4(p-1)\kappa_0^{1-\sigma}}\kappa^\sigma + d\right)|Du|^2$$

and, consequently, we get the assertion with $\omega_{1,p}$ as in the assertion of the theorem. This finishes the proof. ∎

Remark 7.1.6 The conditions (7.1.10) and (7.1.11) can be a bit relaxed and, under these new assumptions, the estimate (7.1.12) can be proved with positive constants $\omega_{k,p}$ ($k = 1, 2, 3$). More precisely, it suffices to assume that

$$\sup_{x\in\mathbb{R}^N}\frac{d(x)}{(\kappa(x))^\sigma} < +\infty,$$

to have (7.1.12) with $k = 1$,

$$\sup_{x\in\mathbb{R}^N}\frac{d(x) + L_2r(x)}{(\kappa(x))^\sigma} < +\infty,$$

(with $L_2 = 2/\sqrt{5}$, if $N = 1$, and $L_2 = N^{3/2}/\sqrt{8}$, if $N > 1$) to have (7.1.12) with $k = 2$,

$$\sup_{x\in\mathbb{R}^N} \frac{3d(x) + \sqrt{N^3(N+1)}r(x)}{3(\kappa(x))^\sigma} < +\infty,$$

to have (7.1.12) with $k = 3$.

7.2 The second type of pointwise gradient estimates

In this section we prove a second type of pointwise estimates. For this purpose, we first prove a lemma.

Lemma 7.2.1 *If $g_n, g : [0,T] \times \mathbb{R}^N$ $(n \in \mathbb{N})$, are continuous functions such that $\|g_n\|_\infty \le M$ for any $n \in \mathbb{N}$ and $\lim_{n\to+\infty} g_n = g$, uniformly in $[\varepsilon, T-\varepsilon] \times B(k)$ for any $k > 0$ and any $\varepsilon \in (0,T)$, then*

$$\lim_{n\to+\infty} T_n(t)g_n(t) = T(t)g(t),$$

uniformly in $[\varepsilon, T-\varepsilon] \times B(k)$ for any $k > 0$ and any $\varepsilon \in (0,T)$.

If $f_\varepsilon, f_0 \in C_b(\mathbb{R}^N)$ $(\varepsilon > 0)$ are such that $\|f_\varepsilon\|_\infty \le M$ for any $\varepsilon > 0$ and $\lim_{\varepsilon\to 0^+} f_\varepsilon = f_0$ uniformly in $B(k)$ for any $k > 0$, then, for any $t > 0$, we have

$$\lim_{\varepsilon\to 0^+} T(t-\varepsilon)f_\varepsilon = T(t)f_0,$$

uniformly in $B(k)$ for any $k > 0$.

Proof. Throughout the proof, in order to simplify the notation, sometimes we do not explicitly write the dependence on x of the functions that we will consider. All the estimates that we write are meant pointwise in x.

Let g_n, g be as above and observe that

$$|T(t)f - T_n(t)f| \le (T(t) - T_n(t))\,|f|, \qquad t > 0. \tag{7.2.1}$$

To check (7.2.1), we recall that (see Theorem 2.2.5 and Proposition C.3.2)

$$(T_n(t)f)(x) = \int_{B(n)} G_n(t,x,y)f(y)dy, \qquad (T(t)f)(x) = \int_{\mathbb{R}^N} G(t,x,y)f(y)dy$$

for any $t > 0$ and any $x \in B(n)$, where G_n and G are positive, respectively, in $(0,+\infty) \times B(n) \times B(n)$ and in $(0,+\infty) \times \mathbb{R}^N \times \mathbb{R}^N$. Moreover $G_n \le G$ in

$(0,+\infty) \times B(n) \times B(n)$. Hence,

$$
\begin{aligned}
|(T(t)f)(x) - (T_n(t)f)(x)| &= \left| \int_{\mathbb{R}^N} \big(G(t,x,y) - G_n(t,x,y)\chi_{B(n)}\big) f(y)dy \right| \\
&\le \int_{\mathbb{R}^N} (G(t,x,y) - G_n(t,x,y)\chi_{B(n)})|f(y)|dy \\
&= \left((T(t) - T_n(t))|f|\right)(x),
\end{aligned}
$$

for any $t > 0$ and any $x \in B(n)$. Hence, taking (7.2.1) into account, we get

$$
\begin{aligned}
|T_n(t)g_n(t) - T(t)g(t)| &\le |T_n(t)(g_n(t) - g(t))| + |(T(t) - T_n(t))g(t)| \\
&\le T_n(t)\big(|g_n(t) - g(t)|\big) + (T(t) - T_n(t))|g(t)| \\
&\le T(t)\big(|g_n(t) - g(t)|\big) + M(T(t) - T_n(t))\mathbb{1}, \quad (7.2.2)
\end{aligned}
$$

for any $n \in \mathbb{N}$.

Let us observe that the second term in the last side of (7.2.2) converges to zero uniformly in $[\varepsilon, T] \times B(k)$ for any $\varepsilon, k > 0$, by Theorem 2.2.1. To estimate the first term we observe that, for any $t \in [\varepsilon, T-\varepsilon]$ and any $x \in \mathbb{R}^N$, we have

$$
\big(T(t)\big(|g_n(t) - g(t)|\big)\big)(x) \le \big(T(t)\big(\sup_{t\in[\varepsilon,T-\varepsilon]} |g_n(t) - g(t)|\big)\big)(x). \qquad (7.2.3)
$$

Since $\sup_{t\in[\varepsilon,T-\varepsilon]} |g_n(t) - g(t)|$ is a bounded and continuous function in $\mathbb{R}^N$, converging to 0, locally uniformly in $\mathbb{R}^N$, from Proposition 2.2.9 we deduce that the right-hand side of (7.2.3) converges to 0, locally uniformly, as n tends to $+\infty$, and the conclusion follows.

Now, we prove the second part of the assertion. Let f_ε, f_0 be as in the statement of the lemma; we have

$$
|T(t-\varepsilon)f_\varepsilon - T(t)f_0| \le |T(t-\varepsilon)f_\varepsilon - T(t-\varepsilon)f_0| + |T(t-\varepsilon)f_0 - T(t)f_0|, \quad (7.2.4)
$$

for any $\varepsilon > 0$. By Proposition 2.2.9, for any fixed $t > 0$, the first term in the right-hand side of (7.2.4) converges to 0 as ε tends to 0, uniformly in $[0,T] \times B(k)$ for any $T, k > 0$. The second term in the right-hand side of (7.2.4) converges to zero uniformly in $[0,T] \times B(k)$, for any $k > 0$, as well, since the function $T(t)f_0$ is continuous in $[0,+\infty) \times \mathbb{R}^N$ and, consequently, uniformly continuous in $[0,T] \times B(k)$ for any $T, k > 0$. ∎

We can now prove our estimates. For notational convenience, we set

$$
\hat{\omega}_k = \hat{\omega}_{k,p} = \begin{cases} 0, & \text{if Hypotheses 6.1.1 hold} \\ \omega_{k,p}, & \text{if Hypotheses 7.1.3 hold} \end{cases}, \qquad k = 1,2,3,
$$

where $\omega_{k,p}$ are defined in the proof of Theorem 7.1.5. Moreover, we denote by $M_k = M_{k,p}$ both the constants in (7.1.1) and in (7.1.12). Finally, we set

$\tilde{L}=L_2$ if Hypotheses 6.1.1 are satisfied, and $\tilde{L}=L$ if Hypotheses 7.1.3 are satisfied.

Theorem 7.2.2 *Let $k\in\{1,2,3\}$ and suppose that either Hypotheses 6.1.1(i)–6.1.1(iii) and 6.1.1(iv-k) or Hypotheses 7.1.3(i) and 7.1.3(ii-k) hold. Then, for any $f\in C_b^{k-1}(\mathbb{R}^N)$ we have*

$$
\begin{aligned}
&|\left(D^kT(t)f\right)(x)|^p\\
&\le\left(\frac{\hat{\omega}_{k,p\wedge 2}}{1-e^{-\hat{\omega}_{k,p\wedge 2}t}}\,\psi_{k,p\wedge 2}(t)\right)^{\max\{1,\frac{p}{2}\}}\left(T(t)\left(\sum_{j=0}^{k-1}|D^jf|^2\right)^{\frac{p}{2}}\right)(x),
\end{aligned}
\tag{7.2.5}
$$

for any $t>0$, any $x\in\mathbb{R}^N$ and any $p>\hat{p}$, where $\hat{p}=1$, under Hypotheses 6.1.1, and $\hat{p}=p_0$, under Hypothesis 7.1.3, whereas

$$\psi_{1,r}(t)=b_{r,1}t^{1-\frac{p}{2}}(1+t)^{\frac{p}{2}},\qquad t>0,$$

for any $r\in(1,2]$, and

$$\psi_{k,r}(t)=b_{r,k}\left(1+\frac{e^{\hat{\omega}_{k-1}t}-1}{\hat{\omega}_{k-1}}\right)^{\frac{r}{2}}\left(t+\frac{e^{\hat{\omega}_{k-1}t}-1}{\hat{\omega}_{k-1}}\right)^{1-\frac{r}{2}},$$

for any $t>0$, any $r\in(1,2]$ and any $k=1,2,3$, $b_{k,r}$ ($r\in(1,2]$, $k=2,3$) being positive constants that can be explicitly determined from the data (see (7.2.23) and (7.2.25)). When $\hat{\omega}_{k-1}=0$ we agree that $(1-e^{-\hat{\omega}_{k-1}t})/\hat{\omega}_{k-1}=t$. Finally, $a=1$, under Hypotheses 6.1.1, and $a=\alpha$ under Hypotheses 7.1.3.

Proof. We first consider the case when $k=3$ and $p\in(\hat{p},2)$. We fix δ,t, $n\in\mathbb{N}$ and let $\theta_n:\mathbb{R}^N\to\mathbb{R}$ be the cut-off function defined by $\vartheta_n(x)=\varrho(|x|/n)$, where $\varrho\in C^\infty([0,+\infty))$ is any nonincreasing function such that $\chi_{(0,1/2)}\le\varphi\le\chi_{(0,1)}$. For any $\alpha,\beta>0$ and any $f\in C_b^2(\mathbb{R}^N)$ we define the function $g_\delta:[0,t]\to C(B(n))$ by

$$
\begin{aligned}
&g_\delta(s)\\
&=T_n(s)\Big\{\left(\alpha|T_n(t-s)f|^2+\beta\vartheta_n^2|DT_n(t-s)f|^2+\vartheta_n^4|D^2T_n(t-s)f|^2+\delta\right)^{\frac{p}{2}}\\
&\qquad-\delta^{\frac{p}{2}}\Big\},
\end{aligned}
\tag{7.2.6}
$$

for any $0\le s\le t$, where $\{T_n(\cdot)\}$ is the semigroup generated in $C_b(\overline{B}(n))$ by the realization A_n of the operator $\mathcal{A}$ with homogeneous Dirichlet boundary conditions (see Section C). To simplify the notation, throughout the remainder of the proof, when there is no damage of confusion, we drop out the dependence of the functions considered on n. Moreover, we set $\varphi(r)=\varphi_n(r):=T_n(t-r)f$ for any $r\in[0,t]$.

As it is easily seen the function

$$\left(\alpha|\varphi(r)|^2+\beta\vartheta^2|D\varphi(r)|^2+\vartheta^4|D^2T\varphi(r)|^2+\delta\right)^{\frac{p}{2}}-\delta^{\frac{p}{2}}$$

belongs to $D(A_n)$ for any $r \in [0,t)$. Recalling that $\mathcal{A}(\delta^{p/2}) = 0$, we get

$$
\begin{aligned}
&g'_\delta(s)\\
&= pT_n(s)\Bigg[\Bigg(\alpha\sum_{i,j=1}^N q_{ij}D_i\varphi D_j\varphi + \beta\vartheta^2\sum_{i,j,h=1}^N q_{ij}D_{ih}\varphi D_{jh}\varphi\\
&\qquad +\vartheta^4\sum_{i,j,h,k=1}^N q_{ij}D_{ihk}\varphi D_{jhk}\varphi + \frac{\beta}{2}\mathcal{A}(\vartheta^2)|D\varphi|^2 + \frac{1}{2}\mathcal{A}(\vartheta^4)|D^2\varphi|^2\\
&\qquad +4\beta\,\vartheta\sum_{i,j,h=1}^N q_{ij}D_i\vartheta D_h\varphi D_{jh}\varphi + 8\vartheta^3\sum_{i,j,h,k=1}^N q_{ij}D_j\vartheta D_{hk}\varphi D_{ihk}\varphi\\
&\qquad -\beta\vartheta^2\sum_{i,j,h=1}^N D_hq_{ij}D_h\varphi D_{ij}\varphi - \beta\vartheta^2\sum_{j,h=1}^N D_hb_jD_j\varphi D_h\varphi\\
&\qquad -2\vartheta^4\sum_{i,j,h,k=1}^N D_hq_{ij}D_{hk}\varphi D_{ijk}\varphi - 2\vartheta^4\sum_{j,h,k=1}^N D_hb_jD_{jk}\varphi D_{hk}\varphi\\
&\qquad -\vartheta^4\sum_{i,j,h,k=1}^N D_{hk}q_{ij}D_{ij}\varphi D_{hk}\varphi - \vartheta^4\sum_{j,h,k=1}^N D_{hk}b_jD_j\varphi D_{hk}\varphi\Bigg)\\
&\qquad\qquad \times\left(\alpha|\varphi|^2 + \beta\vartheta^2|D\varphi|^2 + \vartheta^4|D^2\varphi|^2 + \delta\right)^{\frac{p}{2}-1}\Bigg]\\
&\quad -\frac{p(2-p)}{4}T_n(s)\Bigg[\sum_{i,j=1}^N q_{ij}D_i\left(\alpha\varphi^2 + \beta\vartheta^2|D\varphi|^2 + \vartheta^4|D^2\varphi|^2\right)\\
&\qquad\qquad \times D_j\left(\alpha\varphi^2 + \beta\vartheta^2|D\varphi|^2 + \vartheta^4|D^2\varphi|^2\right)\\
&\qquad\qquad \times\left(\alpha|\varphi|^2 + \beta\vartheta^2|D\varphi|^2 + \vartheta^4|D^2\varphi|^2 + \delta\right)^{\frac{p}{2}-2}\Bigg].
\end{aligned}
$$

Arguing as in the proof of (7.1.4) and observing that

$$(a+b)^2 \le (1+\varepsilon)a^2 + (1+\varepsilon^{-1})b^2$$

for any $a, b, \varepsilon > 0$, we easily deduce that

$$
\begin{aligned}
&\frac{1}{4}\sum_{i,j=1}^N q_{ij}D_i\left(\alpha\varphi^2 + \beta\vartheta^2|D\varphi|^2 + \vartheta^4|D^2\varphi|^2\right)\\
&\qquad\qquad \times D_j\left(\alpha\varphi^2 + \beta\vartheta^2|D\varphi|^2 + \vartheta^4|D^2\varphi|^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1+\varepsilon)\Bigg[\alpha|\varphi|\bigg(\sum_{i,j=1}^N q_{ij}D_i\varphi D_j\varphi\bigg)^{\frac12}\\
&\qquad +\beta\vartheta^2|D\varphi|\bigg(\sum_{i,j,h=1}^N q_{ij}D_{ih}\varphi D_{jh}\varphi\bigg)^{\frac12}\\
&\qquad +\vartheta^4|D^2\varphi|\bigg(\sum_{i,j,h,k=1}^N q_{ij}D_{ihk}\varphi D_{jhk}\varphi\bigg)^{\frac12}\Bigg]^2\\
&\quad +\left(1+\frac{1}{\varepsilon}\right)\left(\beta\vartheta|D\varphi|^2+2\vartheta^3|D^2\varphi|^2\right)^2\sum_{i,j=1}^N q_{ij}D_i\vartheta D_j\vartheta\\
&\leq (1+\varepsilon)\Bigg[\alpha\sum_{i,j=1}^N q_{ij}D_i\varphi D_j\varphi+\beta\vartheta^2\sum_{i,j,h=1}^N q_{ij}D_{ih}\varphi D_{jh}\varphi\\
&\qquad +\vartheta^4\sum_{i,j,h,k=1}^N q_{ij}D_{ihk}\varphi D_{jhk}\varphi\Bigg]\\
&\quad \times\left(\alpha|\varphi|^2+\beta\vartheta^2|D\varphi|^2+\vartheta^4|D^2\varphi|^2\right)\\
&\quad +\left(1+\frac{1}{\varepsilon}\right)\left(\beta|D\varphi|^2+4\vartheta^2|D^2\varphi|^2\right)\\
&\qquad\qquad \times\left(\beta\vartheta^2|D\varphi|^2+\vartheta^4|D^2\varphi|^2\right)\sum_{i,j=1}^N q_{ij}D_i\vartheta D_j\vartheta.
\end{aligned}
$$

Hence, taking $\varepsilon=(p-1)/(4-2p)$ we get

$$
\begin{aligned}
&g_\delta'(s)\\
&\geq pT_n(s)\Bigg\{\Bigg[\frac{p-1}{2}\alpha\sum_{i,j=1}^N q_{ij}D_i\varphi D_j\varphi+\frac{p-1}{2}\beta\vartheta^2\sum_{i,j,h=1}^N q_{ij}D_{ih}\varphi D_{jh}\varphi\\
&\qquad +\frac{p-1}{2}\vartheta^4\sum_{i,j,h,k=1}^N q_{ij}D_{ihk}\varphi D_{jhk}\varphi+\frac{\beta}{2}\mathcal{A}(\vartheta^2)|D\varphi|^2\\
&\qquad +\frac12\mathcal{A}(\vartheta^4)|D^2\varphi|^2+4\beta\vartheta\sum_{i,j,h=1}^N q_{ij}D_i\vartheta D_h\varphi D_{jh}\varphi\\
&\qquad +8\vartheta^3\sum_{i,j,h,k=1}^N q_{ij}D_j\vartheta D_{hk}\varphi D_{ihk}\varphi-\beta\vartheta^2\sum_{i,j,h=1}^N D_hq_{ij}D_h\varphi D_{ij}\varphi\\
&\qquad -\beta\vartheta^2\sum_{j,h=1}^N D_hb_jD_j\varphi D_h\varphi-2\vartheta^4\sum_{i,j,h,k=1}^N D_hq_{ij}D_{hk}\varphi D_{ijk}\varphi
\end{aligned}
$$

$$
\begin{aligned}
&-2\vartheta^4 \sum_{j,h,k=1}^{N} D_h b_j D_{jk}\varphi D_{hk}\varphi - \vartheta^4 \sum_{i,j,h,k=1}^{N} D_{hk} q_{ij} D_{ij}\varphi D_{hk}\varphi \\
&-\vartheta^4 \sum_{j,h,k=1}^{N} D_{hk} b_j D_j\varphi D_{hk}\varphi \\
&\left.-\frac{3-p}{p-1}(2-p)\left(\beta|D\varphi|^2 + 4\vartheta^2|D^2\varphi|^2\right) \sum_{i,j=1}^{N} q_{ij} D_i\vartheta D_j\vartheta\right] \\
&\qquad \left.\times \left(\alpha|\varphi|^2 + \beta\vartheta^2|D\varphi|^2 + \vartheta^4|D^2\varphi|^2 + \delta\right)^{\frac{p}{2}-1}\right\}. \qquad (7.2.7)
\end{aligned}
$$

Taking Hypothesis 6.1.1(iii) and (6.1.24) into account, one can easily check that

$$
\sum_{i,j=1}^{N} q_{ij}(x) D_i\vartheta(x) D_j\vartheta(x) \le C_1\kappa(x), \qquad (7.2.8)
$$

for any $x \in B(n)$ and some positive constant C_1, independent of n. Moreover, in view of (6.1.23)-(6.1.25) and (7.2.8) we can write

$$
\mathcal{A}(\vartheta^2) \ge -C_2\kappa, \qquad (7.2.9)
$$

$$
\mathcal{A}(\vartheta^4) = 2\vartheta^2\mathcal{A}(\vartheta^2) + 2\sum_{i,j=1}^{N} q_{ij} D_i\vartheta^2 D_j\vartheta^2 \ge 2\vartheta^2\mathcal{A}(\vartheta^2) \ge -2C_2\vartheta^2\kappa, \quad (7.2.10)
$$

$$
\left|\vartheta \sum_{i,j,h=1}^{N} q_{ij} D_i\vartheta D_h\varphi D_{jh}\varphi\right| \le \frac{C_3 N}{4\varepsilon_1}\kappa|D\varphi|^2 + \varepsilon_1 C_3\vartheta^2\kappa|D^2\varphi|^2, \qquad (7.2.11)
$$

$$
\left|\vartheta^3 \sum_{i,j,h,k=1}^{N} q_{ij} D_j\vartheta D_{hk}\varphi D_{ihk}\varphi\right| \le \frac{C_3 N}{4\varepsilon_2}\vartheta^2\kappa|D^2\varphi|^2 + \varepsilon_2 C_3\vartheta^4\kappa|D^3\varphi|^2, \qquad (7.2.12)
$$

for any $\varepsilon > 0$, where C_2, C_3 are two positive constants independent of n. Now, we observe that from Hypotheses 6.1.1, or Hypotheses 7.1.3, and (7.2.8), (7.2.9)-(7.2.12), we get

$$
\begin{aligned}
&g_\delta'(s)\\
&\geq pT_n(s)\Bigg\{\Bigg[\Bigg(\Bigg(\frac{p-1}{2}\alpha-\frac{C_2}{2}\beta-\beta\frac{C_3N}{\varepsilon_1}+\beta(p-2)C_1\frac{3-p}{p-1}\Bigg)\kappa-\beta^2\frac{CN^2}{4\varepsilon}\kappa^a\\
&\qquad -\vartheta^2\left(\beta d+r\frac{N^2}{4\varepsilon}\right)\Bigg)|D\varphi|^2\\
&\qquad +\Bigg(\Bigg(\beta\left(\frac{p-1}{2}-4\varepsilon_1C_3\right)-C_2-\frac{2C_3N}{\varepsilon_2}+4\frac{3-p}{p-1}(p-2)C_1\Bigg)\kappa\\
&\qquad\quad -\left(C\varepsilon N+\frac{CN^2}{2\varepsilon_3}+K_1^+\right)\kappa^a-\vartheta^2\left(2d+r\varepsilon N\right)\Bigg)\vartheta^2|D^2\varphi|^2\\
&\qquad +\left[\left(\frac{p-1}{2}-8\varepsilon_2C_3\right)\kappa-2C\varepsilon_3N\kappa^a\right]\vartheta^4|D^3\varphi|^2\Bigg]\\
&\qquad\qquad \times\left(\alpha|\varphi|^2+\beta\vartheta^2|D\varphi|^2+\vartheta^4|D^2\varphi|^2+\delta\right)^{\frac{p}{2}-1}\Bigg\},
\end{aligned}
$$

where $a=1$, under Hypotheses 6.1.1, and $a=\sigma$, under Hypotheses 7.1.3. We now choose $\varepsilon=2\tilde{L}/N$, $\varepsilon_1=2\varepsilon_2=(p-1)/(16C_3)$, $\varepsilon_3=(p-1)\kappa_0^{1-a}/(16CN)$ and $\beta\geq N^3/(8\tilde{L}^2)$. With this choice of $\varepsilon,\varepsilon_1,\varepsilon_2,\varepsilon_3$ and β we get

$$
\begin{aligned}
g_\delta'(s)\geq pT_n(s)\Bigg\{&\kappa\left(H_1(\beta,p)|D\varphi|^2+H_2(\beta,p)\vartheta^2|D^2\varphi|^2+\frac{p-1}{8}\kappa_0^{1-a}\vartheta^4|D^3\varphi|^2\right)\\
&\times(\alpha|\varphi|^2+\beta\vartheta^2|D\varphi|^2+\vartheta^4|D^2\varphi|^2+\delta)^{\frac{p}{2}-1}\Bigg\},
\end{aligned}
$$

with

$$
\begin{aligned}
H_1(\alpha,\beta,p):=&\ \frac{p-1}{2}\alpha-\frac{C_2}{2}\beta-16\frac{C_3^2N}{p-1}\beta+\frac{3-p}{p-1}(p-2)C_1\beta\\
&-\frac{CN^3}{8\tilde{L}}\kappa_0^{a-1}\beta^2-A\beta,
\end{aligned}\tag{7.2.13}
$$

$$
\begin{aligned}
H_2(\beta,p):=&\ \frac{p-1}{4}\beta-C_2-\frac{64C_3^2N}{p-1}+4\frac{3-p}{p-1}(p-2)C_1\\
&-\left(\frac{8C^2N^3}{p-1}\kappa_0^{a-1}+K_1^++2C\tilde{L}\right)\kappa_0^{a-1}-2A,
\end{aligned}\tag{7.2.14}
$$

where $A=(L_3)^+$, under Hypotheses 6.1.1, and $A=0$ otherwise.

It is immediate to notice that, both in the case when $a=1$ and $a=\sigma$, we can fix α,β sufficiently large so that $H_1(\alpha,\beta,p)$ and $H_2(\beta,p)$ are both positive and $\beta\geq N^3/(8\tilde{L}^2)$. Therefore, for such α's and β's we get

$$
\begin{aligned}
g_\delta'(s)\geq c_pT_n(s)\Bigg(&\vartheta^4\left(|D\varphi|^2+|D^2\varphi|^2+|D^3\varphi|^2\right)\\
&\times\left(\alpha\,\varphi^2+\beta\,\vartheta^2|D\varphi|^2+\vartheta^4|D^2\varphi|^2+\delta\right)^{\frac{p}{2}-1}\Bigg),
\end{aligned}\tag{7.2.15}
$$

where

$$c_p = c_p(\alpha,\beta) := p\kappa_0 \min\left\{H_1(\alpha,\beta,p), H_2(\beta,p), \frac{p-1}{8}\kappa_0^{1-a}\right\}. \tag{7.2.16}$$

Then, integrating (7.2.15) in $[\varepsilon, t-\varepsilon]$ ($\varepsilon > 0$), and recalling that $\{T_n(t)\}$ is a positive semigroup, we get

$$\begin{aligned}
c_p \int_\varepsilon^{t-\varepsilon} \Big(T_n(s)\Big(&\vartheta^4(|D\varphi|^2 + |D^2\varphi|^2 + |D^3\varphi|^2) \\
&\times(\alpha\,\varphi^2 + \beta\,\vartheta^2|D\varphi|^2 + \vartheta^4|D^2\varphi|^2 + \delta)^{\frac{p}{2}-1}\Big)\Big)(x)\,ds \\
\le \big(T_n(t-\varepsilon)\big((\alpha|T_n(\varepsilon)f|^2 &+ \beta\,\vartheta^2|DT_n(\varepsilon)f|^2 + \vartheta^4|D^2T_n(\varepsilon)f|^2 + \delta)^{\frac{p}{2}} - \delta^{\frac{p}{2}}\big)\big)(x).
\end{aligned} \tag{7.2.17}$$

Now, we observe that the same arguments as in Remark 6.1.5 show that $D^\gamma\varphi_n$ converges to $D^\gamma T(t-\cdot)f$ uniformly in $[\varepsilon, t-\varepsilon]\times B(k)$, as n tends to $+\infty$, for any $\varepsilon \in (0, t/2)$, any $k \in \mathbb{N}$ and any $|\gamma| \le 3$. Therefore, applying Lemma 7.2.1 we get

$$\begin{aligned}
\lim_{n\to\infty} \big(T_n(s)\big(&\vartheta_n^4(|D\varphi_n(s)|^2 + |D^2\varphi_n(s)|^2 + |D^3\varphi_n(s)|^2) \\
&\times\big(\alpha|D\varphi_n(s)|^2 + \beta\vartheta_n^2|D\varphi_n(s)|^2 + \vartheta_n^4|D^2\varphi_n(s)|^2 + \delta\big)^{\frac{p}{2}-1}\big)\big)(x) \\
= \big(T(s)\big((&|DT(t-s)f|^2 + |D^2T(t-s)f|^2 + |D^3T(t-s)f|^2) \\
&\times\big(\alpha|DT(t-s)f|^2 + \beta|DT(t-s)f|^2 \\
&+|D^2T(t-s)f|^2 + \delta\big)^{\frac{p}{2}-1}\big)\big)(x),
\end{aligned}$$

the convergence being uniform in $[\varepsilon, t-\varepsilon]\times B(k)$ for any $k > 0$, and

$$\begin{aligned}
&\lim_{n\to+\infty} (T_n(t-\varepsilon)((\alpha|T_n(\varepsilon)f|^2 + \beta\vartheta_n^2|DT_n(\varepsilon)f|^2 + \vartheta_n^4|D^2T_n(\varepsilon)f|^2)^{\frac{p}{2}}))(x) \\
&= (T(t-\varepsilon)((\alpha|T(\varepsilon)f|^2 + \beta|DT(\varepsilon)f|^2 + |D^2T(\varepsilon)f|^2)^{\frac{p}{2}}))(x),
\end{aligned}$$

uniformly for $s \in [\varepsilon, t-\varepsilon]$. Then, from (7.2.17) we get

$$\begin{aligned}
c_p \int_\varepsilon^{t-\varepsilon} \Big(T(s)\Big((&|DT(t-s)f|^2 + |D^2T(t-s)f|^2 + |D^3T(t-s)f|^2) \\
&\times\big(\alpha|T(t-s)f|^2 + \beta|DT(t-s)f|^2 + |D^2T(t-s)f|^2 + \delta)^{\frac{p}{2}-1}\Big)\Big)(x)\,ds \\
\le \Big(T(t-\varepsilon)\Big(&\alpha|T(\varepsilon)f|^2 + \beta|DT(\varepsilon)f|^2 + |D^2T(\varepsilon)f|^2)^{\frac{p}{2}}\Big)\Big)(x),
\end{aligned} \tag{7.2.18}$$

for any $x \in \mathbb{R}^N$. Now, from Proposition 7.1.1 it follows that

$$\lim_{\varepsilon\to 0^+} D^jT(\varepsilon)f = D^jf, \qquad j = 0,1,2,$$

uniformly in $B(k)$ for any $k>0$. Therefore, from Lemma 7.2.1 we have

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\Big(T(t-\varepsilon)\Big((\alpha|T(\varepsilon)f|^2+\beta|DT(\varepsilon)f|^2+|D^2T(\varepsilon)f|^2)^{\frac{p}{2}}\Big)\Big)(x)\\&=\big(T(t)\big((\alpha|f|^2+\beta|Df|^2+|D^2f|^2)^{\frac{p}{2}}\big)\big)(x),\end{aligned}\tag{7.2.19}$$

for any $x\in\mathbb{R}^N$.

As far as the first integral term in (7.2.18) is concerned, we observe that, by virtue of Theorem 7.1.2, for any $x\in\mathbb{R}^N$ the function

$$s\mapsto\Big(T(s)\Big(\big(|DT(t-s)f|^2+|D^2T(t-s)f|^2+|D^3T(t-s)f|^2\big)^{\frac{p}{2}}\Big)\Big)(x)$$

behaves as $\widehat{C}(t-s)^{-p/2}$ for some positive constant $\widehat{C}$, independent of x. Therefore, the dominated convergence theorem and (7.2.19) imply that we can take the limit as ε tends to 0^+ in (7.2.18), getting

$$\begin{aligned}&c_p\int_0^t\Big(T(s)\Big(\big(|DT(t-s)f|^2+|D^2T(t-s)f|^2+|D^3T(t-s)f|^2\big)\\&\qquad\times\big(\alpha|T(t-s)f|^2+\beta|DT(t-s)f|^2+|D^2T(t-s)f|^2+\delta\big)^{\frac{p}{2}-1}\Big)\Big)(x)ds\\&\le\Big(T(t)\Big((\alpha f^2+\beta|Df|^2+|D^2f|^2)^{\frac{p}{2}}\Big)\Big)(x),\end{aligned}\tag{7.2.20}$$

for any $x\in\mathbb{R}^N$.

For notational convenience we now set $\varphi=T(t-s)f$. Let us observe that, from (7.1.1) or (7.1.12) (with $k=3$), from the Young inequality and the inequality

$$T(t)(fg)\le(T(t)(f^q))^{\frac{1}{q}}(T(t)(g^r))^{\frac{1}{r}},$$

holding for any positive $f,g\in C_b(\mathbb{R}^N)$ and any $r,q>0$ such that $1/r+1/q=1$ (see Theorem 2.2.5), we deduce that, for any $\gamma\in\mathbb{R}$,

$$\begin{aligned}&\big(|DT(t)f|^2+|D^2T(t)f|^2+|D^3T(t)f|^2\big)^{\frac{p}{2}}\\&=\big(|DT(s)(T(t-s)f)|^2+|D^2T(s)(T(t-s)f)|^2+|D^3T(s)(T(t-s)f)|^2\big)^{\frac{p}{2}}\\&\le M_3e^{\hat\omega_3 s}T(s)\left[\left(\frac{a-\sigma}{1-\sigma}|\varphi|^2+|D\varphi|^2+|D^2\varphi|^2+|D^3\varphi|^2\right)^{\frac{p}{2}}\right]\\&\le M_3e^{\hat\omega_3 s}T(s)\bigg[\left(\frac{a-\sigma}{1-\sigma}|\varphi|^2+|D\varphi|^2+|D^2\varphi|^2+|D^3\varphi|^2\right)^{\frac{p}{2}}\\&\qquad\times\big(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\big)^{-\gamma}\\&\qquad\qquad\times\big(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\big)^{\gamma}\bigg]\end{aligned}$$

$$
\begin{aligned}
&\le M_3 e^{\hat\omega_3 s}\left\{T(s)\left(\left(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\right)^{\frac{2\gamma}{2-p}}\right)\right\}^{1-\frac{p}{2}}\\
&\qquad\times\left\{T(s)\left[\left(\frac{a-\sigma}{1-\sigma}|\varphi|^2+|D\varphi|^2+|D^2\varphi|^2+|D^3\varphi|^2\right)\right.\right.\\
&\qquad\qquad\left.\left.\times\left(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\right)^{-\frac{2\gamma}{p}}\right]\right\}^{\frac{p}{2}}.
\end{aligned}
$$

Choosing $\gamma = p(2-p)/4$, using the Jensen and Young inequalities, we get

$$
\begin{aligned}
&\left(|DT(t)f|^2+|D^2T(t)f|^2+|D^3T(t)f|^2\right)^{\frac{p}{2}}\\
&\le M_3 e^{\hat\omega_3 s}\left\{T(s)\left(\left(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\right)^{\frac{p}{2}}\right)\right\}^{1-\frac{p}{2}}\\
&\qquad\times\left\{T(s)\left[\left(\frac{a-\sigma}{1-\sigma}|\varphi|^2+|D\varphi|^2+|D^2\varphi|^2+|D^3\varphi|^2\right)\right.\right.\\
&\qquad\qquad\left.\left.\times\left(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\right)^{\frac{p}{2}-1}\right]\right\}^{\frac{p}{2}}\\
&\le M_3 e^{\hat\omega_3 s}\left\{\frac{p}{2}\varepsilon^{\frac{2}{p}}T(s)\left[\left(\frac{a-\sigma}{1-\sigma}|\varphi|^2+|D\varphi|^2+|D^2\varphi|^2+|D^3\varphi|^2\right)\right.\right.\\
&\qquad\qquad\left.\times\left(\alpha\varphi^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\right)^{\frac{p}{2}-1}\right]\\
&\qquad\left.+\left(1-\frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}T(s)\left[\left(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\right)^{\frac{p}{2}}\right]\right\}\\
&\le M_3 e^{\hat\omega_3 s}\left\{\frac{p}{2}\varepsilon^{\frac{2}{p}}T(s)\left[\left(|D\varphi|^2+|D^2\varphi|^2+|D^3\varphi|^2\right)\right.\right.\\
&\qquad\qquad\left.\times\left(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\right)^{\frac{p}{2}-1}\right]\\
&\qquad+\left(1-\frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}T(s)\left[\left(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\right)^{\frac{p}{2}}\right]\\
&\qquad\left.+\frac{p}{2}\varepsilon^{\frac{2}{p}}\frac{a-\sigma}{1-\sigma}\frac{1}{\alpha}T(s)\left[\left(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^3\varphi|^2+\delta\right)^{\frac{p}{2}}\right]\right\}.
\end{aligned}
\tag{7.2.21}
$$

Hence, recalling that

$$
\begin{aligned}
&(|\varphi|^2+|D\varphi|^2+|D^2\varphi|^2+\delta)^{\frac{p}{2}}\\
&\le M_2 e^{\hat\omega_2(t-s)}T(t-s)\left((f^2+|Df|^2+|D^2f|^2+\delta)^{\frac{p}{2}}\right)+\delta^{\frac{p}{2}}\\
&\quad+\frac{1-a}{1-\sigma}T(t-s)(|f|^p),
\end{aligned}
$$

(see (7.1.8) and (7.1.17)), from (7.2.21) we get, for any $\varepsilon > 0$,

$$
\begin{aligned}
&\big(|DT(t)f|^2+|D^2T(t)f|^2+|D^3T(t)f|^2\big)^{\frac{p}{2}}\\
&\le M_3 e^{\hat\omega_3 s}\bigg\{\frac{p}{2}\varepsilon^{\frac{2}{p}}T(s)\bigg[\,\big(|D\varphi|^2+|D^2\varphi|^2+|D^3\varphi|^2\big)\\
&\qquad\qquad\times\big(\alpha|\varphi|^2+\beta|D\varphi|^2+|D^2\varphi|^2+\delta\big)^{\frac{p}{2}-1}\bigg]\\
&\quad+\Big(1-\frac{p}{2}\Big)\max(1,\alpha,\beta)^{\frac{p}{2}}M_2\varepsilon^{\frac{2}{p-2}}e^{\hat\omega_2(t-s)}\\
&\qquad\times T(t)\Big[\big(|f|^2+|Df|^2+|D^2f|^2+\delta\big)^{\frac{p}{2}}\Big]\\
&\quad+\Big(1-\frac{p}{2}\Big)\max(1,\alpha,\beta)^{\frac{p}{2}}\varepsilon^{\frac{2}{p-2}}\delta^{\frac{p}{2}}\\
&\quad+\Big(1-\frac{p}{2}\Big)\max(1,\alpha,\beta)^{\frac{p}{2}}\frac{1-a}{1-\sigma}\varepsilon^{\frac{2}{p-2}}T(t)(|f|^p)\\
&\quad+\frac{p}{2}\varepsilon^{\frac{2}{p}}\frac{a-\sigma}{1-\sigma}\frac{1}{\alpha}\max(1,\alpha,\beta)^{\frac{p}{2}}M_2e^{\hat\omega_2(t-s)}\\
&\qquad\times T(t)\Big[\big(f^2+|Df|^2+|D^2f|^2+\delta\big)^{\frac{p}{2}}\Big]\\
&\quad+\frac{p}{2}\varepsilon^{\frac{2}{p}}\frac{a-\sigma}{1-\sigma}\frac{1}{\alpha}\delta^{\frac{p}{2}}\max(1,\alpha,\beta)^{\frac{p}{2}}\bigg\}
\end{aligned}
\tag{7.2.22}
$$

Now, we multiply the first and the last side of (7.2.22) by $e^{-\hat\omega_3 s}$, integrate in $(0,t)$, pointwise with respect to $x\in\mathbb{R}^N$, taking (7.2.20) into account and taking $\alpha\ge 1$. Finally, we let δ tend to 0^+. We get

$$
\begin{aligned}
&\frac{1-e^{-\hat\omega_3 t}}{\hat\omega_3}\big(|DT(t)f|^2+|D^2T(t)f|^2+|D^3T(t)f|^2\big)^{\frac{p}{2}}\\
&\le M_3\max(\alpha,\beta)^{\frac{p}{2}}\bigg\{\frac{p}{2}\bigg(\frac{1}{c_p}+\frac{a-\sigma}{1-\sigma}M_2\frac{e^{\hat\omega_2 t}-1}{\hat\omega_2}\bigg)\varepsilon^{\frac{2}{p}}\\
&\qquad+\Big(1-\frac{p}{2}\Big)\bigg(M_2\frac{e^{\hat\omega_2 t}-1}{\hat\omega_2}+\frac{1-a}{1-\sigma}t\bigg)\varepsilon^{\frac{2}{p-2}}\bigg\}\\
&\quad\times T(t)\big((f^2+|Df|^2+|D^2f|^2)^{\frac{p}{2}}\big)
\end{aligned}
$$

Minimizing with respect to $\varepsilon>0$ gives

$$
\begin{aligned}
&\frac{1-e^{-\hat\omega_3 t}}{\hat\omega_3}\big(|DT(t)f|^2+|D^2T(t)f|^2+|D^3T(t)f|^2\big)^{\frac{p}{2}}\\
&\le M_3\max\{\alpha,\beta\}^{\frac{p}{2}}\bigg(\frac{1}{c_p}+\frac{a-\sigma}{1-\sigma}M_2\frac{e^{\hat\omega_2 t}-1}{\hat\omega_2}\bigg)^{\frac{p}{2}}\bigg(M_2\frac{e^{\hat\omega_2 t}-1}{\hat\omega_2}+\frac{1-a}{1-\sigma}t\bigg)^{1-\frac{p}{2}}\\
&\qquad\times T(t)\big((f^2+|Df|^2+|D^2f|^2)^{\frac{p}{2}}\big).
\end{aligned}
$$

Now, (7.2.5) with $p \in (1,2)$ follows taking

$$\begin{aligned} b_{3,p} = M_3 \max\left\{M_2, \frac{1-a}{1-\sigma}\right\}^{1-\frac{p}{2}} \\ \times \min\left\{\max(\alpha,\beta)^{\frac{p}{2}} \max\left(\frac{1}{c_p(\alpha,\beta)}, \frac{a-\sigma}{1-\sigma}M_2\right)^{\frac{p}{2}} :\right. \\ \left. \alpha \geq 1,\ H_1(\alpha,\beta,p), H_2(\beta,p) > 0\right\}, \end{aligned} \tag{7.2.23}$$

where $H_1(\alpha,\beta,p)$ and $H_2(\beta,p)$ are given by (7.2.13) and (7.2.14). Note that the previous minimum exists since the function

$$(\alpha,\beta) \mapsto \max\{\alpha,\beta\}(c_p(\alpha,\beta))^{-\frac{p}{2}}$$

with domain $D = \{(\alpha,\beta) \in \mathbb{R}^2 : H_j(\alpha,\beta) > 0,\ j = 1,2\}$ tends to $+\infty$ both as $\|(\alpha,\beta)\|$ tends to $+\infty$ and as (α,β) tends to any point on ∂D.

The case when $p = 2$ is similar and even simpler. It suffices to apply the previous arguments to the function g_0 defined by (7.2.6) (with $\delta = 0$ and $p = 2$).

In the case when $p > 2$, (7.2.5) (with $k = 3$) can be obtained from the case when $p = 2$ observing that

$$\begin{aligned} |(D^3T(t)f)(x)|^p &= \left(|(D^3T(t)f)(x)|^2\right)^{\frac{p}{2}} \\ &\leq \left(\frac{\hat{\omega}_{3,2}}{1-e^{-\hat{\omega}_{3,2}t}}\psi_{3,2}(t)\right)^{\frac{p}{2}} \\ &\quad \times \left(\left(T(t)\left(f^2 + |Df|^2 + |D^2f|^2\right)\right)(x)\right)^{\frac{p}{2}} \end{aligned}$$

and, then, applying the Hölder inequality.

The proof of (7.2.5) in the case when $k = 1,2$ is completely similar. In the case when $k = 2$ one can apply the previous arguments to the function g_δ in (7.2.6), obtained disregarding the terms whose coefficients do not depend on α and β and setting $\beta = 1$. After some computations one finds that g_δ' is still given by (7.2.7) with the obvious changes induced by the changes in the definition of the function g_δ. Therefore, taking $\varepsilon = (p-1)\kappa_0^{1-a}/(8CN)$ and $\varepsilon_1 = (p-1)/(16C_3)$, one gets

$$\begin{aligned} g_\delta'(s) \geq pT_n(s)\Bigg\{ \left(H_3(\alpha,p)\kappa|D\varphi|^2 + \frac{p-1}{8}\kappa_0^{1-a}\kappa^a\vartheta^2|D^2\varphi|^2\right) \\ \times(\alpha|\varphi|^2 + \vartheta^2|D\varphi|^2 + \delta)^{\frac{p}{2}-1}\Bigg\}, \end{aligned}$$

where

$$H_3(\alpha,p) = \frac{p-1}{2}\alpha - \frac{C_2}{2} - \frac{16C_3^2N}{p-1} + (p-2)C_1\frac{3-p}{p-1} - \frac{2C^2N^3}{p-1}\kappa_0^{2a-2} - A,$$

where, again, $A = (L_3)^+$, under Hypotheses 6.1.1, and $A = 0$ otherwise. Repeating the same arguments as above, one can now prove that

$$\begin{aligned}&\left(\sum_{j=1}^{2}|(D^jT(t)f)(x)|^2\right)^{\frac{p}{2}}\\ &\le\left(\frac{\omega_{2,p\wedge 2}}{1-e^{-\omega_{2,p\wedge 2}t}}\psi_{2,p\wedge 2}(t)\right)^{\max\{1,\frac{p}{2}\}}\left(T(t)\left(\sum_{j=0}^{1}|D^jf|^2\right)^{\frac{p}{2}}\right)(x),\end{aligned}\tag{7.2.24}$$

for any $t>0$, any $x\in\mathbb{R}^N$ and any $p\in(1,+\infty)$, where $\psi_{2,p\wedge 2}$ is as in the statement of the theorem with

$$\begin{aligned}b_{2,p}={}&M_2\max\left(M_1,\frac{1-a}{1-\sigma}\right)^{1-\frac{p}{2}}\\ &\times\min\left\{\alpha^{p/2},\max\left(\frac{1}{c_p'(\alpha)},\frac{a-\sigma}{1-\sigma}M_1\right)^{\frac{p}{2}}:\alpha\ge 1,\ H_3(\alpha,p)>0\right\}.\end{aligned}\tag{7.2.25}$$

Here,

$$c_p'(\alpha)=c_p'(\alpha,p):=p\kappa_0\min\left\{H_3(\alpha,p),\frac{p-1}{8}\right\}.$$

In the case when $k=1$, the previous arguments actually show that

$$b_{1,p}=M_1\max\left(\frac{1}{p(p-1)},\frac{a-\sigma}{1-\sigma}\right)^{\frac{p}{2}}$$

and

$$|(DT(t)f)(x)|^p\le\left(\frac{\omega_{1,p\wedge 2}}{1-e^{-\omega_{1,p\wedge 2}t}}\psi_{1,p\wedge 2}(t)\right)^{\max\{1,\frac{p}{2}\}}(T(t)(|f|^p))(x),\tag{7.2.26}$$

for any $t>0$ and any $x\in\mathbb{R}^N$. ■

Corollary 7.2.3 *Let $k=2,3$. Moreover, let Hypotheses 6.1.1(i)–6.1.1(iii) and 6.1.1(iv-k), or Hypotheses 7.1.3(i) and 7.1.3(ii-k) hold. Then, for any $f\in C_b(\mathbb{R}^N)$ and any $p\in(\hat{p},+\infty)$ (see Theorem 7.2.2), we have*

$$\begin{aligned}|(D^kT(t)f)(x)|^p\le{}&\left(\sum_{j=0}^{k-1}k^j\prod_{h=0}^{j}\frac{\widetilde{\omega}_{k-h,p\wedge 2}}{1-e^{-\widetilde{\omega}_{k-h,p\wedge 2}t}}\psi_{k-h,p\wedge 2}(t/k)\right)^{\max\{1,\frac{p}{2}\}}\\ &\times(T(t)(|f|^p))(x),\end{aligned}\tag{7.2.27}$$

for any $t>0$ and any $x\in\mathbb{R}^N$, where $\widetilde{\omega}_{k,r}=\frac{1}{k}\hat{\omega}_{j,r}$, and $\psi_{j,r}$ and $\hat{\omega}_{j,r}$ are as in Theorem 7.2.2.

Proof. We begin by proving (7.2.27) in the case when $k = 3$ and $p \le 2$. We fix $f \in C_b(\mathbb{R}^N)$ and $t > 0$. Applying (7.2.5) to the function $T(2t/3)f$ we get

$$\begin{aligned} &|(D^3T(t)f)(x)|^p \\ &\le \frac{\hat{\omega}_{3,p}}{1-e^{-\hat{\omega}_{3,p}t/3}}\,\psi_{3,p}(t/3) \\ &\qquad \times T(t/3)\left\{\left(|T(2t/3)f|^2+|DT(2t/3)f|^2+|D^2T(2t/3)f|^2\right)^{\frac{p}{2}}\right\}(x), \end{aligned} \tag{7.2.28}$$

for any $x \in \mathbb{R}^N$. Now, taking (7.2.24) into account, with $T(t)f$ replaced with $T(2t/3)f$, we get

$$\begin{aligned} &\left(|T(2t/3)f|^2+|DT(2t/3)f|^2+|D^2T(2t/3)f|^2\right)^{\frac{p}{2}} \\ \le&\frac{\hat{\omega}_{2,p}}{1-e^{-\hat{\omega}_{2,p}t/3}}\,\psi_{2,p}(t/3)T(t/3)\left\{\left(|T(t/3)f|^2+|DT(t/3)f|^2\right)^{\frac{p}{2}}\right\} \\ &+T(2t/3)(|f|^p) \\ \le&\frac{\hat{\omega}_{2,p}}{1-e^{-\hat{\omega}_{2,p}t/3}}\psi_{2,p}(t/3)\left[\frac{\hat{\omega}_{1,p}}{1-e^{-\hat{\omega}_{1,p}t/3}}\,\psi_{1,p}(t/3)+1\right]T(2t/3)(|f|^p) \\ &+T(2t/3)(|f|^p). \end{aligned} \tag{7.2.29}$$

Combining (7.2.28) and (7.2.29) gives (7.2.27) for $p \le 2$. If $p > 2$ and $k = 3$, (7.2.27) follows from the case when $p = 2$ applying the Hölder inequality. The proof of (7.2.27) (with $k = 2$) is similar: we write

$$|(D^2T(t)f)(x)|^2 = |(D^2T(t/2)T(t/2)f)(x)|^2$$

and apply (7.2.5) to $T(t/2)f$ with, respectively, $k = 1$ and $k = 2$. We omit the details. ■

Remark 7.2.4 The estimates in (7.2.5) (with $k = 1$) and in Corollary 7.2.3 allow us to improve the uniform estimates for the derivatives of $T(t)$ in Theorem 6.1.7. Indeed, they show that under Hypotheses 6.1.1(i)–6.1.1(iii) and 6.1.1(ii-k) ($k = 1, 2, 3$), the sup-norm of $D^kT(t)f$ stays bounded as t tends to $+\infty$, whereas it decreases exponentially to 0 as t tends to $+\infty$ if Hypotheses 7.1.3(i) and 7.1.3(ii-k) are satisfied.

The estimate (7.2.5) (with $k = 1$) can also be used to prove a Liouville type theorem. Namely, under Hypotheses 7.1.3(i) and 7.1.3(ii-1) such an estimate allows us to prove that, if $u \in C_b(\mathbb{R}^N)$ satisfies the equation $\mathcal{A}u = 0$, then it is constant. As it has already been stressed in the introduction, such a result fails in general. We refer the reader to [128] for further details.

Theorem 7.2.5 *Suppose that Hypotheses* 7.1.3(*i*) *and* 7.1.3(*ii*-1) *hold. If* $f \in C_b(\mathbb{R}^N)$ *satisfies* $\mathcal{A}f = 0$, *then* u *is constant.*

Proof. We observe that if $f \in C_b(\mathbb{R}^N)$ is such that $\mathcal{A}f = 0$, then, by local elliptic regularity, $u \in D_{\max}(\mathcal{A})$ (see (2.0.1)). Moreover, for any $g \in D_{\max}(\mathcal{A})$ and any $x \in \mathbb{R}^N$, the function $t \mapsto (T(t)g)(x)$ is continuously differentiable in $[0, +\infty)$ and $(D_tT(t)g)(x) = (\mathcal{A}T(t)g)(x) = (T(t)\mathcal{A}g)(x)$ (see Lemma 2.3.3, Propositions 2.3.5, 2.3.6 and 4.1.10). Hence $D_tT(t)f = 0$ for any $t > 0$, so that $T(t)f \equiv f$ for any $t > 0$. From (7.2.5) we deduce that

$$|(DT(t)f)(x)| \le \frac{\widetilde{C}e^{-\omega t}}{\sqrt{t}}\left((T(t)f^2)(x)\right)^{\frac{1}{2}} \le \frac{\widetilde{C}e^{-\omega t}}{\sqrt{t}}\|f\|_\infty, \qquad t > 0, \ x \in \mathbb{R}^N, \tag{7.2.30}$$

for some $\widetilde{C}, \omega > 0$. Letting t go to $+\infty$ in (7.2.30), we get

$$Df(x) = \lim_{t \to +\infty} (DT(t)f)(x) = 0, \qquad x \in \mathbb{R}^N,$$

so that f is constant. ∎

7.3 Further estimates for $\mathcal{A} = \Delta + \sum_{j=1}^N b_j(x)D_j$

Now, we consider the particular case when $q_{ij} \equiv \delta_{ij}$ $(i, j = 1, \ldots, N)$. In this case we show that, under Hypotheses 7.1.3, the gradient estimate (7.1.1) can be proved also for $p = 1$.

Theorem 7.3.1 *Fix $k = 1, 2, 3$ and suppose that $q_{ij} \equiv \delta_{ij}$ $(i, j = 1, \ldots, N)$. Moreover, let Hypotheses* 7.1.3(*i*), 7.1.3(*ii-k*) *hold with the conditions* (7.1.10) *and* (7.1.11) *being replaced with the following one:*

$$d(x) + L_k r(x) \le K, \qquad x \in \mathbb{R}^N, \tag{7.3.1}$$

to be satisfied for some constants $L_k > 0$ and $K \in \mathbb{R}$. Then,

$$|(D^kT(t)f)(x)| \le M_k e^{\omega_k t}\left(T(t)\left(\sum_{j=1}^k |D^j f|^2\right)^{\frac{1}{2}}\right)(x), \qquad t > 0, \ x \in \mathbb{R}^N, \tag{7.3.2}$$

where $M_k > 0$ and ω_k are constants which can be explicitly determined (*see the proof*)*. In particular, $\omega_k < 0$ $(k = 2, 3)$ if $K < 0$.*

Finally, if $k = 1$, assume that the coefficient b_j belongs to $C^{1+\delta}_{\rm loc}(\mathbb{R}^N)$ for some $\delta \in (0, 1)$ and any $j = 1, \ldots, N$, and (7.1.9) *is satisfied with the function d being replaced with a real constant d_0. Then,*

$$|(DT(t)f)(x)| \le e^{d_0 t}(T(t)|Df|)(x), \qquad t > 0, \ x \in \mathbb{R}^N. \tag{7.3.3}$$

Proof. We limit ourselves to sketching the proof of (7.3.2) in the case when $k = 3$, since the proof of (7.3.2) (with $k = 2$) and (7.3.3) are similar and even simpler. The proof that we provide is similar to that of Theorem 7.1.2. For the reader's convenience we go into details.

Let $\delta > 0$, $f \in C_b^1(\mathbb{R}^N)$ and consider the function $w_\delta : (0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$w_\delta(t,x) = \left(\alpha|Du(t,x)|^2 + \beta|D^2u(t,x)|^2 + |D^3u(t,x)|^2 + \delta\right)^{\frac{1}{2}},$$

where, as usual, $u = T(\cdot)f$, and α, β are positive constants to be fixed later on. Arguing as in the proof of Theorem 7.1.2 and taking the Cauchy-Schwarz inequality into account, we can easily show that w_δ turns out to solve the Cauchy problem

$$\begin{cases} D_t w_\delta(t,x) = \mathcal{A}w_\delta(t,x) + g_\delta(t,x), & t > 0,\ x \in \mathbb{R}^N, \\ w_\delta(0,x) = \left(\alpha|Df(x)|^2 + \beta|D^2f(x)|^2 + |D^3f(x)|^2 + \delta\right)^{\frac{1}{2}}, & x \in \mathbb{R}^N, \end{cases}$$

where (see (7.1.6))

$$\begin{aligned} g_\delta \le w_\delta^{-1}\Bigg(\alpha \sum_{j,h=1}^N D_hb_jD_juD_hu &+ 2\beta \sum_{j,h,k=1}^N D_hb_jD_{jk}uD_{hk}u \\ &+3 \sum_{j,h,k,l=1}^N D_hb_jD_{jkl}uD_{hkl}u + \beta \sum_{j,h,k=1}^N D_{hk}b_jD_juD_{hk}u \\ &+3 \sum_{j,h,k,l=1}^N D_{hk}b_jD_{jl}uD_{hkl}u + \sum_{j,h,k,l=1}^N D_{hkl}b_jD_juD_{hkl}u\Bigg). \end{aligned}$$

From Hypotheses 7.1.3(ii-3) and (7.3.1), and arguing as in the proof of (7.1.13), we get for any $\varepsilon > 0$

$$\begin{aligned} g_\delta \le w_\delta^{-1}\Bigg\{ &\left(\alpha d + \frac{N^2}{4\varepsilon}\beta r + \frac{N^3}{4\varepsilon_1}r\right)|Du|^2 + \left(\beta(2d + \varepsilon Nr) + \frac{3N^2}{4\varepsilon_1}r\right)|D^2u|^2 \\ &+(3d + 4\varepsilon_1 rN)|D^3u|^2\Bigg\}. \end{aligned}$$

We choose $\varepsilon = 4\varepsilon_1/3 = L_3/N$ and, then, α and β such that

$$\beta > \frac{N^3}{L_3^2}, \qquad \alpha > \frac{N^3}{24L_3^2}(3\beta + 8N).$$

With these choices of $\varepsilon, \varepsilon_1, \alpha, \beta$ we get

$$\begin{aligned} g_\delta &\le \left(\alpha|Du|^2 + \beta|D^2u|^2 + |D^3u|^2 + \delta\right)^{-\frac{1}{2}} \\ &\qquad\times(\alpha K|Du|^2 + 2K|D^2u|^2 + K|D^3u|^2) \\ &\le \max\{K, 2K/\beta\}w_\delta - K^-\delta^{\frac{1}{2}}. \end{aligned}$$

Repeating the same arguments in the last part of the proofs of Theorems 7.1.2 and 7.1.5, we can easily get (7.3.2) with $\omega_3 = \max\{K, 2K/\beta\}$.

In the case when $k = 2$, arguing likewise, we get $\omega_2 = \max\{K, 2K\}$. ∎

Remark 7.3.2 Using the Jensen inequality as in the proof of Theorem 7.1.2, from (7.3.3) we get

$$|(DT(t)f)(x)|^p \le e^{d_0 pt}(T(t)(|Df|^p))(x), \qquad t > 0,\ x \in \mathbb{R}^N, \tag{7.3.4}$$

for any $p \in [1, +\infty)$.

Moreover, adapting the proof of Theorem 7.2.2 to this case, we can show that, for any $f \in C_b(\mathbb{R}^N)$, any $t > 0$ and any $x \in \mathbb{R}^N$, it holds:

$$|(DT(t)f)(x)|^p \le \frac{2^p p^{1-p/2} d_0}{p-1)^{p/2}(1-e^{-pd_0 t})} t^{1-\frac{p}{2}}(T(t)(|f|^p))(x), \tag{7.3.5}$$

for any $p \in (1, 2]$ and

$$|(DT(t)f)(x)|^p \le \left(\frac{d_0}{1-e^{-2d_0 t}}\right)^{\frac{p}{2}} (T(t)(|f|^p))(x), \tag{7.3.6}$$

for any $p \in (2, +\infty)$.

We will use this estimate in Section 8.3.

In general, the estimates (7.2.5) and (7.2.27) fail for $p = 1$ also in the case when the coefficients of the operator $\mathcal{A}$ are bounded. Here, we provide a simple situation in which this happens, taking $\mathcal{A} = \Delta$.

Example 7.3.3 Let $\{T(t)\}$ be the heat semigroup in $\mathbb{R}$, i.e.,

$$(T(t)f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)dy, \qquad t > 0,\ x \in \mathbb{R}.$$

The space derivative of $T(t)f$ is given by

$$(DT(t)f)(x) = \frac{1}{2t\sqrt{4\pi t}} \int_{\mathbb{R}} (y-x)e^{-\frac{(x-y)^2}{4t}} f(y)dy, \qquad t > 0,\ x \in \mathbb{R}.$$

Fix $R > 0$ and let $f \in C_b(\mathbb{R})$ be such that $0 \le f \le 1$, $f(x) = 0$ for any $x < R - R^{-1}$ and $f(x) = 1$ for any $x > R$. Then,

$$(T(t)f)(0) \le \frac{1}{\sqrt{4\pi t}} \int_{R-R^{-1}}^{+\infty} e^{-\frac{|y|^2}{4t}} dy,$$

$$(DT(t)f)(0) \ge \frac{1}{2t\sqrt{4\pi t}} \int_{R}^{+\infty} ye^{-\frac{|y|^2}{4t}} dy.$$

Therefore, $(DT(t)f)(0) \geq c_R(T(t)f)(0)$, where

$$c_R = \frac{1}{2t}\int_R^{+\infty} ye^{-\frac{|y|^2}{4t}}\,dy\left(\int_{R-R^{-1}}^{+\infty} e^{-\frac{|y|^2}{4t}}\,dy\right)^{-1}.$$

Using De L'Hôspital rule, it is readily seen that c_R tends to $+\infty$ as R goes to $+\infty$. This means that no pointwise estimates similar to (7.2.5) can hold for $p=1$.

Chapter 8

Invariant measures μ and the semigroup in $L^p(\mathbb{R}^N, \mu)$

8.0 Introduction

In this chapter we deal with the invariant measures of the semigroup $\{T(t)\}$ associated with the operator $\mathcal{A}$ defined on smooth functions by

$$\mathcal{A}u(x) = \sum_{i,j=1}^{N} q_{ij}(x)D_{ij}u(x) + \sum_{j=1}^{N} b_j(x)D_j u(x), \qquad x \in \mathbb{R}^N. \tag{8.0.1}$$

We recall that, according to Remark 2.2.10, $\{T(t)\}$ is a semigroup of contractions in $B_b(\mathbb{R}^N)$.
Throughout the chapter, for any measure μ, we write L^p_μ for $L^p(\mathbb{R}^N, \mu)$ and we denote by $\|\cdot\|_p$ the norm of L^p_μ. Moreover, we write $W^{k,p}_\mu$ for $W^{k,p}(\mathbb{R}^N, \mu)$ for any $k \in \mathbb{N}$ and any $p \in [1, +\infty]$.

By definition, a probability measure μ is an invariant measure for $\{T(t)\}$ if

$$\int_{\mathbb{R}^N} T(t)f \, d\mu = \int_{\mathbb{R}^N} f \, d\mu, \tag{8.0.2}$$

for any $f \in B_b(\mathbb{R}^N)$ or, equivalently, for any $f \in C^\infty_c(\mathbb{R}^N)$ (see Lemma 8.1.3).

In Section 8.1 we show that if there exists an invariant measure of $\{T(t)\}$, then the semigroup can be extended to a semigroup of bounded operators in the space L^p_μ for any $p \in [1, +\infty)$. Such an extension, which we still denote by $\{T(t)\}$, enjoys many interesting properties. First of all, it defines a strongly continuous semigroup for any $p \in [1, +\infty)$. Moreover, it is possible to describe the behaviour of the function $T(t)f$ in L^p_μ when t approaches $+\infty$. More precisely, for any $f \in L^p_\mu$ we have

$$\lim_{t \to +\infty} \|T(t)f - \overline{f}\|_p = 0, \qquad \text{where} \quad \overline{f} = \int_{\mathbb{R}^N} f \, d\mu.$$

Such a result implies a Liouville type theorem, i.e., it implies that if $u \in D(L_p)$ satisfies $L_p u = 0$ for some $p \in (1, +\infty)$, then u is constant. Here, and in all the chapter, by L_p we denote the infinitesimal generator of the semigroup in L^p_μ.

The invariant measure μ of $\{T(t)\}$, whenever existing, is unique and it is equivalent to the Lebesgue measure on the σ-algebra of the Borel sets of $\mathbb{R}^N$, in the sense that a Borel set A is negligible with respect to μ if and only if it is negligible with respect to the Lebesgue measure. Moreover, the density $\rho = d\mu/dx$ satisfies

$$\operatorname*{ess\,inf}_{x\in B(r)} \rho(x) > 0, \qquad r > 0.$$

We will prove the existence of an invariant measure of $\{T(t)\}$ in three different ways and situations. First we show a classical result by Khas'minskii, which guarantees the existence of the invariant measure in terms of a Lyapunov function related to the operator $\mathcal{A}$. Next, we show that an invariant measure exists whenever $T(t)$ is compact in $C_b(\mathbb{R}^N)$ for any $t > 0$. Finally, we consider the case when the operator $\mathcal{A}$ is given, on smooth functions, by

$$\mathcal{A}u(x) = \Delta u(x) - \langle DU(x) + G(x), Du(x)\rangle, \qquad x \in \mathbb{R}^N \tag{8.0.3}$$

and the functions U and G belong, respectively, to $C^{1+\alpha}_{\rm loc}(\mathbb{R}^N)$ and $C^1(\mathbb{R}^N,\mathbb{R}^N)$ for some $\alpha \in (0,1)$. We prove that if $e^{-U} \in L^1(\mathbb{R}^N)$ and $\operatorname{div} G = \langle G, DU\rangle$, then the measure

$$\mu(dx) = K^{-1}e^{-U(x)}dx, \qquad K = \int_{\mathbb{R}^N} e^{-U(x)}dx,$$

is the invariant measure of $\{T(t)\}$. Moreover, the semigroup $\{T(t)\}$ and its infinitesimal generator L_2 are symmetric in L^2_μ and the following formula holds:

$$\int_{\mathbb{R}^N} \langle Df, Dg\rangle d\mu = -\int_{\mathbb{R}^N} gL_2 f\, d\mu, \qquad f \in D(L_2), \quad g \in W^{1,2}_\mu.$$

Notice that, when $f \in C^\infty_c(\mathbb{R}^N)$, such a formula follows immediately from an integration by parts. The proof that μ is an invariant measure of $\{T(t)\}$ is obtained using variational methods in L^p_μ.

In Section 8.2 we show some regularity properties of the invariant measure μ. We first show that if the diffusion coefficients q_{ij} $(i,j = 1,\dots,N)$ are continuously differentiable in $\mathbb{R}^N$, then μ has a density ρ which belongs to $W^{1,p}_{\rm loc}(\mathbb{R}^N)$. In particular, according to the Sobolev embedding theorems, ρ is a continuous function in $\mathbb{R}^N$. Next, we specialize to the case when

$$\mathcal{A}u = \sum_{i,j=1}^N D_i(q_{ij}D_ju) + \sum_{i=1}^N b_iD_iu,$$

on smooth functions. Allowing both the diffusion and the drift coefficients to be unbounded in $\mathbb{R}^N$, we show some global L^p- and $W^{1,p}$-regularity properties of ρ. In particular, we show that if $b_i \in L^p_\mu$ for any $i = 1,\dots,N$, then $\rho \in C_b(\mathbb{R}^N)$. Further, in the particular case when the diffusion coefficients belong to $C_b^1(\mathbb{R}^N)$, we prove some global $W^{1,q}$- and $W^{2,q}$-regularity properties of ρ. Finally, under some additional assumptions on the coefficients of the operator $\mathcal{A}$, we show some pointwise bounds for the function ρ.

In Section 8.3 we prove some estimates for the derivatives of $T(t)f$ (up to the third-order) and for the gradient of the resolvent operator in the L^p_μ-norm. Such estimates follow integrating the pointwise estimates proved in Chapter 7.

In Section 8.4 we consider the operator $\mathcal{A}$ in (8.0.3) when U is a convex function. In this case we can characterize the domain $D(L_2)$ and we obtain further estimates for the derivatives of the resolvent $R(\lambda, L_2)f$.

In Section 8.5 we study the compactness of the embedding $W^{1,p}_\mu \subset L^p_\mu$ in the symmetric case. We show that the embedding is compact whenever

$$\lim_{|x|\to+\infty} |DU(x)| = +\infty$$

and

$$|\Delta U(x)| \le \delta |DU(x)|^2 + M, \qquad x \in \mathbb{R}^N,$$

in the case when $p \ge 2$, or

$$\langle D^2U(x)DU(x), DU(x)\rangle \ge (\delta|DU(x)|^2 + M)|DU(x)|^2, \qquad x \in \mathbb{R}^N,$$

if $p \in (1,2)$. Here $\delta \in (0,1)$ and $M > 0$ are suitable constants.

A consequence of the compactness of the embedding of $W^{1,p}_\mu$ in L^p_μ is the Poincaré inequality in L^p_μ, that is the inequality

$$\int_{\mathbb{R}^N} |f - \overline{f}|^p d\mu \le C \int_{\mathbb{R}^N} |Df|^p d\mu, \qquad f \in W^{1,p}_\mu.$$

In the case when $p = 2$, the Poincaré inequality can be proved even in the non-symmetric case, using the pointwise estimates of Chapter 7. Such an inequality has the following consequences: first it allows us to show that $T(t)f$ converges exponentially to $\overline{f}$ in L^2_μ as t tends to $+\infty$; secondly, it implies that L_2 has the spectral gap property in L^2_μ, that is

$$\sigma(L_2) \setminus \{0\} \subset \{\lambda \in \mathbb{C}, \mathrm{Re}\,\lambda \le -1/C\}.$$

The Poincaré inequality and the spectral gap property are discussed in Section 8.6.

Finally, in Section 8.7 we consider the logarithmic Sobolev inequality, i.e., the inequality

$$\int_{\mathbb{R}^N} f^p \log f d\mu \le \|f\|_p^p \log \|f\|_p + \frac{p}{\lambda} \int_{\mathbb{R}^N} f^{p-2}|Df|^2 d\mu,$$

for positive and regular functions f. Such an inequality was first studied by Leonard Gross in [69]. For $p = 2$, it implies that $f^2 \log f$ is integrable for any positive $f \in W^{1,2}_\mu$. This is a very sharp result, as it can be seen in the case when the invariant measure μ is a Gaussian measure (see Example 8.7.6). In particular, such an example shows that the Sobolev embedding theorems, in general, fail to hold when μ is an invariant measure.

We prove the logarithmic Sobolev inequality using the pointwise estimates of Chapter 7. Then, adapting a proof by Gross, we see that in the symmetric case the logarithmic Sobolev inequality implies the hypercontractivity of $\{T(t)\}$. This means that the operator $T(t)$ is a contraction from L^2_μ to $L^{q(t)}_\mu$ for any $t > 0$, where $q(t) = 1 + e^{\lambda t}$. This is a sharp result too, since in general $T(t)$ is not bounded from L^2_μ to L^q_μ for any $q > q(t)$. The hypercontractivity was first proved by Edward Nelson in [121] for the Ornstein-Uhlenbeck semigroup, and, in fact, it was the reason for which Gross studied the logarithmic Sobolev inequalities.

8.1 Existence, uniqueness and general properties

In this section, we deal with the problem of the existence and uniqueness of the invariant measure of $\{T(t)\}$ and we prove some general properties of the invariant measures. If not otherwise specified, we always assume that the coefficients of the operator $\mathcal{A}$ satisfy the same assumptions as in Chapter 2 that we rewrite here for the reader's convenience.

Hypotheses 8.1.1 (i) $q_{ij} \equiv q_{ji}$ for any $i, j = 1, \ldots, N$ and

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \geq \kappa(x)|\xi|^2, \quad \kappa(x) > 0, \qquad \xi, x \in \mathbb{R}^N;$$

(ii) q_{ij} and b_i $(i, j = 1, \ldots, N)$ belong to $C^\alpha_{\rm loc}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$.

8.1.1 General properties and uniqueness of the invariant measure of $\{T(t)\}$

To begin with we give a characterization of the invariant measures of $\{T(t)\}$ in terms of the weak generator $\widehat{A}$ of the semigroup (see Section 2.3).

Proposition 8.1.2 *A Borel probability measure μ is an invariant measure of $\{T(t)\}$ if and only if*

$$\int_{\mathbb{R}^N} \widehat{A} f d\mu = 0, \qquad f \in D(\widehat{A}). \tag{8.1.1}$$

To prove the proposition, we first need the following lemma.

Lemma 8.1.3 *A Borel probability measure μ such that*

$$\int_{\mathbb{R}^N} T(t) f d\mu = \int_{\mathbb{R}^N} f d\mu, \qquad t > 0, \tag{8.1.2}$$

for any $f \in C_c^\infty(\mathbb{R}^N)$ is an invariant measure of $\{T(t)\}$.

Proof. Fix $f \in B_b(\mathbb{R}^N)$. We claim that we can approximate f with a sequence $\{f_n\} \subset C_c^\infty(\mathbb{R}^N)$ which is bounded with respect to the sup-norm and such that f_n tends to f almost everywhere in $\mathbb{R}^N$ as n tends to $+\infty$. Observe that a convolution argument with standard mollifiers proves the claim when $f \in C_b(\mathbb{R}^N)$. For a general $f \in B_b(\mathbb{R}^N)$ it suffices to observe that, according to [53, Proposition 3.4.2]), we can determine a sequence $\{f_n\} \subset C_b(\mathbb{R}^N)$ such that $\|f_n\|_\infty \le C$ for any $n \in \mathbb{N}$ and $\lim_{n\to+\infty} f_n(x) = f(x)$ for almost any $x \in \mathbb{R}^N$.

Now, the assertion easily follows from the dominated convergence theorem, writing (8.1.2) with f_n instead of f and letting n go to $+\infty$. Indeed, by Remark 2.2.10, $(T(t)f_n)(x)$ converges to $(T(t)f)(x)$ as n tends to $+\infty$, for any $t > 0$ and any $x \in \mathbb{R}^N$, and $\sup_{n\in\mathbb{N}} \|T(t)f_n\|_\infty < +\infty$. ■

Proof of Proposition 8.1.2. First we recall that, according to (2.3.12), for any $f \in D(\widehat{A})$, any $t > 0$ and any $x \in \mathbb{R}^N$, we have

$$(T(t)f)(x) - f(x) = \int_0^t (T(s)\widehat{A}f)(x) ds. \tag{8.1.3}$$

Now, let μ be an invariant measure of $\{T(t)\}$ and fix $f \in D(\widehat{A})$. Then

$$\int_{\mathbb{R}^N} \frac{T(t)f - f}{t} d\mu = 0, \qquad t > 0. \tag{8.1.4}$$

By (8.1.3), recalling that $\{T(t)\}$ is a semigroup of contractions since $c \equiv 0$ (see Theorem 2.2.5), it follows that $t^{-1}|(T(t)f)(x) - f(x)| \le \|\widehat{A}f\|_\infty$ for any $t > 0$ and any $x \in \mathbb{R}^N$. Thus (8.1.1) follows letting t tend to 0 in (8.1.4), using the dominated convergence theorem.

Conversely, assume that μ is a Borel probability measure satisfying (8.1.1), and let $f \in C_c^\infty(\mathbb{R}^N)$. By Lemma 2.3.3 and Proposition 2.3.6, f and $T(s)f$ belong to $D(\widehat{A})$ and $T(s)\widehat{A}f = \widehat{A}T(s)f$ for any $s > 0$. Integrating (8.1.3) in $\mathbb{R}^N$ and using the Fubini theorem, we get

$$\int_{\mathbb{R}^N} (T(t)f - f) d\mu = \int_0^t ds \int_{\mathbb{R}^N} \widehat{A}T(s) f d\mu = 0, \qquad t > 0$$

and, then, the conclusion follows from Lemma 8.1.3. ■

Remark 8.1.4 A probability measure μ which solves the equation (8.1.1) for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ is usually called *infinitesimally invariant*. In general an infinitesimally invariant measure μ is not an invariant measure of the semigroup $\{T(t)\}$. Indeed, in [136, Example 1.12], W. Stannat shows that the measure $d\mu = e^{-x^2}dx$ is infinitesimally invariant for the one-dimensional elliptic operator defined by

$$\mathcal{A}u(x) = u''(x) - (2x + 6e^{x^2})u'(x), \qquad x \in \mathbb{R},$$

on smooth functions, but it is not invariant for the semigroup associated with the operator $\mathcal{A}$.

Checking that the measure μ is infinitesimally invariant for the operator $\mathcal{A}$ is an easy task. According to the forthcoming Proposition 8.1.10, to prove that it is not invariant for the associated semigroup $\{T(t)\}$ it suffices to show that $\{T(t)\}$ is not conservative. According to Theorem 3.2.2, this is the case if $+\infty$ or $-\infty$ are accessible. In fact, we are going to prove that $+\infty$ is an entrance and $-\infty$ is an exit point. For this purpose, we show that the functions $Q, R : \mathbb{R} \to \mathbb{R}$ defined by

$$Q(x) = \frac{1}{W(x)}\int_0^x W(t)dt, \qquad R(x) = W(x)\int_0^x \frac{1}{W(t)}dt, \qquad x \in \mathbb{R},$$

where

$$W(x) = \exp\left(x^2 + 6\int_0^x e^{t^2}dt\right), \qquad x \in \mathbb{R}$$

(see (3.1.3)-(3.1.5)), are such that $Q \in L^1(0,+\infty)$, $Q \notin L^1(-\infty,0)$, $R \in L^1(-\infty,0)$, $R \notin L^1(0,+\infty)$. But this can be easily shown, observing that

$$\lim_{x\to+\infty} x^4 Q(x) = \lim_{x\to-\infty} x^4 R(x) = 0, \qquad \lim_{x\to-\infty} Q(x) = \lim_{x\to+\infty} R(x) = +\infty.$$

Let us now exploit some properties of the invariant measures.

Proposition 8.1.5 *Let μ be an invariant measure of $\{T(t)\}$. Then, μ is equivalent to the Lebesgue measure m on the σ-algebra of the Borel sets of $\mathbb{R}^N$ (in the sense that μ and the Lebesgue measure have the same sets of zero measure). Besides, for any fixed $r > 0$ the density ρ of μ with respect to the Lebesgue measure satisfies*

$$\operatorname*{ess\,inf}_{x\in B(r)} \rho(x) > 0.$$

Proof. For any Borel set $B \subset \mathbb{R}^N$, any $t > 0$ and any $x \in \mathbb{R}^N$ we have

$$(T(t)\chi_B)(x) = \int_B G(t,x,y)dy$$

(see Theorem 2.2.5), and, by (8.0.2),

$$\mu(B) = \int_{\mathbb{R}^N} \chi_B d\mu = \int_{\mathbb{R}^N} T(t)\chi_B d\mu.$$

Therefore, if $m(B) = 0$, then $(T(t)\chi_B)(x) = 0$ for any $t > 0$ and any $x \in \mathbb{R}^N$, since G is strictly positive in $(0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N$. Thus, $\mu(B) = 0$. Conversely, if $m(B) > 0$, then $(T(t)\chi_B)(x) > 0$ for any $x \in \mathbb{R}^N$ and, therefore, $\mu(B) > 0$. Thus μ and m are equivalent.

Now, let $r > 0$ and fix $R > r$. Moreover, let G_R be the Green's function in $B(R)$ associated with the realization of the operator $\mathcal{A}$ with homogeneous Dirichlet boundary conditions in $C(\overline{B}(R))$ (see Proposition C.3.2). Since $G_R(t, x, y) \le G(t, x, y)$ for any $R > 0$, any $t > 0$ and any $x, y \in B(R)$ (see the proof of Theorem 2.2.5), then, for any positive function $f \in B_b(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} f(x)\rho(x)dx &= \int_{\mathbb{R}^N} (T(1)f)(x)\rho(x)dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G(1, x, y)f(y)\rho(x)dy\, dx \\ &\ge \int_{\mathbb{R}^N} f(y) \int_{B(R)} G_R(1, x, y)\rho(x)dx\, dy.\end{aligned}$$

Since f is arbitrary, we have

$$\rho(y) \ge F(y) = \int_{B(R)} G_R(1, x, y)\rho(x)dx,$$

for almost any $y \in B(R)$. Now, since $G_R(1, \cdot, \cdot)$ is positive, bounded and continuous in $B(R) \times B(R)$, it follows that the function F is positive, bounded and continuous in $B(R)$. Therefore, $\rho(y) \ge \inf_{B(r)} F > 0$ for almost any $y \in B(r)$. ∎

As an immediate consequence of the results in Proposition 8.1.5, we get the following corollary.

Corollary 8.1.6 *Any invariant measure μ of $\{T(t)\}$ can be extended to a complete probability measure defined on the σ-algebra of all the Lebesgue measurable sets.*

Moreover, since the Lebesgue measure is regular, Proposition 8.1.5 immediately implies that μ is regular as well. Hence, applying [135, Theorem 3.14], we get the following useful result.

Corollary 8.1.7 *If μ is an invariant measure of $\{T(t)\}$, then $C_c^\infty(\mathbb{R}^N)$ is dense in L^p_μ for any $p \in [1, +\infty)$.*

We can now prove the following result.

Proposition 8.1.8 *Let μ be an invariant measure of $\{T(t)\}$. For any $p \in [1,+\infty)$, $\{T(t)\}$ extends to a strongly continuous semigroup of contractions in L^p_μ.*

Proof. We begin the proof recalling that

$$(T(t)f)(x) = \int_{\mathbb{R}^N} f(y)p(t,x;dy), \qquad t>0,\ \ x\in\mathbb{R}^N,\ \ f\in C_b(\mathbb{R}^N),$$

where $p(t,x;dy)$ are positive measures satisfying $p(t,x;\mathbb{R}^N)\le 1$ for any $t>0$ and any $x\in\mathbb{R}^N$.

Using the Hölder inequality, we easily deduce that

$$\begin{aligned}|(T(t)f)(x)|^p &= \left|\int_{\mathbb{R}^N} f(y)p(t,x;dy)\right|^p \\ &\le \int_{\mathbb{R}^N} |f(y)|^p p(t,x;dy) = (T(t)(|f|^p))(x),\end{aligned}$$

for any $f\in C_b(\mathbb{R}^N)$, any $t>0$ and any $x\in\mathbb{R}^N$. Therefore,

$$\int_{\mathbb{R}^N} |T(t)f|^p d\mu \le \int_{\mathbb{R}^N} T(t)(|f|^p)d\mu = \int_{\mathbb{R}^N} |f|^p d\mu, \qquad t>0.$$

Thus, since $C_b(\mathbb{R}^N)$ is dense in L^p_μ (due to Corollary 8.1.7), then, for any $t>0$, the operator $T(t)$ can be uniquely extended to a contraction in L^p_μ. The strong continuity of $\{T(t)\}$ in L^p_μ now is an immediate consequence of Propositions 2.2.7 and A.1.2. ∎

As it has been pointed out in the introduction to this chapter, for any $p\in[1,+\infty)$, we denote by L_p the generator of $\{T(t)\}$ in L^p_μ. We simply write L for L_2. From (8.0.2) it readily follows that

$$\int_{\mathbb{R}^N} L_p f d\mu = 0, \qquad f\in D(L_p),\ \ p\in[1,+\infty). \tag{8.1.5}$$

Besides, for any $p\in[1,+\infty)$ the resolvent $R(\lambda,L_p)$ is an extension of $R(\lambda)$ (see Theorem 2.1.3) to L^p_μ and

$$\|R(\lambda,L_p)f\|_p \le \lambda^{-1}\|f\|_p, \qquad f\in D(L_p),\ \ \lambda>0,\ \ p\in[1,+\infty). \tag{8.1.6}$$

As the following proposition shows, $D(\widehat{A})$ is a core of L_p.

Proposition 8.1.9 *Let $\{T(t)\}$ admit an invariant measure μ. Then, $D(\widehat{A})$ is a core of L_p, for any $p\in[1,+\infty)$.*

Proof. Let us begin by observing that $D(\widehat{A}) \subset D(L_p)$. This follows easily recalling that $R(\lambda, L_p)_{|C_b(\mathbb{R}^N)} = R(\lambda)$ and $R(\lambda)(C_b(\mathbb{R}^N)) = D(\widehat{A})$.

Now, since $C_c^\infty(\mathbb{R}^N) \subset D(\widehat{A})$ and $C_c^\infty(\mathbb{R}^N)$ is dense in L^p_μ (see Proposition 2.3.6 and Corollary 8.1.7), then $D(\widehat{A})$ is dense in L^p_μ as well. Besides, $T(t)(D(\widehat{A})) \subset D(\widehat{A})$ for any $t > 0$ (see Lemma 2.3.3). Thus, the conclusion follows from Proposition B.1.10. ■

As the following proposition shows, if $\{T(t)\}$ admits an invariant measure, then the semigroup is conservative.

Proposition 8.1.10 *Assume Hypotheses* 8.1.1. *If there exists an invariant measure μ of $\{T(t)\}$, then $\{T(t)\}$ is conservative.*

Proof. Fix $t > 0$. According to Theorem 2.2.5, $0 < T(t)\mathbb{1} \le \mathbb{1}$ for any $t > 0$. Besides, by (8.0.2) with $f \equiv \mathbb{1}$, we have

$$\int_{\mathbb{R}^N} T(t)\mathbb{1} d\mu = \int_{\mathbb{R}^N} \mathbb{1} d\mu = 1, \qquad t > 0,$$

and, therefore, $T(t)\mathbb{1} \equiv \mathbb{1}$ μ-almost everywhere in $\mathbb{R}^N$. Since $T(t)\mathbb{1}$ is a continuous function and μ is equivalent to the Lebesgue measure, then $T(t)\mathbb{1} \equiv \mathbb{1}$ everywhere in $\mathbb{R}^N$, for any $t > 0$. ■

Now, we show that an invariant measure of $\{T(t)\}$ is always ergodic, i.e., that

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t T(s) f ds = \int_{\mathbb{R}^N} f d\mu, \qquad f \in L^2_\mu,$$

where the limit is meant in L^2_μ.

Proposition 8.1.11 *Let μ be an invariant measure of $\{T(t)\}$. Then μ is ergodic.*

To prove the proposition, we need some preliminary results.

Lemma 8.1.12 *Let H be a Hilbert space and let $T \in L(H)$ be a bounded operator such that $\|T^k\|_{L(H)} \le M$ for any $k \in \mathbb{N}$, some constant $M > 0$ and any $k \in \mathbb{N}$. Then, the sequence of bounded operators $\{T_n\}$, defined by*

$$P_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \qquad x \in H,$$

converges pointwise to a bounded operator P_∞ which is a projection on the kernel of the operator $I - T$.

Proof. We begin the proof by observing that a straightforward computation shows that P_nx converges to x in H, as n tends to $+\infty$, for any $x\in \mathrm{Ker}(I-T)$. Similarly, P_nx converges in H for any $x\in (I-T)(H)$. Since the norms of the operators T_n are equibounded by M, by Proposition A.1.2, P_nx converges in H for any $x\in\overline{(I-T)(H)}$.

We now prove that the sequence $\{P_nx\}$ converges for any $x\in H$. Since the previous sequence is bounded, up to a subsequence, we can assume that there exists $y\in H$ such that P_nx converges weakly to y as n tends to $+\infty$. As a consequence, $T(P_n)x$ converges weakly to Ty in H. Since

$$T(P_nx)=P_nx+\frac{1}{n}(T^nx-x),\qquad n\in\mathbb{N},\tag{8.1.7}$$

we immediately deduce that $y\in \mathrm{Ker}\,(I-T)$. We now split $x=y+(x-y)$ and we claim that $x-y\in\overline{(I-T)(X)}$. Once the claim is proved, we will conclude that P_nx converges in the strong topology of H. To prove the claim, we observe that $x-y$ is the weak limit of the sequence $x-P_nx$, and $x-P_nx\in(I-T)(X)$ for any $n\in\mathbb{N}$. Indeed,

$$x-P_nx=\frac{1}{n}\sum_{k=0}^{n-1}(I-T^k)(x)=(I-T)\left(\frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}T^j\right)(x).$$

Since $(I-T)(H)$ is convex, its weak and strong closures coincide. The claim follows.

Now, let us set $P_\infty x=\lim_{n\to+\infty}P_nx$ for any $x\in H$. To conclude the proof, we show that P_∞ is a projection on $\mathrm{Ker}\,(I-T)$. Letting n go to $+\infty$ in (8.1.7) we get $TP_\infty x=P_\infty x$. This implies that

$$T^kP_\infty x=P_\infty x,\qquad x\in H,\quad k\in\mathbb{N},\tag{8.1.8}$$

and, consequently, $P_\infty^2=P_\infty$, so that P_∞ is a projection. Since we already know that $\mathrm{Ker}\,(I-T)\subset P_\infty(H)$, we just need to show that if $P_\infty x=x$, then $Tx=x$. But this is a straightforward consequence of (8.1.8). ■

Proposition 8.1.13 (Von Neumann) *For any $f\in L^2_\mu$ the limit*

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t T(s)f\,ds$$

is well defined. Here, the integral is meant in the L^2_μ-norm. Moreover, if we denote by $P_\infty f$ the previous limit, the operator P_∞ is a projection on the subspace $C:=\{f\in L^2_\mu : T(t)f=f\ \mu-a.e.\ for\ any\ t>0\}$.

The set C enjoys the following properties:

(i) if $f\in C$, then $|f|\in C$;

(*ii*) *if* $f, g \in C$*, then* $f \wedge g$ *and* $f \vee g$ *belong to* C*. In particular,* f^+ *and* f^- *belong to* C*;*

(*iii*) *if* $f \in C$*, then, for any* $\lambda \in \mathbb{R}$*, the function* $\chi_{\{x:\ f(x)>\lambda\}}$ *belongs to* C*.*

Finally,

$$\int_{\mathbb{R}^N} P_\infty f d\mu = \int_{\mathbb{R}^N} f d\mu, \qquad f \in L^2_\mu. \tag{8.1.9}$$

Proof. Let us introduce the operator $P_t \in L(L^2_\mu)$ defined by

$$P_t f = \frac{1}{t}\int_0^t T(s) f ds, \qquad f \in L^2_\mu, \quad t > 0.$$

Denoting, respectively, by $[t]$ and $\{t\}$ the integer and the fractional part of $t > 0$, we can write

$$\begin{aligned} P_t f &= \frac{1}{t}\sum_{k=0}^{[t]-1}\int_0^1 T(s+k) f ds + \frac{1}{t}\int_0^{\{t\}} T(s+[t]) f ds \\ &= \frac{[t]}{t}\frac{1}{[t]}\sum_{k=0}^{[t]-1} T^k(1)(P_1 f) + \frac{\{t\}}{t} T^{[t]}(1)(P_{\{t\}} f). \end{aligned} \tag{8.1.10}$$

Since the L^2_μ-norm of $T^k(1)$ is bounded by 1 for any $k \in \mathbb{N}$, Lemma 8.1.12 implies that there exists a projection P_* such that $\frac{1}{n}\sum_{k=0}^{n-1} T^k(1) f$ converges to $P_* f$, as n tends to $+\infty$, for any fixed $f \in L^2_\mu$. Therefore, letting t go to $+\infty$ in (8.1.10), we easily deduce that P_t converges pointwise to $P_\infty = P_* \circ P_1$ as t tends to $+\infty$.

To prove that P_∞ is a projection, it suffices to show that

$$T(r) \circ P_\infty = P_\infty, \qquad r > 0. \tag{8.1.11}$$

Indeed, this will imply that $P_1 \circ P_\infty = P_\infty$ and $P_* \circ P_\infty = P_\infty$, which, of course, gives $P_\infty^2 = P_\infty$. To prove (8.1.11), we observe that

$$\begin{aligned} T(r) P_t f &= \frac{1}{t} T(r)\int_0^t T(s) f ds = \frac{1}{t}\int_0^t T(r+s) f ds = \frac{1}{t}\int_r^{r+t} T(s) f ds \\ &= P_t f + \frac{1}{t}\int_0^r (T(t) - 1) T(s) f ds. \end{aligned} \tag{8.1.12}$$

Hence, letting t go to $+\infty$ in (8.1.12) gives (8.1.11).

Now, showing that P_∞ is a projection on C is immediate. The inclusion $C \subset P_\infty(L^2_\mu)$ is straightforward, whereas the other inclusion follows from (8.1.11). Indeed, since $P_\infty f = f$ for any $f \in C$, we have $C \subset P_\infty(L^2_\mu)$. Hence, the first part of the assertion follows.

To prove the second part of the proposition, we begin by observing that since $|T(t)g| \le T(t)|g|$ for any $t>0$ and any function $g \in C_b(\mathbb{R}^N)$, then, by density, such an inequality can be extended to any $g \in L^2_\mu$. Fix now $f \in C$. Then, $|f| = |T(t)f| \le T(t)|f|$ for any $t>0$. Since μ is an invariant measure of $\{T(t)\}$ then $\|T(t)|f| - |f|\|_1 = 0$, so that $T(t)|f| = |f|$ μ-almost everywhere for any $t>0$. Therefore, $|f| \in C$. Now, since $f^+ = (f+|f|)/2$ and $f^- = (f-|f|)/2$, we easily deduce that f^+ and f^- belong to C if f does. Finally, since $f \vee g = g + (f-g)^+$ and $f \wedge g = g + (f-g)^-$, we easily deduce that $f \vee g$ and $f \wedge g$ belong to C if f and g do.

To prove the property (iii), we fix $\lambda \in \mathbb{R}$, $f \in C$ and we set

$$f_n = (n(f-\lambda)^+) \wedge 1, \qquad n \in \mathbb{N}.$$

As it is immediately seen, f_n converges pointwise to $\chi_{\{x:\ f(x)>\lambda\}}$ as n tends to $+\infty$. By Remark 2.2.10 it follows that $T(\cdot)f_n$ tends to $T(\cdot)\chi_{\{x:\ f(x)>\lambda\}}$ pointwise in $(0,+\infty)\times\mathbb{R}^N$. Since each function f_n belongs to C, the function $\chi_{\{x:\ f(x)>\lambda\}}$ is in C as well.

Finally, since

$$\int_{\mathbb{R}^N} P_t f\, d\mu = \frac{1}{t}\int_{\mathbb{R}^N} d\mu \int_0^t T(s)f\, ds = \frac{1}{t}\int_0^t ds \int_{\mathbb{R}^N} T(s)f\, d\mu = \int_{\mathbb{R}^N} f\, d\mu,$$

for any $t>0$, letting t go to $+\infty$, the dominated convergence theorem yields (8.1.9). ■

Proof of Proposition 8.1.11. We begin the proof observing that, if A is a Borel set, then χ_A belongs to C if and only if $\mu(A)=0$ or $\mu(A)=1$. Indeed, suppose that $\chi_A \in C$ and $\mu(A)>0$. Then, since $\{T(t)\}$ is irreducible and strong Feller (see Proposition 2.2.12) the function $T(t)\chi_A$ is continuous in $\mathbb{R}^N$ and it is strictly positive for any $t>0$ (see Remark 2.2.13). Therefore, $T(t)\chi_A \ne \chi_A$ on $\mathbb{R}^N \setminus A$. But since $\chi_A \in C$, then $T(t)\chi_A = \chi_A$ μ-almost everywhere in $\mathbb{R}^N$, which implies that $\mu(\mathbb{R}^N\setminus A)=0$ or, equivalently, $\mu(A)=1$.

We can now show that C contains only the constant functions. For this purpose, let $f \in C$ be a nonconstant function. Then, by Proposition 8.1.13(iii) the characteristic function of the set $A_\lambda = \{x:\ f(x)>\lambda\}$ is in C, for any $\lambda \in \mathbb{R}$. Therefore, $\mu(A_\lambda)=0$ or $\mu(A_\lambda)=1$. Observe that there should exist $\lambda \in \mathbb{R}$ such that $\mu(A_\lambda)=1$. Otherwise, since

$$\mathbb{R}^N = \{x \in \mathbb{R}^N : f(x) = \pm\infty\} \cup \bigcup_{n \in \mathbb{N}\cup\{+\infty\}} A_{-n},$$

it should follow that $\mu(\mathbb{R}^N)=0$. Similarly, if $\mu(A_\lambda)=1$ for any $\lambda \in \mathbb{R}$, then $f(x)=+\infty$ for μ-almost any $x \in \mathbb{R}^N$, which cannot be the case. Set $\lambda_0 = \sup\{\lambda : \mu(A_\lambda)=1\}$. We infer that $f=\lambda_0$ μ-almost everywhere in $\mathbb{R}^N$. Indeed, by definition of λ_0, $f \ge \lambda_0$ μ-almost everywhere in $\mathbb{R}^N$. Moreover, since for any $\lambda > \lambda_0$, $\mu(A_\lambda) = 0$ it follows that $\mu(A_{\lambda_0}) = 0$, so that f is μ-almost everywhere constant in $\mathbb{R}^N$.

Since C contains only the constant functions, its dimension is 1. Therefore, we can find out a linear operator $S \in (L^2_\mu)'$ such that $P_\infty f = S(f)\mathbb{1}$ for any $f \in L^2_\mu$. According to the Riesz-Fisher representation theorem, there exists $g \in L^2_\mu$ such that

$$Sf = \int_{\mathbb{R}^N} fgd\mu, \qquad f \in L^2_\mu.$$

To complete the proof we just need to show that $g = 1$ μ-almost everywhere. For this purpose we observe that from (8.1.9) we get

$$\int_{\mathbb{R}^N} fd\mu = \int_{\mathbb{R}^N} P_\infty fd\mu = \int_{\mathbb{R}^N} fgd\mu, \qquad f \in L^2_\mu.$$

By the arbitrariness of f, it follows that $g = 1$ μ-almost everywhere, and we are done. ■

Remark 8.1.14 The results of Proposition 8.1.13 imply, in particular, that there exists a subsequence $\{t_n\}$ diverging to $+\infty$ such that $P_{t_n}f$ converges pointwise, as n tends to $+\infty$, to $\overline{f}$. As the Chacon-Ornstein theorem show (see [124, Chapter 3, Section 8]), the pointwise limit is $\overline{f}$, for any $f \in L^1_\mu$, and we can take $t_n = n$ for any $n \in \mathbb{N}$.

Now we can prove that $\{T(t)\}$ admits at most one invariant measure. The uniqueness of the invariant measure has been proved in a more general context by Doob. We refer the reader to [42] for further details.

Theorem 8.1.15 *There exists at most one invariant measure of* $\{T(t)\}$.

Proof. Let μ_1 and μ_2 be two invariant measures of $\{T(t)\}$. Let us prove that $\mu_1 = \mu_2$. According to Proposition 8.1.11, μ_1 and μ_2 are ergodic measures. Therefore, for any Borel set A, we can find out two Borel sets M_1 and M_2 such that $\mu_j(M_j) = 1$ for $j = 1, 2$ and

$$\lim_{n\to+\infty} \frac{1}{n} \int_0^n (T(s)\chi_A)(x)ds = \int_{\mathbb{R}^N} \chi_A d\mu_j = \mu_j(A), \qquad x \in M_j, \quad j = 1, 2$$

(see Remark 8.1.14). Since the measures μ_1 and μ_2 are equivalent (see Proposition 8.1.5), then $M_1 \cap M_2 \neq \varnothing$. Therefore, $\mu_1(A) = \mu_2(A)$ and we are done. ■

To conclude this subsection we deal with the behaviour of $T(t)$ as t tends to $+\infty$ and we prove a Liouville type theorem. The following result is due to Doob. For a proof, see, e.g., [42, Theorems 3.4.2 & 4.2.1].

Theorem 8.1.16 *For any $f \in L^p_\mu$ we have*

$$\lim_{t\to+\infty} \|T(t)f - \overline{f}\|_p = 0, \tag{8.1.13}$$

where $\overline{f} = \int_{\mathbb{R}^N} f d\mu$.

As a straightforward consequence of Theorem 8.1.16 we get the following Liouville type theorem.

Theorem 8.1.17 *Suppose that the semigroup $\{T(t)\}$ admits an invariant measure μ. Moreover, let $p \in [1,+\infty)$ and let $u \in D(L_p)$ be such that $L_p u = 0$. Then, u is constant.*

Proof. Let u be as in the statement of the theorem. Since $\{T(t)\}$ is a strongly continuous semigroup, then the function $t \mapsto T(t)u$ is continuously Fréchet differentiable in $[0,+\infty)$ with values in L^p_μ and $D_t T(t)u = T(t)L_p u = 0$ for any $t > 0$ and any $p \in [1,+\infty)$ (see Section B.1). This implies that the function $t \mapsto T(t)u$ is constant and, consequently, $T(t)u = u$ for any $t > 0$. Letting t tend to $+\infty$, from (8.1.13), we immediately deduce that $u = \int_{\mathbb{R}^N} u d\mu$, so that u is constant. ∎

8.1.2 Existence by Khas'minskii theorem

The main result concerning the existence of an invariant measure of a Markov semigroup is the Khas'minskii theorem. To prove it, we need some preliminaries.

Theorem 8.1.18 (Prokhorov) *A family $\mathcal{F}$ of Borel probability measures on $\mathbb{R}^N$ is tight (see Definition 5.1.2) if and only if, for any sequence $\{\mu_n\} \subset \mathcal{F}$, there exists a subsequence weakly* convergent in the dual space $C_b(\mathbb{R}^N)'$ to a probability measure.*

Proof. To begin the proof, we assume that the family $\mathcal{F}$ is tight and we fix a sequence $\{\mu_n\} \subset \mathcal{F}$. For any $n \in \mathbb{N}$, let us consider the restriction $\mu_{n,1}$ of the measure μ_k to $\overline{B}(1)$. Since $C(\overline{B}(1))$ is separable the weak* topology of $(C(\overline{B}(1))'$ is metrizable. Therefore, up to a subsequence, $\mu_{n,1}$ weakly* converges to some measure μ_1. Applying the same argument to the restrictions of the sequence $\{\mu_n\}$ to the ball $B(k)$ $(k \in \mathbb{N})$ and using a diagonal procedure, we can determine a subsequence $\{\mu_{k_n}\} \subset \{\mu_n\}$ such that for any $m \in \mathbb{N}$, μ_{k_n} converges weakly* to a Borel measure $\mu^{(m)}$ on $C(\overline{B}(m))$.

Let us now observe that, for any positive function $f \in C_b(\mathbb{R}^N)$ and any $m \in \mathbb{N}$, one has

$$\int_{\overline{B}(m)} f d\mu^{(m)} = \lim_{n\to+\infty} \int_{\overline{B}(m)} f d\mu_{k_n} \le \lim_{n\to+\infty} \int_{\overline{B}(m+1)} f d\mu_{k_n}. \tag{8.1.14}$$

Now, fix a Borel set B and a bounded sequence $\{f_j\} \subset C_b(\mathbb{R}^N)$ of positive functions converging pointwise to χ_B as j tends to $+\infty$. Taking (8.1.14) into account, we can write

$$\begin{aligned}\mu^{(m)}(B\cap \overline{B}(m)) &= \int_{\overline{B}(m)} \chi_B d\mu^{(m)} = \lim_{j\to+\infty}\int_{\overline{B}(m)} f_j d\mu^{(m)}\\ &\le \lim_{j\to+\infty}\int_{\overline{B}(m+1)} f_j d\mu^{(m+1)} = \int_{\overline{B}(m+1)} \chi_B d\mu^{(m+1)}\\ &= \mu^{(m+1)}(B\cap\overline{B}(m)).\end{aligned}$$

Since the sequence $\{\mu^{(m)}(B\cap\overline{B}(m))\}$ is positive, increasing and bounded by 1, we can define a function μ on the σ algebra of the Borel sets of $\mathbb{R}^N$ by setting

$$\mu(B) := \lim_{m\to+\infty}\mu^{(m)}(B\cap\overline{B}(m)),$$

for any Borel set B. As it is immediately seen, μ is a Borel measure.

Let us now prove that μ is a probability measure and μ_{k_n} converges weakly* to μ. For this purpose, we fix $\varepsilon>0$ and take $h\in\mathbb{N}$ such that

$$\mu_{k_n}(\mathbb{R}^N\setminus B(h))\le\varepsilon, \qquad n\in\mathbb{N}.$$

Moreover, for any $m>h$, we denote by g_m any continuous function such that $\chi_{\overline{B}(m)\setminus B(h+1)}\le g_m\le\chi_{\overline{B}(m+1)\setminus B(h)}$. Then,

$$\begin{aligned}\mu(\overline{B}(m)\setminus B(h+1)) &\le \int_{\overline{B}(m)} g_m d\mu\\ &= \lim_{n\to+\infty}\int_{\overline{B}(m)} g_m d\mu_{k_n}\\ &\le \limsup_{n\to+\infty}\mu_{k_n}(\overline{\mathbb{R}}^N\setminus B(h))\\ &\le\varepsilon.\end{aligned}$$

Letting m tend to $+\infty$ gives $\mu(\mathbb{R}^N\setminus B(h+1))\le\varepsilon$ and we are almost done. Indeed, if $f\in C_b(\mathbb{R}^N)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^N} f d\mu - \int_{\mathbb{R}^N} f d\mu_{k_n}\right| &\le \left|\int_{\overline{B}(h+1)} f d\mu - \int_{\overline{B}(h+1)} f d\mu_{k_n}\right|\\ &\quad+\left|\int_{\mathbb{R}^N\setminus\overline{B}(h+1)} f d\mu - \int_{\mathbb{R}^N\setminus\overline{B}(h+1)} f d\mu_{k_n}\right|\\ &\le \left|\int_{\overline{B}(h+1)} f d\mu - \int_{\overline{B}(h+1)} f d\mu_{k_n}\right| + 2\varepsilon\|f\|_\infty.\end{aligned} \tag{8.1.15}$$

Now, observing that $\mu(\overline{B}(h+1)) = \mu^{(m)}(B(h+1))$ for any $m > h+1$ and letting n go to $+\infty$, (8.1.14) and (8.1.15) give

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^N} f d\mu - \int_{\mathbb{R}^N} f d\mu_{k_n}\right| \le 2\varepsilon\|f\|_\infty, \tag{8.1.16}$$

and the arbitrariness of $\varepsilon > 0$ and $f \in C_b(\mathbb{R}^N)$ allow us to conclude that μ_{k_n} weakly* converges to μ. Moreover, taking $f \equiv \mathbb{1}$ in (8.1.16), we obtain that μ is a probability measure.

Conversely, let us assume that $\mathcal{F}$ is relatively weak* compact and let us prove that it is also tight. By contradiction, assume that there exist $\varepsilon > 0$ and a sequence $\{\mu_n\} \subset \mathcal{F}$ such that

$$\mu_n(B(n)) < 1-\varepsilon, \qquad n \in \mathbb{N}.$$

Since $\mathcal{F}$ is relatively weak* compact we can assume, up to a subsequence, that μ_n converges to some probability measure μ. Let us prove that

$$\mu(B(k)) \le \liminf_{n\to+\infty} \mu_n(B(k)) \le \liminf_{n\to+\infty} \mu_n(B(n)) \le 1-\varepsilon, \tag{8.1.17}$$

for any k sufficiently large. Of course, (8.1.17) will lead us to a contradiction, since $\mu(B(k))$ converges to $\mu(\mathbb{R}^N) = 1$ as k tends to $+\infty$.

So, let us prove the claim. As it is easily seen, it suffices to show that

$$\limsup_{n\to+\infty} \mu_n(\mathbb{R}^N \setminus B(k)) \le \mu(\mathbb{R}^N \setminus B(k)). \tag{8.1.18}$$

For this purpose, we fix $\varepsilon > 0$ and $m \in \mathbb{N}$ sufficiently large such that

$$\mu(\mathbb{R}^N \setminus B(k-1/m)) \le \mu(\mathbb{R}^N \setminus B(k)) + \varepsilon.$$

Further, we introduce the function $f : \mathbb{R}^N \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & |x| \le k - \frac{1}{m}, \\ m|x| + 1 - mk, & k - \frac{1}{m} < |x| < k, \\ 1, & |x| \ge k, \end{cases}$$

and we observe that

$$\mu_n(\mathbb{R}^N \setminus B(k)) = \int_{\mathbb{R}^N\setminus B(k)} f d\mu_n \le \int_{\mathbb{R}^N\setminus B(k-1/m)} f d\mu_n.$$

Letting n go to $+\infty$, we deduce that

$$\begin{aligned}\limsup_{n\to+\infty} \mu_n(\mathbb{R}^N \setminus B(k)) &\le \int_{\mathbb{R}^N\setminus B(k-1/m)} f d\mu \le \mu(\mathbb{R}^N \setminus B(k-1/m)) \\ &\le \mu(\mathbb{R}^N \setminus B(k)) + \varepsilon.\end{aligned}$$

The arbitrariness of $\varepsilon > 0$ yields the inequality (8.1.18), and we are done. ∎

Let $\{p(t,x;dy)\ :\ t>0,\ x\in\mathbb{R}^N\}$ be the same transition family as in Theorem 2.2.5, and define the family of probability measures $\{r(t,x;dy),\ t>0,\ x\in\mathbb{R}^N\}$ by setting

$$r(t,x;B)=\frac{1}{t}\int_0^t p(s,x;B)ds, \qquad B\in\mathcal{B}(\mathbb{R}^N).$$

We recall that $p(s,x;B)=(T(s)\chi_B)(x)$.

Let us prove the following preliminary theorem ([83, Theorem 3.2.1]). In the sequel we will use only the first part of it.

Theorem 8.1.19 (Krylov-Bogoliubov) *If for some $t_0>0$ and some $x_0\in\mathbb{R}^N$ the family of measures $\{r(t,x_0;dy),\ t>t_0\}$ is tight, then the semigroup $\{T(t)\}$ admits an invariant measure μ.*

Conversely, if there exists an invariant measure μ, then

$$\lim_{R\to+\infty}\liminf_{t\to+\infty} r(t,x;\mathbb{R}^N\setminus B(R))=0,$$

for any $x\in\mathbb{R}^N\setminus E$, where E is a negligible set with respect to the Lebesgue measure.

Proof. Suppose that the family of measures $\{r(t,x_0;dy),\ t>t_0\}$ is tight. By Prokhorov theorem, there exists a sequence $\{t_n\}$ diverging to $+\infty$ as n tends to $+\infty$ such that $r(t_n,x_0;dy)$ converges weakly* to a probability measure μ. We claim that μ is an invariant measure of $\{T(t)\}$. Indeed, for any $f\in B_b(\mathbb{R}^N)$ and any $t>0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N}(T(t)f)(y)r(t_n,x_0;dy) &= \frac{1}{t_n}\int_0^{t_n}(T(t+s)f)(x_0)ds\\
&= \frac{1}{t_n}\int_t^{t+t_n}(T(s)f)(x_0)ds\\
&= \int_{\mathbb{R}^N} f(y)r(t_n,x_0;dy)+\frac{1}{t_n}\int_{t_n}^{t_n+t}(T(s)f)(x_0)ds\\
&\quad -\frac{1}{t_n}\int_0^t (T(s)f)(x_0)ds.
\end{aligned}$$

Since $\|T(t)\|_{L(B_b(\mathbb{R}^N))}\le 1$ for any $t\ge 0$, letting n tend to $+\infty$ we get

$$\int_{\mathbb{R}^N} T(t)f\,d\mu=\int_{\mathbb{R}^N} f\,d\mu,$$

that is μ is an invariant measure of $\{T(t)\}$.

Conversely, assume that μ is an invariant measure of $\{T(t)\}$. For any fixed $x\in\mathbb{R}^N$ the function

$$R\mapsto \liminf_{t\to+\infty} r(t,x;\mathbb{R}^N\setminus B(R))$$

is decreasing. Therefore, we can define

$$r(x)=\lim_{R\to+\infty}\liminf_{t\to+\infty} r(t,x;\mathbb{R}^N\setminus B(R)),\qquad x\in\mathbb{R}^N.$$

Now, we observe that

$$\begin{aligned}\mu(\mathbb{R}^N\setminus B(R)) &= \frac{1}{t}\int_0^t \mu(\mathbb{R}^N\setminus B(R))ds\\ &= \frac{1}{t}\int_0^t ds\int_{\mathbb{R}^N} T(s)\chi_{\mathbb{R}^N\setminus B(R)}d\mu\\ &= \frac{1}{t}\int_0^t ds\int_{\mathbb{R}^N} p(s,x;\mathbb{R}^N\setminus B(R))d\mu\\ &= \int_{\mathbb{R}^N} r(t,x;\mathbb{R}^N\setminus B(R))d\mu,\end{aligned}$$

for any $t>0$, and then, by Fatou's lemma, we get

$$\begin{aligned}\mu(\mathbb{R}^N\setminus B(R)) &= \liminf_{t\to+\infty}\int_{\mathbb{R}^N} r(t,x;\mathbb{R}^N\setminus B(R))d\mu\\ &\geq \int_{\mathbb{R}^N}\liminf_{t\to+\infty} r(t,x;\mathbb{R}^N\setminus B(R))d\mu.\end{aligned}$$

Letting R tend to $+\infty$, by the monotone convergence theorem we get

$$0\geq \lim_{R\to+\infty}\int_{\mathbb{R}^N}\liminf_{t\to+\infty} r(t,x;\mathbb{R}^N\setminus B(R))d\mu=\int_{\mathbb{R}^N} r d\mu,$$

which implies that $r=0$, μ-almost everywhere. Since μ is equivalent to the Lebesgue measure, the conclusion follows. ∎

We can now prove the main result of this section, that is the Khas'minskii theorem, which gives a sufficient condition for the tightness of the measures $\{r(t,x_0;dy),\ t>1\}$ and, hence, for the existence of an invariant measure for the semigroup $\{T(t)\}$.

Theorem 8.1.20 (Khas'minskii) *Suppose that there exists a function $\varphi\in C^2(\mathbb{R}^N)$ such that*

$$\varphi\geq 0,\qquad \lim_{|x|\to+\infty}\mathcal{A}\varphi(x)=-\infty.$$

Then, for any fixed $x_0\in\mathbb{R}^N$, the measures $\{r(t,x_0;dy),\ t>1\}$ are tight. As a consequence, the semigroup $\{T(t)\}$ admits an invariant measure.

Proof. Let X be the Markov process associated with $\{T(t)\}$ (see Section 2.4), and let $\tau_n = \tau_{B(n)}$ be the first exit time of X from the ball $B(n)$, as defined in (2.4.6). Fix $t > 0$ and $x \in \mathbb{R}^N$; from the formula (2.4.7) with $\tau' = t \wedge \tau_n$ we have

$$\mathbb{E}_x \varphi(X_{t\wedge\tau_n}) = \varphi(x) + \mathbb{E}_x \int_0^{t\wedge\tau_n} \mathcal{A}\varphi(X_s)ds, \tag{8.1.19}$$

for any $n \in \mathbb{N}$. Now, for any $R > 0$, let

$$-M_R = \sup_{|x|>R} \mathcal{A}\varphi(x),$$

and let $K \geq 0$ be such that $\mathcal{A}\varphi(x) \leq K$ for any $x \in \mathbb{R}^N$. Then, clearly

$$\mathcal{A}\varphi(x) \leq K - \chi_{\{|x|>R\}} M_R, \qquad x \in \mathbb{R}^N,$$

and, by (8.1.19), we get

$$\mathbb{E}_x \varphi(X_{t\wedge\tau_n}) - \varphi(x) \leq Kt - M_R \mathbb{E}_x \int_0^{t\wedge\tau_n} \chi_{\{|X_s|>R\}} ds, \qquad t > 0, \;\; x \in \mathbb{R}^N.$$

Hence,

$$M_R \mathbb{E}_x \int_0^{t\wedge\tau_n} \chi_{\{|X_s|>R\}} ds \leq Kt + \varphi(x) - \mathbb{E}_x \varphi(X_{t\wedge\tau_n}) \leq Kt + \varphi(x), \tag{8.1.20}$$

for any $t > 0$ and any $x \in \mathbb{R}^N$. Now, since $\{T(t)\}$ is conservative, τ_n tends to $+\infty$ almost surely, and then

$$\begin{aligned}
\lim_{n\to+\infty} \mathbb{E}_x \int_0^{t\wedge\tau_n} \chi_{\{|X_s|>R\}} ds &= \mathbb{E}_x \int_0^t \chi_{\{|X_s|>R\}} ds \\
&= \int_0^t \mathbb{E}_x \chi_{\{|X_s|>R\}} ds \\
&= t\, r(t, x; \mathbb{R}^N \setminus B(R)).
\end{aligned}$$

Thus, letting n tend to $+\infty$ in (8.1.20), we obtain

$$r(t, x; \mathbb{R}^N \setminus B(R)) \leq \frac{K + \varphi(x)}{M_R},$$

for any $t > 1$. Since $\lim_{R\to+\infty} M_R = +\infty$, it follows that the measures $\{r(t, x; dy), \; t > 1\}$ are tight, for any $x \in \mathbb{R}^N$. ∎

In [116] Khas'minskii theorem is proved in a similar way, taking advantage of the Krylov-Bogoliubov theorem, but without using the formula (2.4.7). For the sake of completeness we give this proof. In Chapters 12 and 13 we will adapt such a proof, to prove the existence of an invariant measure for the semigroups therein considered.

A second proof of Theorem 8.1.20. We are going to prove that for any $t_0>0$ and any $x_0\in\mathbb{R}^N$ the family of measures $\{r(t,x_0;dy): t>t_0\}$ is tight.

Let $\{\psi_n\}\in C^\infty([0,+\infty))$ be a sequence of smooth functions with the following properties:

(i) $\psi_n(t)=t$ for any $t\in[0,n]$;

(ii) $\psi_n(t)=n+\frac{1}{2}$ for any $t\geq n+1$;

(iii) $\psi_n'(t)\in[0,1]$ and $\psi_n''(t)\leq 0$ for any $t\geq 0$.

As it is immediately seen the function $\varphi_n=\psi_n\circ\varphi$ belongs to $D_{\max}(\mathcal{A})$ for any $n\in\mathbb{N}$ (see (2.0.1)). Since the semigroup $\{T(t)\}$ is conservative (see Remark 4.0.3 and Theorem 4.1.3), Propositions 2.3.5 and 2.3.6 imply that the function $(t,x)\mapsto u_n(t,x):=(T(t)\varphi_n)(x)$ is differentiable with respect to t in $[0,+\infty)\times\mathbb{R}^N$ and

$$
\begin{aligned}
D_tu_n(t,x)&=(T(t)\mathcal{A}\varphi_n)(x)\\
&=\int_{\mathbb{R}^N}(\mathcal{A}\varphi_n)p(t,x;dy)\\
&=\int_{\mathbb{R}^N}\left(\psi_n'(\varphi)\mathcal{A}\varphi+\psi_n''(\varphi)\sum_{i,j=1}^N q_{ij}D_i\varphi D_j\varphi\right)p(t,x;dy), \qquad (8.1.21)
\end{aligned}
$$

for any $t\in[0,+\infty)$, any $x\in\mathbb{R}^N$ and any $n\in\mathbb{N}$. We now fix $t>0$ and $x\in\mathbb{R}^N$, and integrate (8.1.21) with respect to $s\in[0,t]$. Recalling that $u_n\geq 0$ in $[0,+\infty)\times\mathbb{R}^N$ and $\psi_n''\leq 0$ in $[0,+\infty)$, we get

$$
\begin{aligned}
-\varphi_n(x)&\leq\int_0^t ds\int_{\mathbb{R}^N}\psi_n'(\varphi)(\mathcal{A}\varphi)p(s,x;dy)\\
&=\int_0^t ds\int_E\psi_n'(\varphi)(\mathcal{A}\varphi)p(s,x;dy)\\
&\quad+\int_0^t ds\int_{\mathbb{R}^N\setminus E}\psi_n'(\varphi)(\mathcal{A}\varphi)p(s,x;dy), \qquad (8.1.22)
\end{aligned}
$$

where $E=\{y\in\mathbb{R}^N:\mathcal{A}\varphi(y)\geq 0\}$. Since E is a bounded set, the dominated convergence theorem implies that the first integral term in the last side of (8.1.22) converges to

$$
\int_0^t ds\int_E(\mathcal{A}\varphi)p(s,x;dy),
$$

as n tends to $+\infty$. Further, since the sequence $\{\psi_n'(s)\}$ is increasing to 1 for any $s\in[0,+\infty)$, and $\mathcal{A}\varphi\leq 0$ in $\mathbb{R}^N\setminus E$, using the monotone convergence theorem, it is immediate to check that the second integral term converges to

$$\int_0^t ds \int_{\mathbb{R}^N \setminus E} (\mathcal{A}\varphi)p(s,x;dy),$$

which is finite due to (8.1.22). Therefore, again from (8.1.22), we get

$$-\varphi(x) \le \int_0^t ds \int_{\mathbb{R}^N} (\mathcal{A}\varphi)p(s,x;dy). \tag{8.1.23}$$

Now, fix ε and let $\rho > 0$ be such that $\mathcal{A}\varphi < -\varepsilon^{-1}$ outside the ball $B(\rho)$. From (8.1.23) we obtain that

$$\begin{aligned}
-\varphi(x) &\le \int_0^t ds \int_{B(\rho)} (\mathcal{A}\varphi)p(s,x;dy) + \int_0^t ds \int_{\mathbb{R}^N \setminus B(\rho)} (\mathcal{A}\varphi)p(s,x;dy) \\
&\le Mt - \frac{1}{\varepsilon}\int_0^t p(s,x;\mathbb{R}^N \setminus B(\rho))ds \\
&= Mt - \frac{t}{\varepsilon} r(t,x;\mathbb{R}^N \setminus B(\rho)),
\end{aligned}$$

for any $x \in \mathbb{R}^N$, where $M = \sup_{\mathbb{R}^N} \mathcal{A}\varphi$. Consequently,

$$r(t,x,\mathbb{R}^N \setminus B(\rho)) \le \varepsilon \frac{Mt + \varphi(x)}{t},$$

which, of course, implies that for any $t_0 > 0$ and any $x \in \mathbb{R}$ the family of measures $\{r(t,x;dy) : t > t_0\}$ is tight. ∎

8.1.3 Existence by compactness in $C_b(\mathbb{R}^N)$

The compactness of $\{T(t)\}$ in $C_b(\mathbb{R}^N)$, which we studied in Section 5.1, implies the existence of an invariant measure as well as other remarkable properties. To begin with, we consider the following lemma.

Lemma 8.1.21 *If for some $t_0 > 0$ and some $x_0 \in \mathbb{R}^N$ the family of measures $\{p(t,x_0;dy), t > t_0\}$ is tight, then the family of measures $\{r(t,x_0;dy),\ t > t_0\}$ is tight, as well.*

Proof. For any $t > t_0$ we have

$$\begin{aligned}
r(t,x_0;B) &= \frac{1}{t}\int_0^{t_0} p(s,x_0;B)ds + \frac{1}{t}\int_{t_0}^t p(s,x_0;B)ds \\
&\le \sup_{0\le s\le t_0} p(s,x_0;B) + \sup_{s>t_0} p(s,x_0;B).
\end{aligned}$$

Therefore, it is sufficient to prove that the family of measures $\{p(t,x_0;dy),\ t\in[0,t_0]\}$ is tight. This follows from the Prokhorov theorem 8.1.18. Indeed, consider a sequence $\{p(t_n,x_0;dy)\}$ where $t_n\in[0,t_0]$ for any $n\in\mathbb{N}$. Up to a subsequence, $\{t_n\}$ converges to some $t^*\in[0,t_0]$. Taking Theorem 2.2.5 into account, we get

$$\begin{aligned}\lim_{n\to+\infty}\int_{\mathbb{R}^N} f(y)p(t_n,x_0;dy) &= \lim_{n\to+\infty}(T(t_n)f)(x_0)\\ &=(T(t^*)f)(x_0)\\ &=\int_{\mathbb{R}^N} f(y)p(t^*,x_0;dy),\end{aligned}$$

for any $f\in C_b(\mathbb{R}^N)$. Therefore, the sequence $\{p(t_n,x_0;dy)\}$ is weakly* convergent to the measure $p(t^*,x_0;dy)$ in the dual space $C_b(\mathbb{R}^N)'$. ∎

Theorem 8.1.22 *If $T(t)$ is compact in $C_b(\mathbb{R}^N)$ for any $t>0$, then there exists an invariant measure of $\{T(t)\}$. Moreover, for any $p\in(1,+\infty)$, $T(t)$ and $R(\lambda,L_p)$ are compact operators in L^p_μ for any $t>0$ and any $\lambda>0$, and the spectrum of L_p consists of isolated eigenvalues.*

Proof. Fix $t_0>0$, $x_0\in\mathbb{R}^N$ and $\varepsilon>0$. Since $\{T(t)\}$ is conservative, by Proposition 5.1.3 there exists $R>0$ such that

$$p(t_0,y;\mathbb{R}^N\setminus B(R))\le\varepsilon,\qquad y\in\mathbb{R}^N.$$

Recalling that $p(t,x;dy)$ is a probability measure for any $t>0$ and any $x\in\mathbb{R}^N$, it follows that, for any $t>t_0$,

$$p(t,x_0;\mathbb{R}^N\setminus B(R))=\int_{\mathbb{R}^N}p(t_0,y;\mathbb{R}^N\setminus B(R))p(t-t_0,x_0;dy)\le\varepsilon,$$

namely, the family of measures $\{p(t,x_0;dy),\ t>t_0\}$ is tight. Then, the existence of an invariant measure of $\{T(t)\}$ follows from Theorem 8.1.19 and Lemma 8.1.21.

We now fix an arbitrary $t>0$ and we prove that $T(t)$ is compact in L^p_μ. For this purpose, we observe that, since μ is equivalent to the Lebesgue measure on the σ-algebra of the Borel sets of $\mathbb{R}^N$ (see Proposition 8.1.5), then $L^\infty_\mu\subset L^\infty(\mathbb{R}^N,dx)$. Moreover, since $\{T(t)\}$ is strong Feller (see Proposition 2.2.12), then it maps L^∞_μ into $C_b(\mathbb{R}^N)$. Therefore, writing $T(t)=T(t/2)\circ T(t/2)$ and taking Proposition A.1.1(ii) into account, it is immediate to check that $T(t)$ is compact from L^∞_μ into $C_b(\mathbb{R}^N)$ and, since $C_b(\mathbb{R}^N)$ is continuously embedded in L^∞_μ, $T(t)$ is compact from L^∞_μ into itself.

Let now $\overline{B}(1)$ be the closed ball in L^∞_μ with centre at 0 and radius 1 and fix $t\in(0,+\infty)$. Since $T(t)(\overline{B}(1))$ is compact in L^∞_μ and the set of all the bounded simple functions is dense in L^∞_μ then, for any $\varepsilon > 0$, there exists a finite number of simple functions $f_1, \ldots, f_m \in L^\infty_\mu$ ($m \in \mathbb{N}$) such that

$$T(t)(\overline{B}(1)) \subset \bigcup_{j=1}^{m} \{f \in L^\infty_\mu : \|f - f_j\|_{L^\infty_\mu} \le \varepsilon\}.$$

Without loss of generality, we can assume that $f_j = \sum_{i=1}^k \alpha_i^{(j)} \chi_{F_i}$ where F_i are disjoint measurable sets with positive measures such that $\sum_{i=1}^k F_i = \mathbb{R}^N$ and $\alpha_i^{(j)} \in \mathbb{R}$ for any $i = 1, \ldots, k$ and any $j = 1, \ldots, M$. We now consider the operator P_ε defined on measurable functions by

$$P_\varepsilon f = \sum_{i=1}^{k} \left(\frac{1}{\mu(F_j)} \int_{F_j} f d\mu \right) \chi_{F_j}.$$

As it is easily seen,

$$\|P_\varepsilon\|_{L(L^\infty_\mu)} = \|P_\varepsilon\|_{L(L^1_\mu)} = 1,$$

and $P_\varepsilon f_j \equiv f_j$ for any $j = 1, \ldots, N$. Fix $f \in L^\infty_\mu$ such that $\|f\|_{L^\infty_\mu} = 1$ and let $j \in \{1, \ldots, m\}$ be such that $\|T(t)f - f_j\|_{L^\infty_\mu} \le \varepsilon$. Then,

$$\|P_\varepsilon T(t)f - T(t)f\|_{L^\infty_\mu} \le \|P_\varepsilon(T(t)f - f_j)\|_{L^\infty_\mu} + \|f_j - T(t)f\|_{L^\infty_\mu} \le 2\varepsilon.$$

Therefore,

$$\|P_\varepsilon T(t) - T(t)\|_{L(L^\infty_\mu)} \le 2\varepsilon. \tag{8.1.24}$$

Similarly,

$$\|P_\varepsilon T(t) - T(t)\|_{L(L^1_\mu)} \le \|P_\varepsilon T(t)\|_{L(L^1_\mu)} + \|T(t)\|_{L(L^1_\mu)} \le 2\|T(t)\|_{L(L^1_\mu)}. \tag{8.1.25}$$

Hence, by (8.1.24), (8.1.25) and the Riesz-Thorin interpolation theorem (see Theorem A.4.9), we get

$$\begin{aligned}\|P_\varepsilon T(t) - T(t)\|_{L(L^p_\mu)} &\le \|P_\varepsilon T(t) - T(t)\|_{L(L^\infty_\mu)}^{1-\frac{1}{p}} \|P_\varepsilon T(t) - T(t)\|_{L(L^1_\mu)}^{\frac{1}{p}} \\ &\le 2\|T(t)\|_{L(L^1_\mu)}^{\frac{1}{p}} \varepsilon^{1-\frac{1}{p}},\end{aligned}$$

for any $p \in (1, +\infty)$. Since the operator $P_\varepsilon T(t)$ has finite rank for any $\varepsilon > 0$, then it is compact in L^p_μ and, consequently, $T(t)$ is compact in L^p_μ as well.

Now, we show that $R(\lambda, L_p)$ is compact for any $\lambda \in \rho(L_p)$. For this purpose, we prove that the semigroup $\{T(t)\}$ is norm continuous in $(0, +\infty)$. Indeed, once this property is checked, taking (B.1.3) into account, we will get

$$R(\lambda, L_p) = \int_0^{+\infty} e^{-\lambda t} T(t) dt,$$

for any λ with positive real part, where the integral is meant in L^p_μ. Proposition A.1.1(i) then will allow us to conclude that $R(\lambda, L_p)$ is compact for any λ as above. Then the resolvent identity (see the formula (A.3.2)), together with Proposition A.1.1(ii), will yield the compactness of $R(\lambda, L_p)$ for any $\lambda \in \rho(L_p)$.

So, let us prove that $\{T(t)\}$ is norm-continuous in $(0,+\infty)$. Let us fix $t_0 > 0$ and let us set $K = \overline{T(t_0)(B(1))}$, where now $B(1)$ denotes the open unit ball in L^p_μ with centre at 0. Since $T(t_0)$ is a compact operator, then K is compact in L^p_μ. According to Remark B.1.2, $T(t)f$ tends to f as t tends to 0^+, uniformly with respect to f on compact subsets of L^p_μ. Therefore,

$$\lim_{t\to 0^+} T(t+t_0)f = T(t_0)f,$$

uniformly with respect to $f \in B(1)$, and this implies that $T(t)$ tends to $T(t_0)$ in $L(L^p_\mu)$ from the right. Similarly, if $t < 0$, we can fix $\delta \in (0, t_0)$ and write

$$\begin{aligned}\|T(t_0+t)f - T(t_0)f\|_{L^p_\mu} &\le \|T(t_0+t-\delta)(T(\delta)f - T(\delta-t)f)\|_{L^p_\mu}\\ &\le \|T(\delta)f - T(\delta-t)f\|_{L^p_\mu}.\end{aligned} \tag{8.1.26}$$

Letting t tend to 0^-, we deduce that the last side of (8.1.26) vanishes. Hence, $T(\cdot)f$ is norm continuous in $(0,+\infty)$.

Finally, we observe that, since $R(1, L_p)$ is compact (and invertible), then its spectrum consists of isolated eigenvalues. Therefore, the spectrum of L_p consists of isolated eigenvalues as well. ∎

Remark 8.1.23 Applying [7, Proposition 2.6], one can also show that, under the assumptions of Theorem 8.1.22, the spectrum of the operator L_p is independent of p and it coincides with the spectrum of $(A, D_{\max}(\mathcal{A}))$ in $C_b(\mathbb{R}^N)$ (see (2.0.1)).

To conclude this subsection, we show that, under the assumptions of Theorem 8.1.22, the spectral gap for the operator L_p holds for any $p \in (1,+\infty)$, i.e., there exists $\delta > 0$ such that $\sigma(L_p) \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < -\delta\}$.

Theorem 8.1.24 *Under the assumptions of Theorem* 8.1.22*, the spectral gap holds for the operator L_p, for any $p \in (1,+\infty)$. Moreover, $T(t)f$ tends exponentially to $\overline{f}$ in L^2_μ, as t tends to $+\infty$.*

Proof. Fix $p \in (1,+\infty)$ and denote, as in Subsection 8.1.1, by P_∞ the operator defined in L^p_μ by

$$P_\infty f = \int_{\mathbb{R}^N} f d\mu, \qquad f \in L^p_\mu.$$

Since P_∞ is a projection, then $L^p_\mu = H \oplus K$, where $H = \mathrm{Ker}(P_\infty)$ and $K = \mathrm{Ker}(I - P_\infty)$. As it is immediately seen, $T(t)$ commutes with P_∞ for any $t > 0$. Therefore, H and K are both invariant under the action of the semigroup and the restrictions of the semigroup to H and K give rise to two strongly continuous semigroups, whose infinitesimal generators are the restrictions of L_p to H and K, respectively. In particular, for any $t > 0$, $T(t)$ coincides with the identity operator on K. As a consequence, for any $f \in L^p_\mu$ such that $f = f_1 + f_2$ with $f_1 \in H$ and $f_2 \in K$, we have

$$T(t)f = T(t)_{|H} f_1 + f_2.$$

Now, we are going to prove that $T(t)_{|H}$ decreases exponentially as t tends to $+\infty$. As a first step, we claim that $\sigma(T(1)_{|H}) \subset B(1)$. Since $\{T(t)\}$ is a semigroup of contractions, then $\sigma(T(1)) \subset \overline{B}(1)$. Let us show that, if $\lambda \in \partial B(1)$, then $\lambda \in \rho(T(1)_{|H})$. On the contrary, we assume that there exists $\lambda \in \sigma(T(1)_{|H}) \cap \partial B(1)$. Since $T(1)$ is a compact operator, λ belongs to the point spectrum of $T(1)$. Therefore, there exists $f \in H$, with $f \neq 0$, such that $T(1)f = \lambda f$. Using the semigroup rule, we obtain that $T(n)f = \lambda^n f$ for any $n \in \mathbb{N}$. From Theorem 8.1.16 it follows that $T(n)f$ tends to $P_\infty f = 0$ in L^p_μ. Therefore, $\lambda^n f$ should tend to 0 as n tends to $+\infty$, but, of course, this cannot be the case. The claim follows and, in particular, we deduce that there exists $\delta < 1$ such that

$$\lim_{n\to+\infty} \|T(1)^n\|_{L(H)}^{\frac{1}{n}} = \lim_{n\to+\infty} \|T(n)\|_{L(H)}^{\frac{1}{n}} = \delta.$$

Using the semigroup law it is now immediate to check that there exists $\omega < 0$ such that

$$\|T(t)\|_{L(H)} \le Ce^{\omega t}, \qquad t > 0.$$

Since $\{T(t)_{|H}\}$ is a strongly continuous semigroup in H, then $\inf\{\mathrm{Re}\lambda : \lambda \in \sigma((L_p)_{|H})\} \le \omega$ (see Section B.1).

To conclude the proof of the first part of the theorem, it suffices to show that

$$\sigma(L_p) = \sigma((L_p)_{|H}) \cup \sigma((L_p)_{|K}) = \sigma((L_p)_{|H}) \cup \{0\}. \tag{8.1.27}$$

Note that the second equality in (8.1.27) is immediately checked since $T(t)_{|K} = I$ for any $t > 0$.

As a first step to prove (8.1.27), we observe that $0 \in \sigma(L)$ and $0 \notin \sigma((L_p)_{|H})$. Indeed, Theorem 8.1.17 implies that $Lu = 0$ if and only if u is constant. Therefore, $0 \in \sigma(L_p)$. But, since 0 is the only constant in H, then $0 \notin \sigma((L_p)_{|H})$.

Now, let $\lambda \in \rho(L)$, $\lambda \neq 0$, and let $f \in H$. Set $u = R(\lambda, L_p)f$, so that $\lambda u - L_p u = f$. Since $\overline{f} = 0$ and $\overline{L_p u} = 0$, by (8.1.5), then $\overline{u} = 0$ as well, so that $u \in H$. This implies that $\lambda \in \rho((L_p)_{|H})$ and $R(\lambda,(L_p)_{|H}) = R(\lambda, L_p)_{|H}$. Therefore, $\sigma((L_p)_{|H}) \cup \{0\} \subset \sigma(L_p)$.

To prove the other inclusion in (8.1.27), we fix $\lambda \in \rho((L_p)_{|H})$, $\lambda \neq 0$, and we show that $\lambda \in \rho(L_p)$. For this purpose, we observe that, for any $f \in L^p_\mu$, the function $u = R(\lambda,(L_p)_{|H})(f - \overline{f}) + \lambda^{-1}\overline{f}$ is a solution to the equation $\lambda u - L_p u = f$. Note that $\overline{f} \in D(L_p)$, since the constants belong to $D_{\max}(\mathcal{A})$, which is a core of L_p (see Propositions 2.3.6, 4.1.10 and 8.1.9). Hence, the operator $\lambda I - L_p$ is surjective in L^p_μ.

To prove that $\lambda I - L_p$ is also one to one, we observe that, since P_∞ commutes with $T(t)$ for any $t > 0$, then it commutes with L_p on $D(L_p)$. Therefore, if $\lambda u = L_p u$ for some $u \in D(L_p)$, then both $u_1 = P_\infty u$ and $u_2 = (I - P_\infty)u$ solve the equation $\lambda v = L_p v$. Since $\lambda \notin \sigma((L_p)_{|H}) \cup \sigma((L_p)_{|K})$, then $u_1 \equiv u_2 \equiv 0$ and, consequently, $u \equiv 0$.

To conclude the proof of the theorem, we observe that the function $f - \overline{f}$ belongs to H for any $f \in L^2_\mu$. Therefore,

$$\|T(t)f - \overline{f}\|_2 = \|T(t)f - T(t)\overline{f}\|_2 \leq 2Ce^{\omega t}\|f\|_2, \qquad t > 0,$$

and we are done. ∎

8.1.4 Existence by symmetry

In this subsection we study the case when the operator $\mathcal{A}$ is given by

$$\mathcal{A}u(x) = \Delta u(x) - \langle DU(x) + G(x), Du(x)\rangle, \qquad x \in \mathbb{R}^N, \tag{8.1.28}$$

on smooth functions, under the following hypotheses on U and G.

Hypotheses 8.1.25 (i) $U : \mathbb{R}^N \to \mathbb{R}$ belongs to $C^{1+\alpha}_{\rm loc}(\mathbb{R}^N)$, for some $\alpha \in (0,1)$, and $e^{-U} \in L^1(\mathbb{R}^N)$;

(ii) $G \in C^1(\mathbb{R}^N,\mathbb{R}^N)$, $\operatorname{div} G = \langle G, DU\rangle$ and

$$\int_{\mathbb{R}^N} |G(x)|e^{-U(x)}dx < +\infty.$$

Under Hypotheses 8.1.25 we can define the probability measure

$$\mu(dx) = K^{-1}e^{-U(x)}dx, \qquad K = \int_{\mathbb{R}^N} e^{-U(x)}dx.$$

By means of an integration by parts, we get

$$\int_{\mathbb{R}^N} \mathcal{A}f\, g\, d\mu = -\int_{\mathbb{R}^N} \langle Df, Dg\rangle d\mu, \qquad f \in C^\infty_c(\mathbb{R}^N),\quad g \in W^{1,p}_{\rm loc}(\mathbb{R}^N), \tag{8.1.29}$$

for any $p \in [1,+\infty)$. In particular,

$$\int_{\mathbb{R}^N} \mathcal{A}f\,d\mu = 0, \qquad f \in C_c^\infty(\mathbb{R}^N). \tag{8.1.30}$$

The main result of the section is the following theorem.

Theorem 8.1.26 *Assume Hypotheses* 8.1.25*. Then, the semigroup $\{T(t)\}$ associated with the operator $\mathcal{A}$ in $C_b(\mathbb{R}^N)$ is conservative, μ is the invariant measure of $\{T(t)\}$, and $C_c^\infty(\mathbb{R}^N)$ is a core of the infinitesimal generator L_p in L^p_μ for any $p \in [1,+\infty)$.*

Proof. For any $p \in [1,+\infty)$ consider the operator $A_p : C_c^\infty(\mathbb{R}^N) \to L^p_\mu$ defined by $A_p u = \mathcal{A}u$ for any $u \in C_c^\infty(\mathbb{R}^N)$. As a first step we prove that A_p is dissipative, that is

$$\lambda\|u\|_{L^p_\mu} \le \|\lambda u - A_p u\|_{L^p_\mu}, \qquad \lambda > 0, \quad u \in D(A_p). \tag{8.1.31}$$

For this purpose, let $u \in D(A_p)$ and set $f = \lambda u - A_p u$. Multiplying both the sides of the previous equality by $\mathrm{sign}(u)|u|^{p-1}$, and integrating in $\mathbb{R}^N$, we get

$$\begin{aligned}\lambda\int_{\mathbb{R}^N} |u|^p d\mu - \int_{\mathbb{R}^N} A_p u\,\mathrm{sign}(u)|u|^{p-1}d\mu &= \int_{\mathbb{R}^N} f\,\mathrm{sign}(u)|u|^{p-1}d\mu\\ &\le \int_{\mathbb{R}^N} |f||u|^{p-1}d\mu. \end{aligned}\tag{8.1.32}$$

Now, if $p \ge 2$ integrating by parts we get

$$\begin{aligned}\int_{\mathbb{R}^N} A_p u\,\mathrm{sign}(u)|u|^{p-1}d\mu &= -\int_{\mathbb{R}^N} \langle Du, D(\mathrm{sign}(u)|u|^{p-1})\rangle d\mu\\ &= -(p-1)\int_{\mathbb{R}^N} |Du|^2|u|^{p-2}d\mu \le 0. \end{aligned}\tag{8.1.33}$$

Hence, from (8.1.32) and (8.1.33) we easily get (8.1.31).

In the case when $p \in [1,2)$ we consider the sequence $\{\varphi_n\} \subset C^\infty(\mathbb{R})$ defined by $\varphi_n(x) = x(x^2+n^{-1})^{(p-2)/2}$ for any $x \in \mathbb{R}^N$. Since any φ_n is an increasing function in $\mathbb{R}^N$, converging pointwise to $\mathrm{sign}(x)|x|^{p-1}$ as n tends to $+\infty$, by dominated convergence and by (8.1.29) we have

$$\begin{aligned}\int_{\mathbb{R}^N} A_p u\,\mathrm{sign}(u)|u|^{p-1}d\mu &= \lim_{n\to+\infty}\int_{\mathbb{R}^N} A_p u\,\varphi_n(u)d\mu\\ &= -\lim_{n\to+\infty}\int_{\mathbb{R}^N} \varphi_n'(u)|Du|^2 d\mu \le 0,\end{aligned}$$

and, then, (8.1.31) follows from (8.1.32).

As a second step we show that the range of $\lambda I - A_p$ is dense in L^p_μ for some $\lambda > 0$. To prove this, let $g \in L^q_\mu(\mathbb{R}^N)$ (where $1/p + 1/q = 1$, or $q = +\infty$ if $p = 1$) be such that

$$\int_{\mathbb{R}^N} (\lambda u - A_p u) g d\mu = 0,$$

for any $u \in D(A_p)$. First we assume that $p \in (1,+\infty)$. By classical regularity results for weak solutions of elliptic equations (see Theorem C.1.3(ii)), g belongs to $W^{1,q}_{\rm loc}(\mathbb{R}^N)$. If $p = 1$, then $g \in L^2_{\rm loc}(\mathbb{R}^N)$. Therefore, still from Theorem C.1.3(ii), we deduce that $g \in W^{1,2}_{\rm loc}(\mathbb{R}^N)$. Now, from the formula (8.1.29) we get

$$\lambda \int_{\mathbb{R}^N} ug\, d\mu + \int_{\mathbb{R}^N} \langle Du, Dg\rangle d\mu = 0, \qquad u \in C^\infty_c(\mathbb{R}^N). \tag{8.1.34}$$

By density, (8.1.34) can be extended to any compactly supported function $u \in W^{1,p}(\mathbb{R}^N)$, if $p \in (1,+\infty)$ and to any compactly supported function $u \in W^{1,2}(\mathbb{R}^N)$, if $p = 1$. Now, for any $n \in \mathbb{N}$, let $\eta_n \in C^\infty_c(\mathbb{R}^N)$ be a smooth function such that $\eta_n \equiv 1$ in $B(n)$, $0 \le \eta_n \le \eta_{n+1}$, and $|D\eta_n| \le 1/n$. If $1 < p \le 2$, taking $u = \eta_n^2 \mathrm{sign}(g)|g|^{q-1}$ in (8.1.34), we get

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^N} \eta_n^2 |g|^q d\mu + (q-1)\int_{\mathbb{R}^N} \eta_n^2 |g|^{q-2}|Dg|^2 d\mu \\
&= -2\int_{\mathbb{R}^N} \eta_n \mathrm{sign}(g)|g|^{q-1}\langle D\eta_n, Dg\rangle d\mu \\
&\le \frac{1}{n}\int_{\mathbb{R}^N} \eta_n^2 |Dg|^2 |g|^{q-2} d\mu + \frac{1}{n}\|g\|_q^q.
\end{aligned}$$

Hence,

$$\lambda \int_{\mathbb{R}^N} \eta_n^2 |g|^q d\mu \le \frac{1}{n}\|g\|^q_{L^q_\mu},$$

for any $n \ge 2$. Letting n tend to $+\infty$, by monotone convergence we get $g = 0$.

The case when $p = 1$ can be treated as the case when $p = 2$. Suppose now that $p > 2$ and, for any $n, m \in \mathbb{N}$, let $u_{n,m} = \eta_n^2 \varphi_m(g)$, where the sequence $\{\varphi_n\}$ is as above with p being replaced by q. Then, $u_{n,m} \in W^{1,p}(\mathbb{R}^N)$ and has compact support for any $n, m \in \mathbb{N}$. Writing (8.1.34) with $u_{n,m}$ instead of u, we get

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^N} \eta_n^2 g^2 \left(g^2 + \frac{1}{m}\right)^{\frac{q-2}{2}} d\mu \\
&+(q-1)\int_{\mathbb{R}^N} \eta_n^2 g^2 \left(g^2 + \frac{1}{m}\right)^{\frac{q-4}{2}} |Dg|^2 d\mu \\
&\le \frac{2}{n}\int_{\mathbb{R}^N} \eta_n |g| \left(g^2 + \frac{1}{m}\right)^{\frac{q-2}{2}} |Dg| d\mu.
\end{aligned} \tag{8.1.35}$$

Letting m tend to $+\infty$ in (8.1.35), we get

$$\lambda\int_{\mathbb{R}^N}\eta_n^2|g|^q d\mu+(q-1)\int_{\mathbb{R}^N}\eta_n^2|g|^{q-2}|Dg|^2 d\mu \\ \le \frac{2}{n}\int_{\mathbb{R}^N}\eta_n|g|^{q-1}|Dg|d\mu. \qquad (8.1.36)$$

Since the right-hand side of (8.1.36) is finite, the second integral in the left-hand side is finite as well. Therefore, applying the Cauchy-Schwarz inequality gives

$$\lambda\int_{\mathbb{R}^N}\eta_n^2|g|^q d\mu+(q-1)\int_{\mathbb{R}^N}\eta_n^2|g|^{q-2}|Dg|^2 d\mu \\ \le \frac{1}{n}\|g\|_q^q+\frac{1}{n}\int_{\mathbb{R}^N}\eta_n^2|g|^{q-2}|Dg|^2 d\mu.$$

Hence, for $n(q-1)\ge 1$, we get

$$\lambda\int_{\mathbb{R}^N}\eta_n^2|g|^q d\mu\le\frac{1}{n}\|g\|_q^q$$

and, letting n go to $+\infty$, we obtain $g=0$ also in this case.

Now, according to Proposition B.1.8, the closure $\overline{A_p}$ of the operator A_p in L^p_μ is the generator of a strongly continuous semigroup $\{S(t)\}$. By density, (8.1.30) implies that

$$\int_{\mathbb{R}^N}\overline{A_p}f\,d\mu=0, \qquad f\in D(\overline{A_p}).$$

Thus, for any $f\in D(\overline{A_p})$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^N}S(t)f\,d\mu=\int_{\mathbb{R}^N}\overline{A_p}S(t)f\,d\mu=0, \qquad t>0,$$

so that

$$\int_{\mathbb{R}^N}S(t)f\,d\mu=\int_{\mathbb{R}^N}f\,d\mu, \qquad t>0. \qquad (8.1.37)$$

Since $D(\overline{A_p})$ is dense in L^p_μ, the equality (8.1.37) holds for any $f\in L^p_\mu$.

Next, we prove that the semigroup $\{T(t)\}$ can be extended to L^p_μ with a strongly continuous semigroup of contractions and, then, that the so extended semigroup coincides with $\{S(t)\}$. Note that $\{T(t)\}$ is well defined since the coefficients of the operator $\mathcal{A}$ satisfy Hypotheses 8.1.1. For this purpose, for any $R>0$ we denote by $\{T_R(t)\}$ the semigroup associated with the realization of the operator $\mathcal{A}$ with homogeneous Dirichlet boundary conditions in $C(\overline{B}(R))$. Let us observe that the measure μ is a subinvariant measure of $\{T_R(t)\}$ for any $R>0$, that is

$$\int_{B(R)}T_R(t)f\,d\mu\le\int_{B(R)}f\,d\mu, \qquad t>0, \qquad (8.1.38)$$

for any positive function $f \in C_0(B(R))$. Indeed, differentiating the left-hand side of (8.1.38) with respect to t and integrating by parts we get

$$\begin{aligned}\int_{B(R)} \frac{d}{dt}T_R(t)f\,d\mu &= \int_{B(R)} \mathcal{A}T_R(t)f\,d\mu\\ &= K^{-1}\int_{\partial B(R)} \left(\frac{\partial}{\partial\nu}T_R(t)f\right) e^{-U(y)}\,\sigma(dy),\end{aligned}$$

where ν is the unit outward normal vector to $\partial B(R)$ and $\sigma(dy)$ is the surface measure on $\partial B(R)$. Since $T_R(t)f$ is nonnegative in $B(R)$ and it vanishes on $\partial B(R)$ (see Proposition C.3.2), we have $\partial T_R(t)f/\partial\nu \le 0$, and then (8.1.38) follows.

Since $T_R(t)f$ converges to $T(t)f$ in a dominated way, then, letting R tend to infinity, the inequality (8.1.38) holds also when $T_R(t)f$ is replaced with $T(t)f$, for any nonnegative function $f \in C_c(\mathbb{R}^N)$ and $B(R)$ is replaced by $\mathbb{R}^N$. Using an approximation argument we can show that (8.1.38) holds for any nonnegative function $f \in C_b(\mathbb{R}^N)$. Arguing as in the proof of Proposition 8.1.8, we can extend $\{T(t)\}$ to a strongly continuous semigroup in L^p_μ for any $p \in [1,+\infty)$.

Now, let L_p be the generator of $\{T(t)\}$ in L^p_μ. Since $\overline{A_p}$ and L_p coincide with $\mathcal{A}$ on $C_c^\infty(\mathbb{R}^N)$, which is a core of $\overline{A_p}$, then L_p is an extension of $\overline{A_p}$. But since $\overline{A_p}$ and L_p are both generators of strongly continuous semigroups of contractions, then for any $\lambda > 0$ we have

$$L^p_\mu = (\lambda - \overline{A_p})(D(\overline{A_p})) = (\lambda - L_p)(D(\overline{A_p})) \subset (\lambda - L_p)(D(L_p)) = L^p_\mu,$$

which implies that $(\lambda - L_p)(D(\overline{A_p})) = (\lambda - L_p)(D(L_p))$ or, equivalently, that $D(\overline{A_p}) = D(L_p)$, so that $\overline{A_p}$ and L_p coincide. Therefore, $S(t) \equiv T(t)$ in L^p_μ for any $t > 0$ (see Proposition B.1.4). Thus, by (8.1.37), μ is the invariant measure of $\{T(t)\}$. Hence, by Proposition 8.1.10, it follows that $\{T(t)\}$ is conservative. ■

Proposition 8.1.27 *$D(L)$ is continuously and densely embedded in $W^{1,2}_\mu$. Further,*

$$\int_{\mathbb{R}^N} \langle Dg, Dh\rangle d\mu = -\int_{\mathbb{R}^N} Lg\,h\,d\mu, \qquad g \in D(L),\quad h \in W^{1,2}_\mu. \tag{8.1.39}$$

Finally, for any $\lambda > 0$,

$$\|DR(\lambda,L)f\|_2 \le \sqrt{\frac{2}{\lambda}}\|f\|_2, \qquad f \in L^2_\mu. \tag{8.1.40}$$

Proof. The proof of the first part of the assertion is a straightforward consequence of the formula (8.1.29) and Theorem 8.1.26. Indeed, fix $\psi \in D(L)$. Since $C_c^\infty(\mathbb{R}^N)$ is a core of $D(L)$, there exists a sequence $\{\psi_n\} \subset C_c^\infty(\mathbb{R}^N)$ such that ψ_n and $L\psi_n$ tend, respectively, to ψ and $L\psi$ as n tends to $+\infty$. Writing (8.1.29) with $f = g = \psi_n - \psi_m$ ($n, m \in \mathbb{N}$) we get

$$\|D\psi_n - D\psi_m\|_2^2 \le \|\psi_n - \psi_m\|_2 \|L\psi_n - L\psi_m\|_2.$$

Therefore, $\{\psi_n\}$ is a Cauchy sequence in $W_\mu^{1,2}$. Since it converges to ψ in L_μ^2, we deduce that $\psi \in W_\mu^{1,2}$ and, then, writing (8.1.29) with $f = g = \psi$, we obtain that $D(L)$ is continuously embedded in $W_\mu^{1,2}$.

To conclude the proof of the first part of the proposition, we have to show that $D(L)$ is dense in $W_\mu^{1,2}$. But this follows immediately from the forthcoming Proposition 8.7.3.

Now, the formula (8.1.39) can be easily proved by a density argument, recalling that $L\varphi = \mathcal{A}\varphi$ for any $\varphi \in C_c^\infty(\mathbb{R}^N)$.

Finally, to prove the formula (8.1.40) it suffices to apply the formula (8.1.39) with $g = h = R(\lambda, L)f$, recalling that

$$\|R(\lambda, L)f\|_2 \le |\lambda|^{-1}\|f\|_2 \qquad \text{and} \qquad \|LR(\lambda, L)\|_2 \le 2\|f\|_2.$$

■

Remark 8.1.28 In Section 8.4, we will improve the results in Proposition 8.1.27 in the case when the drift term is the gradient of a convex function.

To conclude this section, let us recall the following result.

Proposition 8.1.29 *Under Hypotheses* 8.1.25*, $T(t)f$ belongs to $W_\mu^{1,2}$ for any $f \in L_\mu^2$ and any $t > 0$. Moreover,*

$$\|DT(t)f\|_2^2 \le \frac{1}{2t}\left(\|f\|_2^2 - \|T(t)f\|_2^2\right), \qquad t > 0.$$

Proof. See [62, Lemma 1.3.3]. ■

8.2 Regularity properties of invariant measures

In this section we give some conditions on the coefficients of the operator $\mathcal{A}$ implying global regularity properties of the invariant measure μ.

In Proposition 8.1.5 we have seen that, if the semigroup $\{T(t)\}$ admits an invariant measure μ, then μ is absolutely continuous with respect to the

Lebesgue measure and its density is positive. Now, we specialize to the case when $\mathcal{A}$ is given on smooth functions u by

$$\mathcal{A}u = \sum_{i,j=1}^N D_i(q_{ij}D_j)u + \sum_{i=1}^N b_iD_iu := \mathcal{A}_0u + \sum_{i=1}^N b_iD_iu. \tag{8.2.1}$$

Let us state the main hypotheses on the coefficients q_{ij} and b_i $(i,j=1,\ldots,N)$ which will be assumed throughout this section.

Hypotheses 8.2.1 (i) $q_{ij}=q_{ji}\in C^{1+\alpha}_{\rm loc}(\mathbb{R}^N)$ for any $i,j=1,\ldots,N$ $(N\geq 2)$ and some $\alpha\in(0,1)$ and there exists a positive constant κ_0 such that

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j\geq\kappa_0|\xi|^2, \qquad x,\xi\in\mathbb{R}^N; \tag{8.2.2}$$

(ii) the functions $x\mapsto(1+|x|^2)^{-1}q_{ij}(x)$ and $x\mapsto(1+|x|)^{-1}D_iq_{ij}(x)$ belong to L^1_μ for any $i,j=1,\ldots,N$;

(iii) $b_i\in C^\alpha_{\rm loc}(\mathbb{R}^N)$ for any $i=1,\ldots,N$.

Remark 8.2.2 Hypothesis 8.2.1(ii) is satisfied for instance in the case when the function $x\mapsto(1+|x|^2)^{-1}q_{ij}(x)$ and $x\mapsto(1+|x|)^{-1}D_iq_{ij}(x)$ $(i,j=1,\ldots,N)$ are bounded in $\mathbb{R}^N$. Another sufficient condition for the integrability of the previous functions (with respect to the measure μ) can be easily given under the assumptions of the forthcoming Proposition 8.2.14, which guarantees the integrability of the function $x\mapsto\exp(\delta|x|^\beta)$ for some positive constants β and δ.

In order to present the main results of this section (proved very recently by G. Metafune, D. Pallara and A. Rhandi in [114]) we recall the following theorem, which has been proved in [20, 21] and gives a local regularity property of the function ρ.

Theorem 8.2.3 *Under Hypotheses* 8.2.1(*i*) *and* 8.2.1(*iii*), *the function ρ belongs to $W^{1,p}_{\rm loc}(\mathbb{R}^N)$ for any $p\in[1,+\infty)$. As a consequence, ρ is continuous in $\mathbb{R}^N$.*

Proof. We split the proof into three steps.

Step 1. Here, we show that $\rho\in L^p_{\rm loc}(\mathbb{R}^N)$ for any $p<N/(N-1)$. For this purpose, we begin by observing that, according to Proposition 8.1.2,

$$\int_{\mathbb{R}^N}\sum_{i,j=1}^N q_{ij}D_{ij}\varphi d\mu = -\int_{\mathbb{R}^N}\sum_{i=1}^N \overline{b}_iD_i\varphi d\mu, \qquad \varphi\in C^\infty_c(\mathbb{R}^N), \tag{8.2.3}$$

where

$$\overline{b}_i := b_i + \sum_{j=1}^N D_i q_{ij}, \qquad i = 1, \ldots, N.$$

By density, it is immediate to extend (8.2.3) to any function $\varphi \in C_c^2(\mathbb{R}^N)$.

Now, we fix $R > 0$ and a function $\vartheta \in C_c^\infty(B(R))$ such that $\chi_{B(R/2)} \le \vartheta \le \chi_{B(R)}$. For any $\psi \in C^2(\overline{B}(R))$ we write (8.2.3) with $\varphi = \psi\vartheta$. We get

$$\begin{aligned}\left| \int_{B(R)} \sum_{i,j=1}^N q_{ij} D_{ij}\psi\, \vartheta d\mu \right| \le{}& \left| 2\int_{B(R)} \sum_{i,j=1}^N q_{ij} D_i\vartheta D_j\psi d\mu \right. \\ &+ \int_{B(R)} \psi \sum_{i,j=1}^N q_{ij} D_{ij}\vartheta d\mu + \int_{\mathbb{R}^N} \vartheta \sum_{i=1}^N \overline{b}_i D_i\psi d\mu \\ &\left. + \int_{\mathbb{R}^N} \psi \sum_{i=1}^N \overline{b}_i D_i \vartheta d\mu \right| \\ \le{}& C_1 \|\psi\|_{C^1(B(R))}, \end{aligned} \tag{8.2.4}$$

where C_1 is a positive constant, depending on R, but being independent of ψ.

Let $f \in C_c^\infty(B(R))$ be a smooth function. By [66, Theorem 6.14 and Lemma 9.17], the equation

$$\begin{cases} \displaystyle\sum_{i,j=1}^N q_{ij}(x) D_{ij}u(x) = f(x), & x \in B(R), \\ u(x) = 0, & x \in \partial B(R), \end{cases} \tag{8.2.5}$$

admits a unique solution $u \in C^2(\overline{B}(R))$ satisfying

$$\|u\|_{W^{2,q}(B(R))} \le C_2 \|f\|_{L^q(B(R))},$$

for any $q > N$ and some positive constant $C_2 = C_2(q, R)$, independent of f. Using the Sobolev embedding theorems (see [2, Theorem 5.4]) we deduce that

$$\|u\|_{C^1(\overline{B}(R))} \le C_3 \|f\|_{L^q(B(R))},$$

with C_3 being independent of f. Therefore, plugging $\psi = u$ into (8.2.4), we deduce that

$$\left| \int_{B(R)} f\, \vartheta \rho dx \right| \le C_1 C_3 \|f\|_{L^q(B(R))}.$$

By the arbitrariness of $f \in C_c^\infty(B(R))$ and $q > N$ the function $\rho\vartheta$ belongs to $L^p(B(R))$ for any $p \in [1, N/(N-1))$. Since $\rho\vartheta = \rho$ in $B(R/2)$ and R is arbitrary, the assertion easily follows.

Step 2. We now show that $\rho\in W^{1,p}_{\rm loc}(\mathbb{R}^N)$ for any $p\in[1,N/(N-1))$. So, let us fix $p\in(1,N/(N-1))$ and $M\in\mathbb{N}$. For any $x_0\in\overline{B}(M)$ and any $R>0$ we fix two functions $\eta,\psi\in C^2(x_0+\overline{B}(R))$ such that $\chi_{x_0+B(R/2)}\le\eta\le\chi_{x_0+B(R)}$ and ψ vanishes on $x_0+\partial B(R)$. We are going to prove that we can fix R sufficiently small (depending only on M) such that $\rho\in W^{1,p}(x_0+B(R/2))$ for any $x_0\in B(M)$. The arbitrariness of $x_0\in B(M)$ and $M>0$, then, will imply that $\rho\in W^{1,p}_{\rm loc}(\mathbb{R}^N)$. Without loss of generality, we can assume that $R<1$.

Writing (8.2.3) with $\varphi=\psi\eta$ and using the Poincaré inequality, we get

$$
\begin{aligned}
&\left|\int_{x_0+B(R)}\eta\sum_{i,j=1}^N q_{ij}D_{ij}\psi d\mu\right|\\
&\le 2\left|\int_{x_0+B(R)}\sum_{i,j=1}^N q_{ij}D_i\eta D_j\psi d\mu\right|+\left|\int_{x_0+B(R)}\psi\sum_{i,j=1}^N q_{ij}D_{ij}\eta d\mu\right|\\
&\quad+\left|\int_{x_0+B(R)}\eta\sum_{i=1}^N\overline{b}_iD_i\psi d\mu\right|+\left|\int_{x_0+B(R_0)}\psi\sum_{i=1}^N\overline{b}_iD_i\eta d\mu\right|\\
&\le C_4\int_{x_0+B(R)}(|\psi|+|D\psi|)d\mu\\
&\le C_5\|D\psi\|_{L^{p/(p-1)}(x_0+B(R))},
\end{aligned}
\tag{8.2.6}
$$

for some positive constants C_4 and C_5, depending on R, M and the sup-norm of the coefficients of $\mathcal{A}$ in $B(M+1)$, but independent of ψ and x_0.

For any set of functions $f_i\in C^\infty_c(x_0+B(R))$ $(i=1,\ldots,N)$, let us denote by $u\in C^2(x_0+\overline{B}(R))$ the solution to the elliptic boundary problem (8.2.5) with $\sum_{i=1}^N D_if_i$ instead of f. Plugging $\psi=u$ into (8.2.6), we get

$$
\left|\int_{x_0+B(R)}\eta\sum_{i=1}^N D_if_id\mu\right|\le C_5\|Du\|_{L^{p/(p-1)}(x_0+B(R))}. \tag{8.2.7}
$$

Let us show that

$$
\|Du\|_{L^{p/(p-1)}(x_0+B(R))}\le C_6\sum_{i=1}^N\|D_if_i\|_{W^{-1,p/(p-1)}(x_0+B(R))}, \tag{8.2.8}
$$

for some positive constant C_6, independent of R, x_0 and f, provided that R is sufficiently small. Here, $W^{-1,p/(p-1)}(x_0+B(R))$ denotes the dual of the space $W^{1,p}_0(x_0+B(R))$. From the estimates (8.2.7) and (8.2.8) we will then immediately deduce that $\eta\rho\in W^{1,p}_0(x_0+B(R))$ and, since $\eta=1$ in $x_0+B(R/2)$, that $\rho\in W^{1,p}(x_0+B(R/2))$.

From now on, we denote by C_j positive constants, depending on M, but independent of x_0, R and f.

To prove (8.2.8) we begin by observing that, since

$$\sum_{i,j=1}^N D_i(q_{ij}D_ju) = \sum_{i=1}^N D_if_i + \sum_{i,j=1}^N D_iq_{ij}\, D_ju := g_1 + g_2,$$

then

$$\begin{aligned}&\|Du\|_{L^{p/(p-1)}(x_0+B(R))}\\ &\le C_7\left(\|g_1\|_{W^{-1,p/(p-1)}(x_0+B(R))} + \|g_2\|_{W^{-1,p/(p-1)}(x_0+B(R))}\right);\end{aligned} \tag{8.2.9}$$

see, e.g., [64, Section 4.3] or [80, Section 5.5].

To estimate the norm of the function g_2 we observe that, since $p < N$, then $L^{pN/(N(p-1)+p)}(x_0 + B(R)) \subset W^{-1,p/(p-1)}(x_0 + B(R))$ and

$$\|k\|_{W^{-1,p/(p-1)}(x_0+B(R))} \le C_8\|k\|_{L^{pN/(N(p-1)+p)}(x_0+B(R))}.$$

for any $k \in W^{-1,p/(p-1)}(x_0 + B(R))$. Therefore,

$$\begin{aligned}\|g_2\|_{W^{-1,p/(p-1)}(x_0+B(R))} &\le C_9\|g_2\|_{L^{pN/(N(p-1)+p)}(x_0+B(R))}\\ &\le C_{10}\|h\|_{L^N(x_0+B(R))}\|Du\|_{L^{p/(p-1)}(x_0+B(R))}\\ &\le C_{11}R\|h\|_{L^\infty(B(M+1))}\|Du\|_{L^{p/(p-1)}(x_0+B(R))}.\end{aligned} \tag{8.2.10}$$

Here, $h^2 = \sum_{i,j=1}^N |D_iq_{ij}|^2$. From (8.2.9) and (8.2.10) we deduce that

$$\begin{aligned}&\|Du\|_{L^{p/(p-1)}(x_0+B(R))}\\ &\le C_{12}\left(R\|Du\|_{L^{p/(p-1)}(x_0+B(R))} + \|g_1\|_{W^{-1,p/(p-1)}(x_0+B(R))}\right).\end{aligned} \tag{8.2.11}$$

Taking R sufficiently small, we get the estimate (8.2.8).

Step 3. In this step we conclude the proof, using a bootstrap argument. Since $\rho \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$ for any $p \in [1, N/(N-1))$, the Sobolev embedding theorems imply that $\rho \in L^p_{\rm loc}(\mathbb{R}^N)$ for any $p \in [1, N/(N-2))$. Now, repeating the argument in Step 2, we deduce that $\rho \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$ for any $p \in [1, N/(N-2))$. Iterating this argument, we can show that $\rho \in L^p_{\rm loc}(\mathbb{R}^N)$ for any $p \in [1, +\infty)$.

Now, we observe that we can adapt the arguments in Step 2 to prove that $\rho \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$ also in the case when $p > N$. As it is immediately seen, we just need to modify the estimate (8.2.10). For this purpose, we observe that, if $p > N$, then $L^1(x_0 + B(R))$ is continuously embedded in $W^{-1,p/(p-1)}(x_0 + B(R))$ and

$$\|k\|_{W^{-1,p/(p-1)}(x_0+B(R))} \le C_{13}R^{\frac{p-N}{p}}\|k\|_{L^1(x_0+B(R))},$$

for any $k \in W^{-1,p/(p-1)}(x_0 + B(R))$. Hence, in this case we have

$$\begin{aligned}\|g_2\|_{W^{-1,p/(p-1)}(x_0+B(R))} &\le C_{14}R^{\frac{p-N}{p}}\|g_2\|_{L^1(x_0+B(R))}\\ &\le C_{15}R^{\frac{p-N}{p}}\|h\|_{L^p(x_0+B(R))}\|Du\|_{L^{p/(p-1)}(x_0+B(R))}\\ &\le C_{16}R\|h\|_{L^\infty(B(M+1))}\|Du\|_{L^{p/(p-1)}(x_0+B(R))}.\end{aligned}$$

Taking R sufficiently small, we get (8.2.8) and, consequently, we obtain that $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$ also in this case. ∎

Remark 8.2.4 (i) The previous theorem applies also to the case when μ is just an infinitesimally invariant measure. See Remark 8.1.4 for the definition of infinitesimally invariant measures.

(ii) The result in Theorem 8.2.3 holds also in the case when $N=1$. To see it, we denote by x the variables in $\mathbb{R}$ and we introduce the operator $\mathcal{B}$ defined on smooth functions of two variables (x,y) by

$$\mathcal{B}u(x,y)=\mathcal{A}u(x,y)+D_{yy}u(x,y)-yD_yu(x,y), \qquad (x,y)\in\mathbb{R}^2.$$

Moreover, we denote by $\widetilde{\mu}$ the measure in $\mathbb{R}^2$ whose density with respect to the Lebesgue measure is given by

$$\widetilde{\rho}(x,y)=\frac{1}{\sqrt{2\pi}}\rho(x)e^{-\frac{1}{2}y^2}, \qquad (x,y)\in\mathbb{R}^2.$$

Of course, the coefficients of the operator $\mathcal{B}$ satisfy the assumptions of Theorem 8.2.3. Moreover, for any $\varphi\in C^\infty_c(\mathbb{R}^2)$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\mathcal{B}\varphi\, d\widetilde{\mu} &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}y^2}dy\\ &\qquad\times\int_{\mathbb{R}}(\mathcal{A}\varphi(x,y)+\varphi_{yy}(x,y)-y\varphi_y(x,y))\rho(x)dx\\ &=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}y^2}\left(\int_{\mathbb{R}}(\varphi_{yy}(x,y)-y\varphi_y(x,y))\rho(x)dx\right)dy\\ &=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}y^2}(D_{yy}-yD_y)\left(\int_{\mathbb{R}}\varphi(x,y)\rho(x)dx\right)dy\\ &=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(D_{yy}e^{-\frac{1}{2}y^2}+D_y(ye^{-\frac{1}{2}y^2}))\left(\int_{\mathbb{R}^2}\varphi(x,y)\rho(x)dx\right)dy\\ &=0.\end{aligned}$$

Therefore, from the results in Theorem 8.2.3, the function $\widetilde{\rho}$ belongs to $W^{1,p}_{\rm loc}(\mathbb{R}^2)$ and, hence, $\rho\in W^{1,p}_{\rm loc}(\mathbb{R})$ for any $p\in[1,+\infty)$.

(iii) The reason why, throughout this section, we decide to deal only with the case when $N\geq 2$, is based on the fact that, when $N=1$, an explicit representation formula for the density of the infinitesimally invariant measure μ is available. Indeed, in such a case, a straightforward integration by parts shows that

$$\int_{\mathbb{R}}\{q(x)\rho'(x)-b(x)\rho(x)\}\varphi'(x)dx=0,$$

for any $\varphi \in C_c^\infty(\mathbb{R})$. Hence,

$$q(x)\rho'(x) - b(x)\rho(x) = C, \qquad x \in \mathbb{R},$$

for some real constant C.

8.2.1 Global L^q-regularity of the density ρ

In this subsection we are devoted to prove some L^q-regularity results for the function ρ. By virtue of Remark 8.2.4(iii), we limit ourselves to dealing with the case when $N \geq 2$.

Theorem 8.2.5 *Assume Hypothesis* 8.2.1. *Further, suppose that* $b_i \in L^p_\mu$ *for any* $i = 1, \ldots, N$ *and some* $p \in [2, +\infty)$. *Then, the following properties are met:*

(*i*) *if* $N \geq 3$ *and* $p \in [2, N)$, *then* $\rho \in L^q(\mathbb{R}^N)$ *for any* $q \leq \frac{N}{N-p}$;

(*ii*) *if* $p = N$, *then* $\rho \in L^q(\mathbb{R}^N)$ *for any* $q \in [1, +\infty)$;

(*iii*) *if* $p > N$, *then* $\rho \in C_b(\mathbb{R}^N)$.

Remark 8.2.6 As it has been already noticed in Remark 8.2.2, one can take advantage of the forthcoming Proposition 8.2.14 for a sufficient condition ensuring that the integrability assumptions on the coefficients b_i ($i = 1, \ldots, N$) are satisfied.

Proof of Theorem 8.2.5. To prove the assertion we use a bootstrap argument. Since the proof is rather long we divide it into three steps. First, in Step 1 we assume that $p = 2$ and we prove that ρ belongs to $L^{N/(N-2)}(\mathbb{R}^N)$, if $N > 2$, and to $L^q(\mathbb{R}^N)$ for any $q \in [1, +\infty)$, if $N = 2$. For this purpose, according to the Sobolev embedding theorems (see [2, Theorem 5.4]), it suffices to show that $\sqrt{\rho} \in W^{1,2}(\mathbb{R}^N)$. Next in Step 2, where we assume that $N > 2$, we prove that, if $\rho \in L^r(\mathbb{R}^N)$ for some $r \in (1, +\infty)$, then actually it belongs to $L^q(\mathbb{R}^N)$, where $q = N(r(p-2)+2)/(p(N-2))$. Finally, in Step 3 we conclude the proof. Throughout the proof, for notational convenience, we set

$$q(\xi, \eta) = \sum_{i,j=1}^N q_{ij}\xi_i\eta_j, \qquad \xi, \eta \in \mathbb{R}^N.$$

We also stress that, throughout the proof, actually we do not require μ to be the invariant measure of $\{T(t)\}$. We just need that μ is infinitesimally invariant, i.e.,

$$\int_{\mathbb{R}^N} \mathcal{A}\varphi \, d\mu = 0, \qquad \varphi \in C_c^\infty(\mathbb{R}^N). \tag{8.2.12}$$

Of course, according to Proposition 8.1.2, if μ is the invariant measure of $\{T(t)\}$, then the previous condition is always satisfied.

Step 1. Integrating by parts (taking Theorem 8.2.3 into account), from the formula (8.2.12) we easily see that

$$\int_{\mathbb{R}^N} q(D\rho, D\varphi)dx = \int_{\mathbb{R}^N} \sum_{j=1}^N b_j D_j\varphi d\mu, \tag{8.2.13}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. Since ρ is continuous (see again Theorem 8.2.3), then $b\rho \in L^2_{\rm loc}(\mathbb{R}^N)$. Therefore, by density, we can extend (8.2.13) to any $\varphi \in W^{1,2}(\mathbb{R}^N)$ with compact support.

Now, for any ε, k such that $0 < \varepsilon < 1 < k$, we introduce the function $\rho_{\varepsilon,k} : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\rho_{\varepsilon,k}(x) = (\rho(x) \vee \varepsilon) \wedge k, \qquad x \in \mathbb{R}^N. \tag{8.2.14}$$

Moreover, let $\{\vartheta_n\}$ be the sequence of smooth functions defined by $\vartheta_n(x) = \vartheta(x/n)$ for any $x \in \mathbb{R}^N$ and any $n \in \mathbb{N}$, where $\vartheta \in C_c^\infty(\mathbb{R}^N)$ is such that $\chi_{B(1/2)} \le \vartheta \le \chi_{B(1)}$. A straightforward computation shows that there exists a positive constant C such that

$$|D\vartheta_n(x)| \le \frac{C}{1+|x|}, \qquad |D^2\vartheta_n(x)| \le \frac{C}{1+|x|^2}, \qquad x \in \mathbb{R}^N. \tag{8.2.15}$$

As it is easily seen, the function $\log(\rho_{\varepsilon,k})$ belongs to $W^{1,p}_{\rm loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for any $n \in \mathbb{N}$. Plugging $\varphi = \vartheta_n^2 \log(\rho_{\varepsilon,k})$ in (8.2.13), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \vartheta_n^2 \chi_{\{\varepsilon<\rho<k\}} \frac{q(D\rho, D\rho)}{\rho} dx &= -2\int_{\mathbb{R}^N} \vartheta_n \log(\rho_{\varepsilon,k}) q(D\rho, D\vartheta_n)dx \\
&\quad + \int_{\mathbb{R}^N} \vartheta_n^2 \sum_{j=1}^N b_j D_j\rho \chi_{\{\varepsilon<\rho<k\}} dx \\
&\quad +2\int_{\mathbb{R}^N} \vartheta_n \rho \log(\rho_{\varepsilon,k}) \sum_{j=1}^N b_j D_j\vartheta_n dx \\
&:= I_n + J_n + K_n.
\end{aligned}$$

Let us estimate the term I_n. An integration by parts shows that

$$\begin{aligned}
I_n = {}& 2\int_{\mathbb{R}^N} \log(\rho_{\varepsilon,k}) q(D\vartheta_n, D\vartheta_n)d\mu + 2\int_{\mathbb{R}^N} \vartheta_n \log(\rho_{\varepsilon,k}) \sum_{i,j=1}^N q_{ij} D_{ij}\vartheta_n d\mu \\
&+2\int_{\mathbb{R}^N} \vartheta_n \log(\rho_{\varepsilon,k}) \sum_{i,j=1}^N D_i q_{ij} D_j \vartheta_n d\mu \\
&+2\int_{\mathbb{R}^N} \vartheta_n \chi_{\{\varepsilon<\rho<k\}} \frac{q(D\rho, D\vartheta_n)}{\rho} d\mu.
\end{aligned}$$

Using the Young inequality, we get

$$|I_n| \le C_{\varepsilon,k}\int_{\mathbb{R}^N}\left(q(D\vartheta_n,D\vartheta_n)+\mathrm{Tr}(QD^2\vartheta_n)+\sum_{i,j=1}^N |D_iq_{ij}D_j\vartheta_n|\right)d\mu$$

$$+2\left(\int_{\mathbb{R}^N}\vartheta_n^2\chi_{\{\varepsilon<\rho<k\}}\frac{q(D\rho,D\rho)}{\rho}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}q(D\vartheta_n,D\vartheta_n)\rho dx\right)^{\frac{1}{2}}$$

$$\le C_{\varepsilon,k}\int_{\mathbb{R}^N}\left(q(D\vartheta_n,D\vartheta_n)+\mathrm{Tr}(QD^2\vartheta_n)+\sum_{i,j=1}^N |D_iq_{ij}D_j\vartheta_n|\right)d\mu$$

$$+\frac{1}{\delta^2}\int_{\mathbb{R}^N}q(D\vartheta_n,D\vartheta_n)d\mu+\delta\int_{\mathbb{R}^N}\vartheta_n^2\chi_{\{\varepsilon<\rho<k\}}\frac{q(D\rho,D\rho)}{\rho}dx, \qquad (8.2.16)$$

where $C_{\varepsilon,k}=|\log(\varepsilon)|\vee\log(k)$. Similarly, we have

$$J_n+K_n\le\delta\int_{\mathbb{R}^N}\vartheta_n^2\chi_{\{\varepsilon<\rho<k\}}\frac{|D\rho|^2}{\rho}dx+\frac{1}{4\delta^2}\int_{\mathbb{R}^N}|b|^2\rho dx+\frac{CC_{\varepsilon,k}}{n}\int_{\mathbb{R}^N}|b|d\mu, \qquad (8.2.17)$$

for any $\delta>0$. From (8.2.16) and (8.2.17) and Hypothesis 8.2.1(i) it follows that

$$\left(1-\delta-\frac{\delta}{\kappa_0}\right)\int_{\mathbb{R}^N}\vartheta_n^2\chi_{\{\varepsilon<\rho<k\}}\frac{q(D\rho,D\rho)}{\rho}dx$$

$$\le C_{\varepsilon,k}\int_{\mathbb{R}^N}\left(q(D\vartheta_n,D\vartheta_n)+\mathrm{Tr}(QD^2\vartheta_n)+\sum_{i,j=1}^N |D_iq_{ij}D_j\vartheta_n|\right)d\mu$$

$$+\frac{1}{\delta^2}\int_{\mathbb{R}^N}q(D\vartheta_n,D\vartheta_n)d\mu+\frac{CC_{\varepsilon,k}}{n}\int_{\mathbb{R}^N}|b|d\mu+\frac{1}{4\delta^2}\int_{\mathbb{R}^N}|b|^2d\mu.$$

Letting n go to $+\infty$ and taking Hypothesis 8.2.1(ii) and the condition (8.2.15) into account, from the dominated convergence theorem we deduce that

$$\left(1-\delta-\frac{\delta}{\kappa_0}\right)\int_{\mathbb{R}^N}\chi_{\{\varepsilon<\rho<k\}}\frac{q(D\rho,D\rho)}{\rho}dx\le\frac{1}{4\delta^2}\int_{\mathbb{R}^N}|b|^2\rho dx,$$

for any $\delta>0$ and any $0<\varepsilon<1<k$. Now, taking δ small enough and letting ε go to 0^+ and k go to $+\infty$, we deduce that $\rho^{-1}q(D\rho,D\rho)$ (and, hence, $\rho^{-1}|D\rho|^2$) belongs to $L^1(\mathbb{R}^N)$ and

$$\kappa_0\int_{\mathbb{R}^N}\frac{|D\rho|^2}{\rho}dx\le\int_{\mathbb{R}^N}\frac{q(D\rho,D\rho)}{\rho}dx\le\widetilde{C}\int_{\mathbb{R}^N}|b|^2d\mu, \qquad (8.2.18)$$

for a suitable positive constant $\widetilde{C}$. Hence, $\sqrt{\rho}\in W^{1,2}(\mathbb{R}^N)$. Note that $\rho\in L^1(\mathbb{R}^N)$ according to Proposition 8.1.5.

Step 2. Now, we assume that $N>2$ and that $b_i\in L^p_\mu$ for any $i=1,\ldots,N$, and we prove that, if $\rho\in L^r(\mathbb{R}^N)$ for some $r\in(1,+\infty)$, then, actually, it belongs to $L^q(\mathbb{R}^N)$, where $q=N(r(p-2)+2)/(p(N-2))$. For this purpose, we prove that $\rho^{(\beta-1)/2}|D\rho|\in L^2(\mathbb{R}^N)$, where $\beta=(r-1)(1-2/p)$, and

$$\kappa_0^2\int_{\mathbb{R}^N}\rho^{\beta-1}|D\rho|^2dx\le\int_{\mathbb{R}^N}|b|^2\rho^{\beta+1}dx. \tag{8.2.19}$$

The estimate (8.2.19) will imply that the function $\rho^{(\beta+1)/2}$ is an element of $W^{1,2}(\mathbb{R}^N)$ (note that, since $\beta+1<2r$ and $\rho\in L^1(\mathbb{R})$, then $\rho\in L^s(\mathbb{R}^N)$ for any $s\in[1,r]$). Again, the Sobolev embedding theorems will yield $\rho\in L^q(\mathbb{R}^N)$.

So, let us prove (8.2.19). We observe that the integral in the right-hand side of (8.2.19) converges since, by the Hölder inequality, we have

$$\int_{\mathbb{R}^N}|b|^2\rho^{\beta+1}dx\le\left(\int_{\mathbb{R}^N}|b|^pd\mu\right)^{\frac{2}{p}}\left(\int_{\mathbb{R}^N}\rho^r\,dx\right)^{1-\frac{2}{p}}, \tag{8.2.20}$$

and the right-hand side of (8.2.20) is finite. The estimate (8.2.19) can be proved by applying the same arguments as in Step 1. Indeed, plugging $\varphi_n=\vartheta_n^2\rho_{\varepsilon,k}^\beta$ into (8.2.13) we get

$$\begin{aligned}\beta\int_{\mathbb{R}^N}\vartheta_n^2\rho^{\beta-1}\chi_{\{\varepsilon<\rho<k\}}q(D\rho,D\rho)dx&=-2\int_{\mathbb{R}^N}\vartheta_n\rho_{\varepsilon,k}^\beta q(D\rho,D\vartheta_n)dx\\&\quad+\beta\int_{\mathbb{R}^N}\vartheta_n^2\rho^\beta\chi_{\{\varepsilon<\rho<k\}}\sum_{i=1}^N b_iD_i\rho dx\\&\quad+2\int_{\mathbb{R}^N}\vartheta_n\rho\rho_{\varepsilon,k}^\beta\sum_{i=1}^N b_iD_i\theta_n dx\\&=I_n+J_n+K_n.\end{aligned} \tag{8.2.21}$$

With the same computations as in Step 1, we can easily show that

$$\begin{aligned}&\left(\beta\left(1-\frac{\delta}{\kappa_0}\right)-\delta\right)\int_{\mathbb{R}^N}\eta_n^2\rho^{\beta-1}\chi_{\{\varepsilon<\rho<k\}}q(D\rho,D\rho)dx\\&\le\frac{1}{4\delta^2}\int_{\mathbb{R}^N}|b|^2\rho^{\beta+1}dx+C(\varepsilon,k,\delta,n),\end{aligned} \tag{8.2.22}$$

where $C(\varepsilon,k,\delta,n)$ is a positive constant which tends to 0 as n tends to $+\infty$, for any fixed ε,k,δ. Therefore, letting first n go to $+\infty$ and, then, ε and k, respectively, to 0 and $+\infty$, we deduce that the function $\rho^{\beta-1}q(D\rho,D\rho)$ (and, consequently, the function $\rho^{\beta-1}|D\rho|^2$) belongs to $L^1(\mathbb{R}^N)$. Now, from (8.2.21) the formula (8.2.19) easily follows.

Step 3. Here we make the bootstrap argument work to prove the assertions (i)-(iii). We first assume that $N>2$. In such a situation the results of Steps 1 and 2 allow us to show that $\rho \in L^{q_n}(\mathbb{R}^N)$ for any $n \in \mathbb{N}$, where q_n is defined recursively by the following relation

$$\begin{cases} q_{n+1} = \dfrac{N(p-2)}{p(N-2)} q_n + \dfrac{2N}{p(N-2)}, & n \geq 1, \\ q_1 = \dfrac{N}{N-2}. \end{cases}$$

Therefore,

$$q_n = \frac{N^n (p-2)^n}{p^{n-1}(N-2)^n (p-N)} + \frac{N}{N-p}, \qquad n \geq 2,$$

if $p \neq N$, whereas

$$q_n = \frac{2n+N-2}{N-2}, \qquad n \geq 2,$$

if $p = N$. In the case when $p \in (2, N)$ it is immediate to check that q_n converges to $N/(N-p)$ as n tends to $+\infty$. Hence, to complete the proof of the assertion (i) it suffices to show that the sequence $\{\|\rho\|_{L^{q_n}(\mathbb{R}^N)}\}$ is bounded. Indeed, once this is proved, we will easily deduce that

$$\|\rho\|_{L^{N/(N-p)}(B(k))} = \lim_{n\to+\infty} \|\rho\|_{L^{q_n}(B(k))} \leq \limsup_{n\to+\infty} \|\rho\|_{L^{q_n}(\mathbb{R}^N)} \leq \sup_{n\in\mathbb{N}} \|\rho\|_{L^{q_n}(\mathbb{R}^N)},$$

for any $k \in \mathbb{N}$, and the Fatou lemma will imply that $\|\rho\|_{L^{N/(N-p)}(\mathbb{R}^N)}$ is finite.

Let us observe that according to the formulas (8.2.19) and (8.2.20) and the Sobolev embedding theorems, we have

$$\|\rho\|_{L^{q_{n+1}}(\mathbb{R}^N)} \leq C_1^{\frac{2N}{q_{n+1}(N-2)}} \|\rho\|_{L^{q_n}(\mathbb{R}^N)}^{\frac{N}{N-2}\frac{p-2}{p}\frac{q_n}{q_{n+1}}},$$

where $C_1 = C\|b\|_p$, C being the constant appearing in the Sobolev embedding theorems. Equivalently, setting $\alpha_n = \log(\|\rho\|_{L^{q_n}(\mathbb{R}^N)})$, we have

$$\alpha_{n+1} \leq \frac{2N}{q_{n+1}(N-2)} |\log(C_1)| + \frac{N}{N-2}\frac{p-2}{p}\frac{q_n}{q_{n+1}} \alpha_n. \tag{8.2.23}$$

Since $N(p-2)/(p(N-2)) < 1$ and $\{q_n\}$ is increasing, from (8.2.23) it follows that there sequence $\{\alpha_n\}$ satisfies

$$\alpha_{n+1} \leq \beta + \gamma \alpha_n, \qquad n \in \mathbb{N},$$

for some $\beta > 0$ and some $\gamma \in (0,1)$, and this implies that $\{\alpha_n\}$ and, consequently, $\{\|\rho\|_{L^{q_n}(\mathbb{R}^N)}\}$ are bounded. On the other hand, in the case when $p \geq N$ the sequence $\{q_n\}$ is increasing and it diverges to $+\infty$. Hence, the assertion (ii) easily follows. To prove that ρ belongs to $L^\infty(\mathbb{R}^N)$ when $p > N$, we will show that the L^{q_n}-norms of ρ are bounded by a constant C, independent of n. This will be enough for our purposes. Indeed, since ρ is continuous then, for any $k \in \mathbb{N}$,

$$\|\rho\|_{C(B(k))} = \lim_{n\to+\infty} \|\rho\|_{L^{q_n}(B(k))} \le \limsup_{n\to+\infty} \|\rho\|_{L^{q_n}(\mathbb{R}^N)} < C.$$

The arbitrariness of k then will imply that $\rho \in L^\infty(\mathbb{R}^N)$.

To prove the boundedness of $\|\rho\|_{L^{q_n}(\mathbb{R}^N)}$ we argue as in the case when $p < N$. From (8.2.23) we easily deduce that

$$\alpha_n \le \beta_{n-1} + \sum_{k=1}^{n-2} \beta_{n-1-k} \prod_{l=n-k}^{n-1} \gamma_l + \alpha_1 \prod_{k=1}^{n-1} \gamma_k, \qquad n \ge 2,$$

where

$$\beta_j = \frac{2N}{q_{j+1}(N-2)} |\log(C_1)|, \qquad \gamma_j = \frac{N}{N-2}\frac{p-2}{p}\frac{q_j}{q_{j+1}} = 1 - \frac{2N}{p(N-2)}\frac{1}{q_{j+1}}, \tag{8.2.24}$$

for any j. Observing that $\gamma_l \le 1$ for any l, we get

$$\alpha_n \le \beta_{n-1} + \sum_{k=1}^{n-2} \beta_k + \alpha_1.$$

Since, $\sum_{k=1}^{+\infty} \beta_k < +\infty$, we immediately deduce that

$$\alpha_n \le \alpha_1 + \sum_{k=1}^{+\infty} \beta_k,$$

for any $n \in \mathbb{N}$ and we are done.

To conclude the proof, let us consider the case when $N = 2$. The property (ii) easily follows from the Sobolev embedding theorems since, as we have shown in Step 1, the function $\sqrt{\rho}$ belongs to $W^{1,2}(\mathbb{R}^N)$.

To prove the property (iii) in the case when $N = 2$ we observe that, since $L^p_\mu(\mathbb{R}^N)$ is continuously embedded in $L^2_\mu(\mathbb{R}^N)$, then $\rho \in L^q(\mathbb{R}^N)$ for any $q \in [1,+\infty)$. Therefore, it belongs to $L^{r_0/(r_0-2)}(\mathbb{R}^N)$, where r_0 is such that $2 < r_0 < p$.

Adapting the proof of Step 2 to this situation, we can easily show that, if $\rho \in L^r(\mathbb{R}^N)$ for some $r > 1$, then, actually, it belongs to $L^q(\mathbb{R}^N)$, where $q = r_0(r(p-2)+2)/(p(r_0-2))$.

Now, applying the same bootstrap argument used in the case when $N \ge 3$, we can show that $\rho \in L^{q_n}(\mathbb{R}^N)$ for any $n \in \mathbb{N}$, where now the sequence $\{q_n\}$ is defined recursively by

$$\begin{cases} q_{n+1} = \dfrac{r_0(p-2)}{p(r_0-2)} q_n + \dfrac{2r_0}{p(r_0-2)}, \ n \ge 2, \\[2ex] q_1 = \dfrac{r_0}{r_0-2}. \end{cases}$$

Since $r_0(p-2)/(p(r_0-2)) > 1$, we can conclude that $\rho \in C_b(\mathbb{R}^N)$ repeating step by step the same arguments used in the case when $N > 2$. ∎

Remark 8.2.7 In fact, the proof of Theorem 8.2.5 shows some differentiability properties of the density ρ. In particular, it shows that, if $b_i \in L^p_\mu$ for any $i = 1, \ldots, N$ and some $p \geq (N+2)/2$, $p > 2$, then $\rho \in W^{1,2}(\mathbb{R}^N)$. To see it, first suppose that $N \geq 3$. By Theorem 8.2.5(i), $\rho \in L^q(\mathbb{R}^N)$ for any $q \in [1, N/(N-p)]$. Hence, in particular, it belongs to $L^{2N/(N-2)}(\mathbb{R}^N)$. Therefore, we can apply Step 2 in the proof of Theorem 8.2.5, taking $\beta = 1$. This shows that $D\rho \in L^2(\mathbb{R}^N)$, and we are done.

In the case when $N = 2$, Theorem 8.2.5 implies that $\rho \in L^q(\mathbb{R}^N)$ for any $q \in [1, +\infty]$. In particular, it belongs to $L^{2(p-1)/(p-2)}(\mathbb{R}^N)$ and, again, we can apply Step 2 in the proof of Theorem 8.2.5 to get the assertion.

A result similar to whose in Theorem 8.2.5(iii) can be proved, without assuming $p > N$, in the case when the function b is much more regular.

We refer the reader to the forthcoming Proposition 8.2.14 for a sufficient condition ensuring that the integrability assumptions on the coefficients b_i ($i = 1, \ldots, N$) in the following theorem are satisfied.

Theorem 8.2.8 *Suppose that $b_i \in W^{1,\infty}_{\rm loc}(\mathbb{R}^N)$ and $b_i, {\rm div}\, b \in L^p_\mu$ for some $p > N/2$, $p \geq 2$ and any $i = 1, \ldots, N$. Then $\rho \in C_b(\mathbb{R}^N)$.*

Proof. The proof is similar to that of Theorem 8.2.5. To make the same arguments as in the proof of the quoted theorem work, we need to show that if $\rho \in L^r(\mathbb{R}^N)$ for some $r > 1$, then the function $\rho^{(\beta+1)/2}$ belongs to $W^{1,2}(\mathbb{R}^N)$, where $\beta = (r-1)(1-1/p)$. For this purpose it suffices to show that

$$\kappa_0(\beta+1)\int_{\mathbb{R}^N} \rho^{\beta-1}|D\rho|^2 dx \leq -\int_{\mathbb{R}^N} \rho^{\beta+1}{\rm div} b\, dx, \tag{8.2.25}$$

since integral in the right-hand side of (8.2.25) converges (compare with (8.2.20)). Multiplying the (distributional) identity $\mathcal{A}_0\rho = {\rm div}\,(\rho b)$ by $\vartheta_n^2\rho_{\varepsilon,k}^\beta$ (where $\rho_{\varepsilon,k}$ is given by (8.2.14) and $\{\vartheta_n\}$ is the sequence of smooth functions defined by $\vartheta_n(x) = \vartheta(x/n)$ for any $x \in \mathbb{R}^N$ and any $n \in \mathbb{N}$, $\vartheta \in C^\infty_c(\mathbb{R}^N)$ being any smooth function such that $\chi_{B(1/2)} \leq \vartheta \leq \chi_{B(1)}$), we obtain again (8.2.21). Here, $\mathcal{A}_0$ is given by (8.2.1). The terms I_n and K_n can be estimated as in Step 1 in the proof of Theorem 8.2.5. As far as the term J_n is concerned, we observe that $\rho^\beta\chi_{\{\varepsilon<\rho<k\}}D\rho = (\beta+1)^{-1}D\rho_{\varepsilon,k}^{\beta+1}$. Therefore, since $\beta+1 < r$, both the functions $\rho_{\varepsilon,k}^{\beta+1}$ and $\rho^{\beta+1}$ belong to $L^1(\mathbb{R}^N)$. Hence, integrating by parts, we can write

$$J_n = -\frac{\beta}{\beta+1}\int_{\mathbb{R}^N} \rho_{\varepsilon,k}^{\beta+1}\sum_{i=1}^N b_i D_i(\vartheta_n^2)dx - \frac{\beta}{\beta+1}\int_{\mathbb{R}^N} \rho_{\varepsilon,k}^{\beta+1}\vartheta_n^2{\rm div} b\, dx.$$

Now, letting n go to $+\infty$, yields

$$\lim_{n\to+\infty} J_n = -\frac{\beta}{\beta+1}\int_{\mathbb{R}^N} \rho_{\varepsilon,k}^{\beta+1}\mathrm{div}b\,dx.$$

Therefore, arguing as in the proof of (8.2.22), we get

$$\beta(1-\delta)\int_{\mathbb{R}^N} \rho^{\beta-1}\chi_{\{\varepsilon<\rho<k\}}\sum_{i,j=1}^N q_{ij}D_i\rho D_j\rho dx \le -\frac{\beta}{\beta+1}\int_{\mathbb{R}^N} \rho_{\varepsilon,k}^{\beta+1}\mathrm{div}b\,dx.$$

Finally, letting ε and δ go to 0^+ and k go to $+\infty$, (8.2.25) follows. ∎

8.2.2 Global Sobolev regularity

In this subsection we prove some Sobolev regularity results for ρ under the following more restrictive assumptions on the diffusion coefficients q_{ij} $(i,j=1,\ldots,N)$.

Hypotheses 8.2.9 (i) Hypotheses 8.2.1(i) and 8.2.1(iii) are satisfied;

(ii) the functions q_{ij} belong to $C_b^1(\mathbb{R}^N)$ for any $i,j=1,\ldots,N$.

Theorem 8.2.10 *Under Hypotheses* 8.2.9 *the following properties are met:*

(*i*) *if* $b_i\in L^1_\mu$ $(i=1,\ldots,N)$, *then* $\rho\in L^q(\mathbb{R}^N)$ *for any* $q\in\left[1,\frac{N}{N-1}\right)$;

(*ii*) *if* $b_i\in L^p_\mu$ $(i=1,\ldots,N)$ *for some* $p\in(1,2)$, *then* $\rho\in W^{1,q}(\mathbb{R}^N)$ *for any* $q\in\left(1,\frac{N}{N-p+1}\right)$;

(*iii*) *if* $b_i\in L^p_\mu$ $(i=1,\ldots,N)$ *for some* $p\in[2,N)$, *then* $\rho\in W^{1,q}(\mathbb{R}^N)$ *for any* $q\in\left[1,\frac{N}{N-p+1}\right]$;

(*iv*) *if* $b_i\in L^N_\mu$ $(i=1,\ldots,N)$, *then* $\rho\in W^{1,q}(\mathbb{R}^N)$ *for any* $q\in[1,N)$;

(*v*) *if* $b_i\in L^p_\mu$ $(i=1,\ldots,N)$ *for some* $p>N$, *then* $\rho\in W^{1,q}(\mathbb{R}^N)$ *for any* $q\in[1,p]$.

Proof. We begin the proof checking the property (i). For this purpose, we observe that, according to (8.2.12), we can write

$$\int_{\mathbb{R}^N}(\varphi-\mathcal{A}_0\varphi)\,d\mu = \int_{\mathbb{R}^N}\left(\varphi+\sum_{i=1}^N b_iD_i\varphi\right)d\mu,$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. Here $\mathcal{A}_0$ is defined by (8.2.1). Then, using the Hölder inequality we get

$$\left|\int_{\mathbb{R}^N} (\varphi - \mathcal{A}_0\varphi)\, d\mu\right| \le C\|\varphi\|_{1,\infty}, \tag{8.2.26}$$

for some positive constant C, independent of φ. Let us now show that (8.2.26) implies that

$$\left|\int_{\mathbb{R}^N} f d\mu\right| \le C\|f\|_{L^q(\mathbb{R}^N)}, \tag{8.2.27}$$

for any $f \in L^q(\mathbb{R}^N)$ and any $q > N$. This, of course, will lead us to the assertion thanks to the Riesz-Fisher representation theorem. Without loss of generality we assume that $f \in C_c^\infty(\mathbb{R}^N)$. According to [66, Theorems 9.15], the equation $w - \mathcal{A}_0 w = f$ admits a unique solution $w \in W^{2,q}(\mathbb{R}^N)$. Moreover, there exists a positive constant C, independent of f, such that $\|w\|_{W^{2,q}(\mathbb{R}^N)} \le C\|f\|_{L^q(\mathbb{R}^N)}$ (see [66, Lemma 9.17]). Since $q > N$, the Sobolev embedding theorems imply that $w \in C_b^1(\mathbb{R}^N)$ and $\|w\|_{C_b^1(\mathbb{R}^N)} \le \widetilde{C}\|f\|_{L^q(\mathbb{R}^N)}$ for some constant $\widetilde{C}$, independent of w.

To get (8.2.27) we just need to show that we can plug w into (8.2.26). For this purpose, we use an approximation argument. Let $\{\vartheta_n\}$ be the sequence of smooth functions defined by $\vartheta_n(x) = \vartheta(x/n)$ for any $x \in \mathbb{R}^N$ and any $n \in \mathbb{N}$, where $\vartheta \in C_c^\infty(\mathbb{R}^N)$ is such that $\chi_{B(1/2)} \le \vartheta \le \chi_{B(1)}$. As it is easily seen, the function $\vartheta_n w$ tends to w in $W^{2,q}(\mathbb{R}^N)$ and, consequently, $\mathcal{A}_0(\vartheta_n w)$ tends to $\mathcal{A}_0 w$ in $C_b(\mathbb{R}^N)$. We now regularize the sequence $\{\vartheta_n w\}$ by convolution in a standard way, introducing the sequence $\{w_{\varepsilon,n}\}$ defined by

$$w_{\varepsilon,n} = \int_{\mathbb{R}^N} \vartheta_n(y)w(y)\xi_\varepsilon(x-y)dy, \qquad x \in \mathbb{R}^N,$$

for any $\varepsilon > 0$ and any $n \in \mathbb{N}$, where $\xi_\varepsilon(x) = \varepsilon^{-N}\xi(\varepsilon^{-1}x)$ and $\xi \in C_c^\infty(\mathbb{R}^N)$ is a smooth function with $\|\xi\|_{L^1(\mathbb{R}^N)} = 1$. Of course, $w_{\varepsilon,n} \in C_c^\infty(\mathbb{R}^N)$ and it tends to $\vartheta_n w$ in $W^{2,q}(\mathbb{R}^N)$ as ε tends to 0. Moreover,

$$\begin{aligned}(\mathcal{A}_0 w_{\varepsilon,n})(x) = &\int_{\mathbb{R}^N} (\mathcal{A}_0(\vartheta_n w))(y)\xi_\varepsilon(x-y)dy\\ &+\sum_{i,j=1}^N \int_{\mathbb{R}^N} (q_{ij}(x) - q_{ij}(x-y))(D_{ij}(\vartheta_n w))(x-y)\xi_\varepsilon(y)dy\\ &+\sum_{i=1}^N \int_{\mathbb{R}^N} (c_i(x) - c_i(x-y))(D_i(\vartheta_n w))(x-y)\xi_\varepsilon(y)dy,\end{aligned} \tag{8.2.28}$$

for any $x \in \mathbb{R}^N$, where $c_i = \sum_{j=1}^N D_j q_{ij}$ for any $i = 1, \ldots, N$. Since $\mathcal{A}_0 w \in BUC(\mathbb{R}^N)$, then $\mathcal{A}_0(\vartheta_n w) \in BUC(\mathbb{R}^N)$ as well. Therefore, the first term in the right-hand side of (8.2.28) tends to $\mathcal{A}(\vartheta_n w)$ uniformly in $\mathbb{R}^N$, as ε tends to

0. Since $q_{ij}\in \mathrm{Lip}(\mathbb{R}^N)$ for any $i,j=1,\dots,N$, a similar argument shows that the second term in the right-hand side of (8.2.28) converges to zero uniformly in $\mathbb{R}^N$ as ε tends to 0. Finally, the third term converges to 0 locally uniformly in $\mathbb{R}^N$. It follows that $\mathcal{A}_0 w_{\varepsilon,n}$ tends to $\mathcal{A}_0(\vartheta_n w)$ locally uniformly in $\mathbb{R}^N$. Moreover, there exists a positive constant $\overline{C}$, independent of ε and n, such that $\|\mathcal{A}_0 w_{\varepsilon,n}\|_\infty \le \overline{C}$ for any $\varepsilon>0$ and any $n\in\mathbb{N}$. We have so proved that there exists a sequence $\{w_n\}\subset C_c^\infty(\mathbb{R}^N)$ converging to w in $W^{2,q}(\mathbb{R}^N)$ and such that $\mathcal{A}_0 w_n$ tends to $\mathcal{A}_0 w$ locally uniformly in $\mathbb{R}^N$ and the sequence $\{\mathcal{A}_0 w_n\}$ is bounded in $C_b(\mathbb{R}^N)$. As a consequence, writing (8.2.26) with w_n instead of φ and letting n go to $+\infty$ easily yields (8.2.27).

Now we can prove the property (ii). We argue as in Step 2 of the proof of Theorem 8.2.5. Suppose that $\rho\in L^q(\mathbb{R}^N)$ for some $q>1$. Then, using (8.2.12) and the Hölder inequality, we can show that

$$\left|\int_{\mathbb{R}^N}\mathcal{A}_0\varphi d\mu\right|\le C\|\varphi\|_{1,r'},$$

where $1/r+1/r'=1$ and

$$\frac{1}{r}=\frac{1}{p}+\left(1-\frac{1}{p}\right)\frac{1}{q}. \tag{8.2.29}$$

According to Theorem C.1.3(i), ρ belongs to $W^{1,r}(\mathbb{R}^N)$ and since $r<N$ (recall that $N\ge 2$) the Sobolev embedding theorems then imply that $\rho\in L^{\widetilde{q}}(\mathbb{R}^N)$ where

$$\frac{1}{\widetilde{q}}=\frac{1}{p}-\frac{1}{N}+\left(1-\frac{1}{p}\right)\frac{1}{q}.$$

Therefore, using a bootstrap argument, we can now easily show that $\rho\in L^{q_n}(\mathbb{R}^N)$ for any $n\in\mathbb{N}$, where $s_n:=1/q_n$ ($n\in\mathbb{N}$) is defined recursively by

$$\begin{cases} s_{n+1}=\dfrac{N-p}{pN}+\dfrac{p-1}{p}s_n,\\[2mm] s_1=\alpha,\end{cases}$$

α being any real number belonging to the interval $((N-1)/N,1)$. Since the sequence $\{s_n\}$ converges to $(N-p)/N$, then q_n tends to $N/(N-p)$ as n tends to $+\infty$. Using (8.2.29) the assertion follows.

Let us now prove the property (iii). The arguments that we are going to use can be also adapted to prove the properties (iv) and (v). According to Theorem 8.2.5(i) we know that $\rho\in L^{N/(N-p)}(\mathbb{R}^N)$. Therefore, by the Hölder inequality and our assumptions, it is immediate to check that the functions $b_i\rho$ belong to $L^{(N/(N-p+1)}(\mathbb{R}^N)$ for any $i=1,\dots,N$. Therefore, from (8.2.12) we deduce that

$$\left|\int_{\mathbb{R}^N}\mathcal{A}_0\varphi d\mu\right|\le C\|\varphi\|_{W^{1,\frac{N}{p-1}}(\mathbb{R}^N)}, \tag{8.2.30}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. Theorem C.1.3(i) now implies that the function ρ belongs to $W^{1,N/(N-p+1)}(\mathbb{R}^N)$. Since $\rho^{-1}|D\rho|^2$ belongs to $L^1(\mathbb{R}^N)$ (see Step 1 in the proof of Theorem 8.2.5), using the Hölder inequality we deduce that $|D\rho| \in L^1(\mathbb{R}^N)$. Therefore, $\rho \in W^{1,1}(\mathbb{R}^N)$, and, hence, it belongs to $W^{1,q}(\mathbb{R}^N)$ for any $q \in [1, N/(N-p+1)]$. ■

Remark 8.2.11 For a sufficient condition ensuring that the integrability assumptions on the coefficients b_i $(i = 1, \ldots, N)$ are satisfied, we refer the reader to the forthcoming Proposition 8.2.14.

We now deal with the regularity of the second-order derivatives of ρ.

Theorem 8.2.12 *In addition to Hypotheses* 8.2.9, *let* $b_i \in W^{1,\infty}_{\rm loc}(\mathbb{R}^N) \cap L^p_\mu$ *for any* $i = 1, \ldots, N$ *and* $\operatorname{div} b \in L^p_\mu$, *for some* $p \geq 2$. *Then, the following properties are met:*

(*i*) *if* $p \in (N/2, N)$, *then* $\rho \in W^{2,q}(\mathbb{R}^N)$ *for any* $q \in \left(1, \frac{pN}{pN-p^2+3p-2}\right]$;

(*ii*) *if* $p = N$, *then* $\rho \in W^{2,q}(\mathbb{R}^N)$ *for any* $q \in \left(1, \frac{N^2}{3N-2}\right)$;

(*iii*) *if* $p \in (N, 2N)$, *then* $\rho \in W^{2,q}(\mathbb{R}^N)$ *for any* $q \in \left(1, \frac{p^2}{3p-2}\right]$;

(*iv*) *if* $p = 2N$, *then* $\rho \in W^{2,q}(\mathbb{R}^N)$ *for any* $q \in \left(1, \frac{p}{2}\right)$;

(*v*) *if* $p > 2N$, *then* $\rho \in W^{2,q}(\mathbb{R}^N)$ *for any* $q \in \left(1, \frac{p}{2}\right]$.

Proof. To prove the first three properties it suffices to show that, if $p > N/2$ and $p \geq 2$, and $D\rho \in L^r(\mathbb{R}^N)$ for some $r \in (1, +\infty)$, then $\rho \in W^{2,q}(\mathbb{R}^N)$ for any $q \in (1, s]$, where

$$\frac{1}{s} = \left(1 - \frac{2}{p}\right)\frac{1}{r} + \frac{2}{p}. \tag{8.2.31}$$

Indeed, once this property is checked, the conditions (i)-(iii) will immediately follow from Theorem 8.2.10.

Let us observe that, using (8.2.12) and integrating by parts, we can write

$$\int_{\mathbb{R}^N} (\mathcal{A}_0\varphi)\rho dx = \int_{\mathbb{R}^N} \left(\rho \operatorname{div} b + \sum_{i=1}^N b_i D_i\rho\right)\varphi dx,$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, where $\mathcal{A}_0$ is defined by (8.2.1). Since the function $f := \rho \operatorname{div} b + \sum_{i=1}^N b_i D_i\rho$ belongs to $L^m_{\rm loc}(\mathbb{R}^N)$ for any $m \in [1, +\infty)$ (see Theorem 8.2.3), according to Theorem C.1.3(iv), $\rho \in W^{2,m}_{\rm loc}(\mathbb{R}^N)$ for any $m \in$

$[1,+\infty)$. Moreover, $\rho\,\mathrm{div}\, b \in L^m(\mathbb{R}^N)$ for any $m \in [1,p]$, since $\rho \in C_b(\mathbb{R}^N)$ (see Theorem 8.2.8). Now, using the Hölder inequality, we can write

$$\begin{aligned}\int_{\mathbb{R}^N} |b|^\alpha |D\rho|^\alpha dx &= \int_{\mathbb{R}^N} |b|^\alpha |D\rho|^{\alpha-\frac{2}{\beta}} |D\rho|^{\frac{2}{\beta}} \rho^{-\frac{1}{\beta}} \rho^{\frac{1}{\beta}}\, dx\\ &\le \left(\int_{\mathbb{R}^N} \frac{|D\rho|^2}{\rho} dx\right)^{\frac{1}{\beta}} \left(\int_{\mathbb{R}^N} |b|^{\alpha\beta} d\mu\right)^{\frac{1}{\beta}}\\ &\quad \times \left(\int_{\mathbb{R}^N} |D\rho|^{\frac{\alpha\beta-2}{\beta-2}} dx\right)^{\frac{\beta-2}{\beta}}, \end{aligned} \tag{8.2.32}$$

for any $\alpha \ge 1$ and any $\beta \ge 2$ and, taking (8.2.18) into account, we easily see that $\alpha = s$ is the largest exponent such that the right-hand side of (8.2.32) is finite. Summing up, we have proved that $g := \sum_{i=1}^N b_i D_i\rho + \rho\mathrm{div}\, b \in L^s(\mathbb{R}^N)$. Since $\mathcal{A}_0\rho = g$, the Calderón-Zygmund estimates imply that $\rho \in W^{2,s}(\mathbb{R}^N)$. Hence, the statement follows with $q = s$.

To prove that $\rho \in W^{2,q}(\mathbb{R}^N)$ for any $q \in (1,s)$, we now observe that since $\rho \in W^{2,s}(\mathbb{R}^N)$, then $D\rho \in L^s(\mathbb{R}^N)$. Moreover, it belongs to $L^1(\mathbb{R}^N)$ by virtue of Theorem 8.2.10(iii) (recall that $p \ge 2$). Therefore, ρ belongs to $L^{r_*}(\mathbb{R}^N)$, where $r_* \in (1,s)$ is chosen so that $q^{-1} = (1-2/p)r_*^{-1} + 2/p$. Now, from the first part of the proof, we get $\rho \in W^{2,q}(\mathbb{R}^N)$.

To conclude the proof, let us check the properties (iv) and (v). To begin with, we show that $D\rho \in L^q(\mathbb{R}^N)$ for any $q < +\infty$ if $p = 2N$ and that $D\rho \in L^\infty(\mathbb{R}^N)$ if $p > 2N$. Iterating the results in the first part of the proof, it is easy to check that $\rho \in W^{2,q_{n+1}}(\mathbb{R}^N)$ for any $n \in \mathbb{N}$ such that $q_n < N$. Here, $\tau_n := 1/q_n$ is defined recursively by

$$\tau_{n+1} = \left(1 - \frac{2}{p}\right)\tau_n + \frac{2N-p+2}{pN}, \qquad n \ge 2.$$

We claim that, if $p > 2N$, there exists $n_0 \in \mathbb{N}$ such that $q_{n_0} > N$. The Sobolev embedding theorems then will imply that $D\rho \in C_b(\mathbb{R}^N)$. To prove the claim, we observe that if $q_n < N$ for any $n \le m$ and some $m \in \mathbb{N}$, then

$$\tau_n = \left(1 - \frac{2}{p}\right)^n \left(\tau_1 - 1 + \frac{p-2}{2N}\right) + 1 + \frac{2-p}{2N},$$

for any $n \le m+1$. Since the sequence $\{\tau_n\}$ converges to $1 + (2-p)(2N)^{-1} < N^{-1}$ as n tends to $+\infty$, there should exist $n \in \mathbb{N}$ such that $q_n > N$. In the case when $p = 2N$, the sequence τ_n converges to N^{-1}. Therefore, $\rho \in W^{2,\alpha}(\mathbb{R}^N)$ for any $\alpha < N$, and, consequently, $D\rho \in L^r(\mathbb{R}^N)$ for any $r \in (1,+\infty)$.

Now we are almost done. Indeed, to prove the properties (iv) and (v) it suffices to prove that the function $g = \sum_{i=1}^N b_i D_i\rho + \rho\mathrm{div}\, b$ belongs to $L^q(\mathbb{R}^N)$ for any $q \in (1,p/2)$, if $p = 2N$, and it belongs also to $L^{p/2}(\mathbb{R}^N)$, if $p > 2N$. Once these properties are checked, the Calderón-Zygmund estimates will imply that $u \in W^{2,q}(\mathbb{R}^N)$. Of course, without loss of generality we can limit ourselves to dealing with the function $|bD\rho|$. Indeed, since $\rho \in C_b(\mathbb{R}^N)$, it is immediate to check that all the other terms in the definition of g belong to $L^q(\mathbb{R}^N)$ for any $q \in (1, p/2)$, if $p = 2N$, and for any $q \in (1, p/2]$, if $p > 2N$. From the Hölder inequality we get

$$\begin{aligned}\int_{\mathbb{R}^N} |b|^q |D\rho|^q dx &= \int_{\mathbb{R}^N} |b|^q |D\rho|^{q-\frac{2q}{p}} |D\rho|^{\frac{2q}{p}} \rho^{-\frac{q}{p}} \rho^{\frac{q}{p}} dx \\ &\le \left(\int_{\mathbb{R}^N} |D\rho|^{\frac{q(p-2)}{p-2q}} dx\right)^{\frac{p-2q}{p}} \left(\int_{\mathbb{R}^N} \frac{|D\rho|^2}{\rho} dx\right)^{\frac{q}{p}} \\ &\quad \times \left(\int_{\mathbb{R}^N} |b|^p d\mu\right)^{\frac{q}{p}}, \end{aligned} \tag{8.2.33}$$

for any $q \in (1, p/2)$, if $p = N/2$, and

$$\begin{aligned}\int_{\mathbb{R}^N} |b|^{\frac{p}{2}} |D\rho|^{\frac{p}{2}} dx &= \int_{\mathbb{R}^N} |b|^{\frac{p}{2}} |D\rho|^{\frac{p}{2}-1} |D\rho| \rho^{-\frac{1}{2}} \rho^{\frac{1}{2}} dx \\ &\le \|D\rho\|_\infty^{\frac{p}{2}-1} \left(\int_{\mathbb{R}^N} \frac{|D\rho|^2}{\rho} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |b|^p d\mu\right)^{\frac{1}{2}}, \end{aligned} \tag{8.2.34}$$

if $p > 2N$. Since the last sides of (8.2.33) and (8.2.34) are finite, the assertion follows. ■

To conclude this subsection we prove some pointwise bounds for the function ρ. For this purpose we need to assume some additional assumptions on the measure μ and on the drift coefficients b_i ($i = 1, \ldots, N$). For notational convenience, for any $\delta, \beta > 0$ we denote by $V_{\delta,\beta}$ the function defined by $V_{\delta,\beta}(x) = \exp(\delta|x|^\beta)$ for any $x \in \mathbb{R}^N$.

Theorem 8.2.13 *In addition to Hypotheses* 8.2.9 *assume that* $V_{\delta,\beta} \in L^1_\mu$ *for some* $\beta, \delta > 0$. *Then, the following properties are met:*

(*i*) *if* $|b(x)| \le C\exp(|x|^\gamma)$ *for some* $C > 0$, $\gamma < \beta$ *and any* $x \in \mathbb{R}^N$, *then there exist* $c_1, c_2 > 0$ *such that*

$$\rho(x) \le c_1 \exp(-c_2|x|^\beta), \qquad x \in \mathbb{R}^N;$$

(*ii*) *if in addition to the previous set of hypotheses,* $b_i \in W^{1,\infty}_{\rm loc}(\mathbb{R}^N)$ ($i = 1, \ldots, N$) *and* $|b(x)| + |{\rm div}\, b(x)| \le C\exp(|x|^\gamma)$ *for some* $C > 0$, *some* $\gamma < \beta$ *and any* $x \in \mathbb{R}^N$, *then there exist* $c_1, c_2 > 0$ *such that*

$$|D\rho(x)| \le c_1 \exp(-c_2|x|^\beta), \qquad x \in \mathbb{R}^N.$$

Proof. We begin the proof by checking the property (i). Let $w : \mathbb{R}^N \to \mathbb{R}$ be the function defined by $w(x) = \exp(c_2|x|^\beta)$ for any $x \in \mathbb{R}^N$, where the positive constant c_2 will be chosen in the sequel. Moreover, set $\varphi = w\psi$, where ψ is any function belonging to $C_c^\infty(\mathbb{R}^N)$. Since μ is the invariant measure of $\{T(t)\}$, then

$$\int_{\mathbb{R}^N} \mathcal{A}_0\varphi d\mu = -\int_{\mathbb{R}^N} \sum_{i=1}^N b_i D_i\varphi d\mu, \tag{8.2.35}$$

or, equivalently, by differentiation,

$$\begin{aligned}&\int_{\mathbb{R}^N} w\rho\mathcal{A}_0\psi dx\\ &= -\int_{\mathbb{R}^N}\left(\psi\mathcal{A}_0 w + 2\sum_{i,j=1}^N q_{ij}D_i\psi D_j w + w\sum_{i=1}^N b_i D_i\psi + \psi\sum_{i=1}^N b_i D_i w\right)d\mu.\end{aligned} \tag{8.2.36}$$

Here $\mathcal{A}_0$ is defined by (8.2.1). Now, we fix $p > N$ and we prove that the right-hand side of (8.2.36) can be estimated by $C\|\psi\|_{W^{1,p'}(\mathbb{R}^N)}$ for a suitable positive constant C, independent of ψ. Here $1/p+1/p' = 1$. Theorem C.1.3(i) then will imply that $\rho w \in W^{1,p}(\mathbb{R}^N)$ and the Sobolev embedding theorems will yield $\rho w \in C_b(\mathbb{R}^N)$. The assertion will follow.
So, let us fix $q > p$ and choose $c_2 < \delta/q$. According to our assumptions, it is immediate to check that the functions w, Dw and $\mathcal{A}_0 w$ belong to L^q_μ and $\rho \in C_b(\mathbb{R}^N)$ (see Theorem 8.2.5(iii)). Therefore, the first two terms in the right-hand side of (8.2.36) can be estimated as wished. Indeed, let us consider, for instance, the first one. We have

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\psi\mathcal{A}_0 w d\mu\right| &= \left|\int_{\mathbb{R}^N}\psi(\mathcal{A}_0 w)\rho dx\right|\\ &\le \|\rho\|_\infty^{1-\frac{1}{p}}\left|\int_{\mathbb{R}^N}\psi(\mathcal{A}_0 w)\rho^{\frac{1}{p}}dx\right|\\ &\le \|\rho\|_\infty^{1-\frac{1}{p}}\left(\int_{\mathbb{R}^N}|\mathcal{A}_0 w|^p d\mu\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^N}|\psi|^{p'}dx\right)^{\frac{1}{p'}}.\end{aligned} \tag{8.2.37}$$

As far as the last two terms in the right-hand side of (8.2.36) are concerned, we observe that the functions wb_i and b_iD_iw belong to $L^p_\mu(\mathbb{R}^N)$ for any $i = 1,\ldots,N$. Indeed, according to our assumptions, the functions b_i ($i = 1,\ldots,N$) belong to $L^r_\mu(\mathbb{R}^N)$ for any $r \in [1,+\infty)$. Hence, in particular, they belong to $L^{pq/(q-p)}_\mu(\mathbb{R}^N)$. The Hölder inequality now implies that wb_i and b_iD_iw belong to $L^p_\mu(\mathbb{R}^N)$ for any $i = 1,\ldots,N$, and, repeating the same arguments as in the proof of (8.2.37), we are done.

To conclude the proof, let us check the property (ii). For this purpose, we begin by observing that, according to Theorem 8.2.12(v), $\rho \in W^{2,p}(\mathbb{R}^N)$ for any $p \in (1,+\infty)$. Therefore, integrating by parts the identity (8.2.35), we easily see that

$$\int_{\mathbb{R}^N}(\mathcal{A}_0\rho)\varphi dx = \int_{\mathbb{R}^N}\{\rho(\operatorname{div} b) + \langle b, D\rho\rangle\}\varphi dx,$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ or, equivalently, $\mathcal{A}_0\rho = \rho(\operatorname{div} b) + \langle b, D\rho\rangle$. It follows that

$$\mathcal{A}_0(\rho w) = w\rho \operatorname{div} b + w\sum_{i=1}^N b_i D_i\rho + \rho(A_0 w) + 2\sum_{i,j=1}^N q_{ij} D_i w D_j \rho. \quad (8.2.38)$$

Now, we replace the constant c_2 with a smaller constant to make ρDw bounded in $\mathbb{R}^N$. Our purpose consists in showing that the right-hand side of (8.2.38) belongs to $L^p(\mathbb{R}^N)$ for some $p > N$. This will imply that ρw belongs to $W^{2,p}(\mathbb{R}^N)$ and the Sobolev embedding theorems then will yield $D(\rho w) \in C_b(\mathbb{R}^N)$ and, consequently, $wD\rho \in C_b(\mathbb{R}^N)$.
As in the proof of the property (i), one sees that $w\rho\,(\operatorname{div} b), \rho(A_0 w) \in L^p(\mathbb{R}^N)$. As far as the terms containing $D\rho$ are concerned, we observe that

$$\begin{aligned}\int_{\mathbb{R}^N} \left| w\sum_{i=1}^N b_i D_i\rho\right|^p dx &\le \int_{\mathbb{R}^N} w^p |b|^p |D\rho|^{p-1} |D\rho| \rho^{-\frac{1}{2}} \rho^{\frac{1}{2}} dx \\ &\le \|D\rho\|_\infty^{p-1} \left(\int_{\mathbb{R}^N} \frac{|D\rho|^2}{\rho} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} w^{2p}|b|^{2p} d\mu\right)^{\frac{1}{2}},\end{aligned}$$

and the last side of the previous chain of inequalities is finite due to our assumptions and to (8.2.18). In a similar way one can estimate the term $|Dw||D\rho|$. ■

A situation in which $V_{\delta,\beta} \in L^1_\mu$, for suitable values of β and δ, is provided by the next proposition. We stress that it holds under Hypotheses 8.2.1.

Proposition 8.2.14 *Assume that*

$$\limsup_{|x|\to+\infty} \left\{ c\lambda_{\max}(x) + |x|^{-\beta}\left(\sum_{i=1}^N b_i(x)x_i + \sum_{i,j=1}^N D_j q_{ij}(x) x_i\right)\right\} < 0, \quad (8.2.39)$$

for some $c, \beta > 0$, where, for any $x \in \mathbb{R}^N$, $\lambda_{\max}(x)$ denotes the maximum eigenvalue of the matrix $(q_{ij}(x))$. Then, the operator $\mathcal{A}$ admits a (unique) invariant measure μ. Moreover, the function $V_{\delta,\beta}$ is integrable with respect to μ for any $\delta < \beta^{-1}c$.

Proof. As it is easily seen,

$$\begin{aligned}\mathcal{A}V_{\delta,\beta}(x) = \delta\beta|x|^{\beta-1}e^{\delta|x|^\beta} \\ \times\Bigg(\frac{\operatorname{Tr}(Q(x))}{|x|} + \frac{\beta-2}{|x|^3}\sum_{i,j=1}^N q_{ij}x_i x_j + \delta\beta|x|^{\beta-3}\sum_{i,j=1}^N q_{ij}(x)x_i x_j \\ + \frac{1}{|x|}\sum_{i=1}^N b_i x_i + \frac{1}{|x|}\sum_{i,j=1}^N D_j q_{ij}(x) x_i\Bigg),\end{aligned}$$

$$\leq \delta\beta|x|^{\beta-2}e^{\delta|x|^\beta}\Bigg((N+|\beta-2|+\delta\beta|x|^\beta)\lambda_{\max}(x) + \sum_{i=1}^N b_i x_i + \sum_{i,j=1}^N D_j q_{ij} x_i\Bigg), \tag{8.2.40}$$

for any $x\in\mathbb{R}^N$, and due to (8.2.39) and the choice of δ and β, the last side of (8.2.40) is negative for large $|x|$, and this implies that $\mathcal{A}V_{\delta,\beta}(x)$ tends to $-\infty$ as $|x|$ tends to $+\infty$. Therefore, according to Khas'minskii theorem 8.1.20, the semigroup associated with the operator $\mathcal{A}$ admits a (unique) invariant measure μ.

Let us now prove that the function $V_{\delta,\beta}\in L^1_\mu$. From (8.2.40) we easily deduce that, if $\beta\geq 1$, then $V_{\delta,\beta}(x)\leq|\mathcal{A}V_{\delta,\beta}(x)|$ for $|x|$ sufficiently large, whereas, if $\beta\in(0,1)$, then, for any $\varepsilon>0$, $|V_{(1-\varepsilon)\delta,\beta}(x)|\leq|\mathcal{A}V_{\delta,\beta}(x)|$ still for sufficiently large $|x|$. Therefore, in both the cases it suffices to show that $\mathcal{A}V_{\delta,\beta}\in L^1_\mu$ for any $\delta<\beta^{-1}c$ to deduce that $V_{\delta,\beta}\in L^1_\mu$.

To prove that $\mathcal{A}V_{\delta,\beta}\in L^1_\mu$ we observe that, by a simple approximation argument, we can show that

$$\int_{\mathbb{R}^N}\mathcal{A}\varphi d\mu=0, \tag{8.2.41}$$

for any $\varphi\in C^2_b(\mathbb{R}^N)$ with compact support. Since $\mathcal{A}\mathbb{1}=0$, we can extend (8.2.41) to any $\varphi\in C^2_b(\mathbb{R}^N)$ which is constant outside a ball. Now, let $\{\psi_n\}\in C^2_b([0,+\infty))$ be a sequence of increasing functions such that $\psi_n(t)=t$ for any $t\in[0,n]$, $\psi(t)=n+1/2$ for any $t\geq n+1$, and $\psi_n''(t)\leq 0$ for any $t\geq 0$. Then, the function $V_n=\psi_n\circ V_{\delta,\beta}$ belongs to $C^2_b(\mathbb{R}^N)$ and it is constant outside a ball. Therefore,

$$\int_{\mathbb{R}^N}\mathcal{A}V_n d\mu=0, \qquad n\in\mathbb{N}. \tag{8.2.42}$$

Fix now M sufficiently large such that $\mathcal{A}V_{\delta,\beta}(x)<0$ for any $x\notin B(M)$ and, then, fix a large enough n so that $V_n=V_{\delta,\beta}$ in $B(M)$. Taking (8.2.42) into account, we can write

$$\int_{\mathbb{R}^N\setminus B(M)}|\mathcal{A}V_n|d\mu=-\int_{\mathbb{R}^N\setminus B(M)}|\mathcal{A}V_n|d\mu=\int_{B(M)}\mathcal{A}V_n d\mu=\int_{B(M)}\mathcal{A}V_{\delta,\beta}d\mu.$$

Now, according to Fatou's lemma, we deduce that the function $\mathcal{A}V_{\delta,\beta}$ is integrable in $\mathbb{R}^N\setminus B(M)$ (and, hence, in $\mathbb{R}^N$) with respect to the measure μ.

∎

Assuming some more regularity on the diffusion coefficients and some bounds on the growth at infinity of the drift term, one can show also lower bounds for ρ.

Theorem 8.2.15 ([114], Theorem 6.3) *Assume that* $q_{ij} \in C_b^3(\mathbb{R}^N)$ *and that* $b_i \in C^2(\mathbb{R}^N)$ $(i,j = 1,\ldots,N)$ *satisfy*

$$|b_i(x)| + |Db_i(x)| + |D^2 b_i(x)| \le C(1+|x|^{\beta-1}), \qquad x \in \mathbb{R}^N, \quad i = 1,\ldots,N,$$

for some $\beta > 0$ *and* $C > 0$. *Then*

$$\rho(x) \ge \exp(-M(1+|x|^\beta)), \qquad x \in \mathbb{R}^N,$$

where $M > 0$ *is a positive constant depending only on* C, κ_0 *and* $\|q_{ij}\|_{C_b^3(\mathbb{R}^N)}$ $(i,j = 1,\ldots,N)$.

8.3 Some consequences of the estimates of Chapter 7

In this section we show some interesting consequences of the estimates proved in Chapter 7, when the semigroup $\{T(t)\}$ admits an invariant measure.

Remark 8.3.1 Note that if there exists a function φ such that

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty, \qquad \lim_{|x|\to+\infty} \mathcal{A}\varphi(x) = -\infty,$$

then both the assumptions of Khas'minskii theorem and Hypothesis 6.1.1(ii) are satisfied.

The following proposition is a consequence of the results in Theorem 7.2.2 and Corollary 7.2.3.

Proposition 8.3.2 *Suppose that Hypotheses* 6.1.1(*i*)–6.1.1(*iii*), *and Hypothesis* 6.1.1(*iv-k*) *or* 7.1.3(*ii-k*) $(k = 1,2,3)$ *are satisfied, and let* $p \in (1,+\infty)$. *Then, for any* $t > 0$, $T(t)$ *maps* L^p_μ *into* $W^{k,p}_\mu$ *and*

$$\|D^k T(t) f\|_p \le \left(k^2 \prod_{j=1}^{k} \frac{\widetilde{\omega}_{k,p\wedge 2}}{1 - e^{-\widetilde{\omega}_{k,p\wedge 2} t}} \psi_{j,r}(t/k) \right)^{\max\{\frac{1}{p},\frac{1}{2}\}} \|f\|_p, \qquad t > 0, \tag{8.3.1}$$

for any $f \in L^p_\mu$. *Here,* $\widetilde{\omega}_{k,p} = 0$ *under Hypothesis* 6.1.1(*iv-k*) *and* $\widetilde{\omega}_{k,p} < 0$ *is given by Theorem* 7.1.5 *if Hypothesis* 7.1.3(*ii-k*) *is satisfied. Moreover,* $\psi_{k,p}$ *is given in Theorem* 7.2.2. *In particular, the* L^p_μ*-norm of* $D^k T(t) f$ *behaves as* $t^{-1/2}$ *as* t *tends to* 0, *whereas, at infinity, it stays bounded, if Hypothesis* 6.1.1(*iv-k*) *is satisfied, and it decreases exponentially if Hypothesis* 7.1.3(*ii-k*) *is satisfied.*

Proof. The proof is a straightforward consequence of Theorem 7.2.2 and Corollary 7.2.3. Indeed, let first $f \in C_b(\mathbb{R}^N)$; integrating (7.2.5) (if $k=1$) or (7.2.27) (if $k=2,3$) in $\mathbb{R}^N$, we deduce that $T(t)f$ belongs to $W^{l,p}_\mu$ and it satisfies (8.3.1) for any $t>0$. In the general case when $f \in L^p_\mu$, consider a sequence $\{f_n\} \subset C_b(\mathbb{R}^N)$ converging to f in L^p_μ. Writing (8.3.1) for $f_n - f_m$ it follows that $\{T(t)f_n\}$ is a Cauchy sequence in $W^{l,p}_\mu$. Therefore, $T(t)f$ belongs to $W^{k,p}_\mu$ and satisfies (8.3.1). ∎

Proposition 8.3.3 *Let Hypotheses* 6.1.1(*i*)–6.1.1(*iii*) *and* 6.1.1(*iv*-1) *be satisfied and let $p>1$. Then, $D(L_p) \subset W^{1,p}_\mu$ and, for any $\omega>0$, there exists a positive constant M_ω such that*

$$\|Df\|_p \le M_{\omega,p}\|f\|_p^{\frac12}\,\|(L_p-\omega)f\|_p^{\frac12}, \qquad f \in D(L_p). \tag{8.3.2}$$

Under Hypotheses 6.1.1(*i*)-6.1.1(*iii*) *and* 7.1.3(*i*), *the estimate* (8.3.2) *holds true with $\omega = 0$.*

Proof. The proof is similar to that of Theorem 6.2.2. Indeed, fix $f \in D(L_p)$, $\lambda>0$ and let ω be as in the statement of the proposition. Moreover, let us set $u = (\lambda+\omega)f - L_pf$. Then, we have

$$f = \int_0^{+\infty} e^{-(\lambda+\omega)t}T(t)u\,dt. \tag{8.3.3}$$

Now, from Proposition 8.3.2 we deduce that in correspondence of ω we can find out a positive constant $C = C_\omega$ such that

$$\|DT(t)f\|_p \le \frac{Ce^{\omega t}}{\sqrt{t}}\|f\|_p, \qquad t>0. \tag{8.3.4}$$

Using (8.3.3) and (8.3.4) it is immediate to check that $Df \in L^p_\mu$ and

$$\begin{aligned}\|Df\|_p &\le \int_0^{+\infty} e^{-(\lambda+\omega)t}\|DT(t)u\|_p\,dt\\ &\le C_{\omega,p}\left(\int_0^{+\infty} t^{-\frac12}e^{-\lambda t}dt\right)\|\lambda f-(L_p-\omega)f\|_p\\ &\le \sqrt{\pi}C_{\omega,p}\left(\sqrt{\lambda}\|f\|_p + \frac{1}{\sqrt{\lambda}}\|(L_p-\omega)f\|_p\right).\end{aligned} \tag{8.3.5}$$

Minimizing with respect to $\lambda \in (0,+\infty)$ the last side of (8.3.5), we get the assertion with $M_{\omega,p} = 2\sqrt{2}C_{\omega,p}$. ∎

In the remainder of this section, we consider the particular case when

$$\mathcal{A}u(x) = \Delta u(x) + \langle b(x), Du(x)\rangle, \qquad x \in \mathbb{R}^N,$$

under the following assumptions on the function b.

Hypotheses 8.3.4 (i) the drift coefficient b belongs to $C^{1+\delta}_{\rm loc}(\mathbb{R}^N,\mathbb{R}^N)$ for some $\delta>0$. Moreover, there exists $d_0\in\mathbb{R}$ such that

$$\langle b(x)-b(y),x-y\rangle\le d_0|x-y|^2,\qquad x,\xi\in\mathbb{R}^N;\tag{8.3.6}$$

(ii) Hypothesis 6.1.1(ii) is satisfied and the semigroup associated with the operator $\mathcal{A}$ admits an invariant measure.

Remark 8.3.5 (i) If the drift term b satisfies Hypotheses 8.3.4, then it satisfies the condition (7.1.9). To check this latter condition, it suffices to write (8.3.6) with $y=x+t\xi$, dividing, then, both the members by t^2 and letting t go to 0. Therefore, the assumptions of Theorem 7.3.1 are satisfied. Therefore, the estimates (7.3.4) and (7.3.5) apply. In particular, these latter two estimates imply that for any $f\in L^p_\mu(\mathbb{R}^N)$ and any $t>0$, it holds that

$$\|DT(t)f\|_p\le\frac{2p^{1/p-1/2}d_0^{1/p}}{(p-1)^{1/2}(1-e^{-pd_0t})^{1/p}}t^{\frac{1}{p}-\frac{1}{2}}\|f\|_p,$$

for any $p\in(1,2]$ and

$$\|DT(t)f\|_p\le\left(\frac{d_0}{1-e^{-2d_0t}}\right)^{\frac{1}{2}}\|f\|_p,$$

for any $p\in(2,+\infty)$.

Finally, Proposition 8.3.3 holds true.

(ii) Hypothesis 8.3.4(ii) is satisfied, for instance, under the assumptions of Remark 8.3.1.

(iii) In the particular case when $d_0<0$, Hypothesis 8.3.4(ii) is a consequence of (8.3.4)(i) and the previous point (ii). Indeed, it suffices to take $\varphi(x)=|x|^2$ for any $x\in\mathbb{R}^N$.

Further, in this situation we can also prove the following formulas.

Proposition 8.3.6 *Assume that Hypotheses* 8.3.4 *hold and let* $p,q>1$ *be such that* $p^{-1}+q^{-1}=1$. *For any* $f\in D(L_p)$ *and any* $g\in D(L_q)$, *the function* fg *belongs to* $D(L_1)$ *and*

$$L_1(fg)=fL_qg+gL_pf+2\langle Df,Dg\rangle.\tag{8.3.7}$$

In particular,

$$\int_{\mathbb{R}^N}gL_pf\,d\mu=-\int_{\mathbb{R}^N}fL_qg\,d\mu-2\int_{\mathbb{R}^N}\langle Df,Dg\rangle d\mu,\tag{8.3.8}$$

and, for any $f\in D(L)$, *we have*

$$\int_{\mathbb{R}^N}fLf\,d\mu=-\int_{\mathbb{R}^N}|Df|^2d\mu.\tag{8.3.9}$$

Proof. Assume first $f,g\in D(\widehat{A})$. Since $D(\widehat{A})=D_{\max}(\mathcal{A})$ (see Propositions 2.3.6 and 4.1.10), then $fg\in D(\widehat{A})$ and

$$A(fg)=f\widehat{A}g+g\widehat{A}f+2\langle Df,Dg\rangle. \tag{8.3.10}$$

Now, let $f\in D(L_p)$ and $g\in D(L_q)$. Since $D(\widehat{A})$ is a core of L_p and L_q (see Proposition 8.1.9), then there exist two sequences $\{f_n\},\{g_n\}\subset D(\widehat{A})$ such that f_n, L_pf_n converge, respectively, to f and L_pf in L^p_μ, whereas g_n, L_qg_n converge, respectively, to g and L_qg in L^q_μ, as n tends to $+\infty$. Then, f_ng_n converges to fg in L^1_μ; besides by (8.3.2), for any $j=1,\ldots,N$, D_jf_n converges to D_jf in L^p_μ and D_jg_n converges to D_jg in L^q_μ, so that $\langle Df_n,Dg_n\rangle$ converges to $\langle Df,Dg\rangle$ in L^1_μ. By (8.3.10) we conclude that $\widehat{A}(f_ng_n)=L_1(f_ng_n)$ converges to the right-hand side of (8.3.7). The closedness of L_1 implies that $fg\in D(L_1)$ and that (8.3.7) holds.

Formula (8.3.8) now easily follows integrating (8.3.7) in $\mathbb{R}^N$ and taking (8.1.5) into account. Formula (8.3.9) then follows from (8.3.8) taking $f=g$. ■

In the symmetric case we can prove the following improvement of the formula (8.3.8), which generalizes the formula 8.1.39.

Proposition 8.3.7 *Assume that Hypotheses* 8.1.25 *hold with* $U\in C^{2+\alpha}_{\rm loc}(\mathbb{R}^N)$ *and* $G\in C^{1+\alpha}_{\rm loc}(\mathbb{R}^N)$. *Further, assume that the function* U *therein defined satisfies*

$$\langle DU(x)+G(x)-DU(y)-G(y),y-x\rangle\le d_0|x-y|^2,\qquad x,y\in\mathbb{R}^N,$$

for some constant $d_0\in\mathbb{R}$. *Finally, let* $p,q>1$ *be such that* $p^{-1}+q^{-1}=1$. *Then, for any* $f\in D(L_p)$ *and any* $g\in W^{1,q}_\mu$ *we have*

$$\int_{\mathbb{R}^N}gL_pf\,d\mu=-\int_{\mathbb{R}^N}\langle Df,Dg\rangle d\mu. \tag{8.3.11}$$

Proof. If $f,g\in D(\widehat{A})$, then (8.3.11) follows from (8.1.39). Indeed, it suffices to observe that since $D(\widehat{A})\subset D(L_2)$, then

$$\int_{\mathbb{R}^N}gL_pf\,d\mu=\int_{\mathbb{R}^N}gL_2f\,d\mu=-\int_{\mathbb{R}^N}\langle Df,Dg\rangle d\mu.$$

Consider now the general case when $f\in D(L_p)$ and $g\in W^{1,q}_\mu$. Let $\{f_n\}\subset D(\widehat{A})$ be a sequence such that f_n and $\widehat{A}f_n$ converge in L^p_μ, respectively, to f and L_pf, and let $\{g_n\}\subset D(\widehat{A})$ converge to g in $W^{1,q}_\mu$. Then, by the estimate (8.3.2), f_n converges to f in $W^{1,q}_\mu$. Thus (8.3.11) follows letting n tend to $+\infty$. ■

8.4 The convex case

In this section we consider the particular case when the operator $\mathcal{A}$ is given by

$$\mathcal{A}u(x) = \Delta u(x) - \langle DU(x), Du(x)\rangle, \qquad x \in \mathbb{R}^N, \tag{8.4.1}$$

under the following hypothesis on the function U.

Hypothesis 8.4.1 U belongs to $C^2(\mathbb{R}^N)$, $e^{-U} \in L^1(\mathbb{R}^N)$ and

$$\sum_{i,j=1}^{N} D_{ij}U(x)\xi_i\xi_j \geq 0, \qquad x, \xi \in \mathbb{R}^N. \tag{8.4.2}$$

Note that since Hypothesis 8.4.1 is stronger than Hypotheses 8.1.25, then all the results of Section 8.1.4 hold and, in particular, the measure

$$\mu(dx) = K^{-1}e^{-U(x)}dx, \qquad K = \int_{\mathbb{R}^N} e^{-U(x)}dx, \tag{8.4.3}$$

is the invariant measure of $\{T(t)\}$.

The results of this section have been proved in [40], in a more general context.

Theorem 8.4.2 *Assume Hypothesis* 8.4.1. *Then*

$$D(L) = \{u \in W^{2,2}_\mu : \langle DU, Du\rangle \in L^2_\mu\} \tag{8.4.4}$$

and

$$\|D^2u\|_2 \leq \|Lu\|_2, \qquad u \in D(L), \tag{8.4.5}$$

$$\|u\|_{W^{2,2}_\mu} \leq \frac{3}{2}\big(\|u\|_2 + \|Lu\|_2\big), \qquad u \in D(L). \tag{8.4.6}$$

Proof. We consider first the case when $u \in C^\infty_c(\mathbb{R}^N)$. Fix $i \in \{1, \ldots, N\}$; by (8.1.39) we have

$$\int_{\mathbb{R}^N} |D(D_iu)|^2 d\mu = -\int_{\mathbb{R}^N} L(D_iu)D_iu d\mu.$$

Writing $L(D_iu) = D_iLu + \langle D(D_iU), Du\rangle$ and summing as i runs from 1 to N we obtain

$$\int_{\mathbb{R}^N} \sum_{i,j=1}^{N} |D_{ij}u|^2 d\mu = -\int_{\mathbb{R}^N} \left\{\sum_{i=1}^{N}(D_iLu)D_iu + \sum_{i,j=1}^{N} D_{ij}UD_iuD_ju\right\}d\mu.$$

Now, by (8.1.39) and (8.4.2) it follows that

$$\int_{\mathbb{R}^N}\sum_{i,j=1}^N |D_{ij}u|^2 d\mu \le -\int_{\mathbb{R}^N}\sum_{i=1}^N (D_iLu)D_iu d\mu = \int_{\mathbb{R}^N}|Lu|^2 d\mu,$$

that is (8.4.5).

The estimate (8.4.6) with $u\in C_c^\infty(\mathbb{R}^N)$ now follows from (8.1.39) and (8.4.5).

Let us now prove the set equality (8.4.4). For this purpose, we fix $u\in D(L)$. By Theorem 8.1.26, $C_c^\infty(\mathbb{R}^N)$ is a core of L; therefore, there exists a sequence $\{u_n\}\subset C_c^\infty(\mathbb{R}^N)$ such that u_n and Lu_n converge in L^2_μ, respectively, to u and Lu. By (8.4.6) applied to u_n-u_m it follows that $u\in W^{2,2}_\mu$ and that u_n converges to u in $W^{2,2}_\mu$. As a straightforward consequence, u satisfies (8.4.5) and (8.4.6). Moreover, since $\langle DU,Du_n\rangle = Lu_n-\Delta u_n$, letting n go to $+\infty$ we get $\langle DU,Du\rangle\in L^2_\mu$. This proves the inclusion "$\subset$" in (8.4.4).

Conversely, let $u\in W^{2,2}_\mu$ be such that $\langle DU,Du\rangle\in L^2_\mu$, and let $f\in W^{1,2}_\mu$; we prove that

$$\int_{\mathbb{R}^N}\langle Du,Df\rangle d\mu = -\int_{\mathbb{R}^N}(\mathcal{A}u)f d\mu, \tag{8.4.7}$$

from which it follows that $u\in D(L)$. Indeed, if $\widetilde{L}$ denotes the realization of the operator $\mathcal{A}$ with domain given by the right-hand side of (8.4.4), the formula (8.4.7) implies that the operator $I-\widetilde{L}$ is injective. Since $(\widetilde{L},D(\widetilde{L}))$ is an extension of $(L,D(L))$, then $I-\widetilde{L}$ is also surjective. Therefore, $1\in\rho(\widetilde{L})\cap\rho(L)$ from which we get

$$L^2_\mu = (I-\widetilde{L})(D(\widetilde{L}))\supset(I-\widetilde{L})(D(L)) = (I-L)(D(L)) = L^2_\mu$$

and, consequently, $D(L)=D(\widetilde{L})$.

So, let us prove the formula (8.4.7). For this purpose, we consider a sequence $\{f_n\}\in C_c^\infty(\mathbb{R}^N)$ converging to f in $W^{1,2}_\mu$ (see the last part of the proof of Proposition 8.1.27). An integration by parts shows that

$$\int_{\mathbb{R}^N}\langle Du,Df_n\rangle d\mu = -\int_{\mathbb{R}^N} f_n\big(\Delta u-\langle DU,Du\rangle\big)d\mu, \qquad n\in\mathbb{N}.$$

Letting n tend to $+\infty$, we get (8.4.7). ■

As a consequence of the results in Theorem 8.4.2, we get the following corollary.

Corollary 8.4.3 *Assume Hypothesis* 8.4.1. *Then, for any* $\lambda>0$

$$\|D^2R(\lambda,L)f\|_2\le 2\|f\|_2, \tag{8.4.8}$$

for any $f\in L^2_\mu$.

Proof. By (8.4.4), the function $R(\lambda, L)f$ belongs to $W^{2,2}_\mu$ for any $f \in L^2_\mu$. Setting $u = R(\lambda, L)f$, by (8.4.5) we deduce that

$$\int_{\mathbb{R}^N} |D^2u|^2 d\mu \leq \|Lu\|_2^2 = \|\lambda u - f\|_2^2 \leq 4\|f\|_2^2,$$

that is (8.4.8). ∎

Remark 8.4.4 In [40] the authors prove some more refined results. Indeed, they consider the operator $\mathcal{A}$ in (8.4.1) assuming that U is a convex function which goes to $+\infty$ as $|x|$ tends to $+\infty$. No regularity assumptions on U are made. When U is not differentiable at $x \in \mathbb{R}^N$, $DU(x)$ is meant as the element with minimal norm in the subdifferential $\partial U(x)$ of U at x, where $\partial U(x) = \{y \in \mathbb{R}^N : U(\xi) \geq U(x) + \langle y, \xi - x\rangle, \ \forall \xi \in \mathbb{R}^N\}$. Under these rather weak assumptions on U, they prove that the realization A in L^2_μ of the operator $\mathcal{A}$ with domain $D(A) = \{u \in W^{2,2}_\mu : \langle DU, Du\rangle \in L^2_\mu\}$ is a dissipative self-adjoint operator. Therefore, it generates a strongly continuous analytic semigroup in L^2_μ. Moreover, they prove that all the results in Theorem 8.4.2 and Corollary 8.4.3 hold also in this situation.

To prove their results, the authors use an approximation argument. In the case when U is not smooth, they approximate U with a sequence of smooth functions by means of the Moreau-Yosida approximations of U (say $\{U_\alpha : \alpha > 0\}$), which are defined as follows:

$$U_\alpha(x) = \inf_{y \in \mathbb{R}^N} \left(U(y) + \frac{1}{2\alpha}|x - y|^2 \right), \qquad x \in \mathbb{R}^N, \ \alpha > 0.$$

Each function U_α is convex differentiable and

$$U_\alpha(x) \leq U(x), \qquad |DU_\alpha(x)| \leq |DU(x)|,$$

$$\lim_{\alpha \to 0^+} U_\alpha(x) = U(x), \qquad \lim_{\alpha \to 0^+} DU_\alpha(x) = DU(x),$$

for any $x \in \mathbb{R}^N$. Moreover, any U_α is Lipschitz continuous in $\mathbb{R}^N$ and its Lipschitz constant is $1/\alpha$.

To the approximated operators $\mathcal{A}_\alpha := \Delta u - \langle DU_\alpha, Du\rangle$ ($\alpha > 0$), the results in Theorem 8.4.2 and Corollary 8.4.3 apply with the same constants appearing in (8.4.4)-(8.4.6) and (8.4.8). Taking advantage of this fact, the authors show that, if $f \in C^\infty_c(\mathbb{R}^N)$, then (up to a subsequence) $R(\lambda, A_\alpha)f$ converges weakly in $W^{2,2}_\mu(\mathbb{R}^N)$ to a function $u \in \{u \in W^{2,2}_\mu : \langle DU, Du\rangle \in L^2_\mu\}$, which turns out to be the (unique) solution of the equation $\lambda u - \mathcal{A}u = f$ in such a space, and it satisfies the estimates in Theorem 8.4.2 and Corollary 8.4.3. Here A_α is the realization of the operator $\mathcal{A}_\alpha$ in $L^2_{\mu_\alpha}$, with domain given by (8.4.4) with μ_α instead of μ, where μ_α is defined accordingly to (8.4.3) with U being replaced by U_α.

Finally, using a density argument, they conclude that the operator A, with domain $D(A)$, generates a strongly continuous analytic semigroup in L^2_μ and it satisfies the estimates in Theorem 8.4.2 and Corollary 8.4.3.

8.5 Compactness of $T(t)$ and of the embedding $W^{1,p}_\mu \subset L^p_\mu$

In this section we prove some compactness result in the symmetric case, i.e., in the case when the operator $\mathcal{A}$ is given by (8.4.1) under the following hypothesis on U.

Hypothesis 8.5.1 The function U belongs to $C^2(\mathbb{R}^N)$ and the function e^{-U} is integrable in $\mathbb{R}^N$.

We recall that in this case the invariant measure of the associated semigroup is given by $d\mu = K^{-1}e^{-U(x)}dx$ (see Theorem 8.1.26). The results of this section are due to G. Metafune.

Lemma 8.5.2 *Assume that the function U satisfies Hypothesis 8.5.1 as well as the following condition*

$$\Delta U(x) \le \delta_1 |DU(x)|^2 + M_1, \qquad x\in\mathbb{R}^N, \tag{8.5.1}$$

for some $\delta_1\in(0,1)$ and $M_1>0$. Then, for any $p\in[2,+\infty)$ there exists a positive constant C such that

$$\int_{\mathbb{R}^N} |f|^p |DU|^2 d\mu \le C\|f\|^p_{W^{1,p}_\mu}, \qquad f\in W^{1,p}_\mu. \tag{8.5.2}$$

If, moreover,

$$\langle D^2U(x)DU(x), DU(x)\rangle \ge (\delta_2|DU(x)|^2+M_2)|DU(x)|^2, \qquad x\in\mathbb{R}^N, \tag{8.5.3}$$

for some $\delta_2>\delta_1-1$ and some $M_2\in\mathbb{R}$, then, for any $p\in[1,2)$, there exists a positive constant C such that

$$\int_{\mathbb{R}^N} |f|^p |DU|^p d\mu \le C\|f\|^p_{W^{1,p}_\mu}, \qquad f\in W^{1,p}_\mu. \tag{8.5.4}$$

Proof. First we prove the assertion in the case when f has compact support. Fix $g\in W^{1,p}_\mu$ with compact support and $\alpha\in(1,p]$. Since $L^q_\mu\subset L^r_\mu$ for any

$q > r$, and g is compactly supported, the function $|g|^\alpha(1+|DU|^2)$ belongs to L^1_μ. Now, integrating by parts we get

$$\int_{\mathbb{R}^N} |g|^\alpha(1+|DU|^2)d\mu = \int_{\mathbb{R}^N} |g|^\alpha d\mu - K^{-1}\int_{\mathbb{R}^N} |g|^\alpha\langle DU, De^{-U}\rangle dx$$
$$= \int_{\mathbb{R}^N} |g|^\alpha(1+\Delta U)d\mu + \alpha\int_{\mathbb{R}^N} |g|^{\alpha-2}g\langle Dg, DU\rangle d\mu. \tag{8.5.5}$$

Assume now that $p \in [1,2)$. Letting α tend to 1 in (8.5.5), we get

$$\int_{\mathbb{R}^N} |g|(1+|DU|^2)d\mu = \int_{\mathbb{R}^N} \{|g|(1+\Delta U) + \mathrm{sign}(g)\langle Dg, DU\rangle\}d\mu, \tag{8.5.6}$$

where we have set $\mathrm{sign}(0) = 0$. Choosing $g = |f|^p(1+|DU|^2)^{(p-2)/2}$ and observing that

$$Dg = p|f|^{p-2}fDf(1+|DU|^2)^{\frac{p-2}{2}} + (p-2)|f|^p(1+|DU|^2)^{\frac{p-4}{2}}D^2UDU,$$

from (8.5.6) we obtain that

$$\int_{\mathbb{R}^N} |f|^p(1+|DU|^2)^{\frac{p}{2}}d\mu = \int_{\mathbb{R}^N} |f|^p(1+|DU|^2)^{\frac{p-2}{2}}(1+\Delta U)d\mu$$
$$+p\int_{\mathbb{R}^N} |f|^{p-2}f(1+|DU|^2)^{\frac{p-2}{2}}\langle Df, DU\rangle d\mu$$
$$+(p-2)\int_{\mathbb{R}^N} |f|^p(1+|DU|^2)^{\frac{p-4}{2}}\langle D^2UDU, DU\rangle d\mu$$
$$:= I_1 + I_2 + I_3.$$

Using (8.5.1), the Hölder inequality and (8.5.3), we get

$$I_1 \le \delta_1\int_{\mathbb{R}^N} |f|^p(1+|DU|^2)^{\frac{p}{2}}d\mu + (M_1+1-\delta_1)\int_{\mathbb{R}^N} |f|^p(1+|DU|^2)^{\frac{p-2}{2}}d\mu$$
$$\le \delta_1\int_{\mathbb{R}^N} |f|^p(1+|DU|^2)^{\frac{p}{2}}d\mu + (M_1+1-\delta_1)\int_{\mathbb{R}^N} |f|^p d\mu,$$

$$I_2 \le p\Big(\int_{\mathbb{R}^N} |f|^p(1+|DU|^2)^{\frac{p}{2}}d\mu\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^N} |Df|^p d\mu\Big)^{\frac{1}{p}}$$
$$\le \varepsilon\int_{\mathbb{R}^N} |f|^p(1+|DU|^2)^{\frac{p}{2}}d\mu + C_{\varepsilon,p}\int_{\mathbb{R}^N} |Df|^p d\mu,$$

for any $\varepsilon > 0$ and some positive constant $C_{\varepsilon,p}$. Finally,

$$\begin{aligned}
I_3 &\le (p-2)\int_{\mathbb{R}^N}|f|^p(1+|DU|^2)^{\frac{p-4}{2}}(\delta_2^-|DU|^2+M_2^-)|DU|^2d\mu\\
&\le (p-2)\int_{\mathbb{R}^N}|f|^p(1+|DU|^2)^{\frac{p-2}{2}}(\delta_2^-|DU|^2+M_2^-)d\mu\\
&\le (p-2)\delta_2^-\int_{\mathbb{R}^N}|f|^p(1+|DU|^2)^{\frac{p}{2}}d\mu+|(p-2)(M_2^--\delta_2^-)|\int_{\mathbb{R}^N}|f|^pd\mu.
\end{aligned}$$

Since $\delta_1-\delta_2^- < 1$, choosing $\varepsilon > 0$ small enough, (8.5.4) follows.

In the case when $p \ge 2$, the estimate (8.5.2) follows for any compactly supported function $f \in W^{1,p}_\mu$ by (8.5.5) with $\alpha = p$ and $g = f$, using (8.5.1) to obtain the inequalities

$$\int_{\mathbb{R}^N}|f|^p(1+\Delta U)d\mu \le \delta_1\int_{\mathbb{R}^N}|f|^p(1+|DU|^2)d\mu+(M_1+1-\delta_1)\int_{\mathbb{R}^N}|f|^pd\mu,$$

$$\begin{aligned}
&\int_{\mathbb{R}^N}|f|^{p-1}|Df|(1+|DU|^2)^{\frac{1}{2}}d\mu\\
&\le \left(\int_{\mathbb{R}^N}|f|^p(1+|DU|^2)^{\frac{p}{2p-2}}d\mu\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}|Df|^pd\mu\right)^{\frac{1}{p}}\\
&\le \left(\int_{\mathbb{R}^N}|f|^p(1+|DU|^2)d\mu\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}|Df|^pd\mu\right)^{\frac{1}{p}}\\
&\le \varepsilon\int_{\mathbb{R}^N}|f|^p(1+|DU|^2)d\mu+C_\varepsilon\int_{\mathbb{R}^N}|Df|^pd\mu, \qquad \varepsilon > 0,
\end{aligned}$$

and then choosing $\varepsilon > 0$ small enough.

The general case when $f \in W^{1,p}_\mu$ follows by approximation: consider a sequence of cut-off functions $\{\vartheta_n\} \subset C^\infty_c(\mathbb{R}^N)$ such that $\vartheta_n \equiv 1$ in $B(n)$ and $\vartheta_n \equiv 0$ outside $B(n+1)$; the function $\eta_n f$ satisfies (8.5.2), if $p \in [2,+\infty)$, or (8.5.4) if $p \in [1,2)$. Then, the conclusion follows from the monotone convergence theorem. ∎

Theorem 8.5.3 *Suppose that the assumptions in Lemma* 8.5.2 *hold and assume that*

$$\lim_{|x|\to+\infty}|DU(x)| = +\infty. \tag{8.5.7}$$

Then, the embedding $W^{1,p}_\mu \subset L^p_\mu$ is compact for any $p \in [1,+\infty)$. In particular $T(t)$ is compact in L^2_μ for any $t > 0$.

Proof. We have to prove that the set

$$E = \left\{ f \in W^{1,p}_\mu : \; \|f\|_{W^{1,p}_\mu} \le 1 \right\}$$

is totally bounded in L^p_μ. For this purpose, fix $r > 0$ and let

$$E_r = \left\{ f_{\big|B(r)} : \; f \in E \right\}, \qquad r > 0.$$

Since U is continuous, the norms of $L^p(B(r), dx)$ and $L^p(B(r), \mu)$ are equivalent on the σ-algebra of the Borel sets of $\mathbb{R}^N$ (in the sense that there exists a positive constant C such that $C^{-1}\mu(A) \le m(A) \le C\mu(A)$ for any Borel set A, where by m we denote the Lebegue measure). Therefore, E_r is bounded in $W^{1,p}(B(r), dx)$ and then, by Rellich-Kondrachov theorem (see [2, Theorem 6.2]), it is totally bounded in $L^p(B(r), dx)$ and in $L^p(B(r), \mu)$ as well. Thus, for any $\varepsilon > 0$ there exists a finite number of functions $g_1, \ldots, g_n$ in $L^p(B(r), \mu)$ such that

$$E_r \subset \bigcup_{i=1}^{n} \{ f : \; \|f - g_i\|_{L^p(B(r),\mu)} < \varepsilon \}.$$

Moreover, by (8.5.2) and (8.5.4) it follows that

$$\int_{\mathbb{R}^N \setminus B(r)} |f|^p d\mu \le \frac{1}{\inf\limits_{\mathbb{R}^N \setminus B(r)} |DU|^{p \wedge 2}} \int_{\mathbb{R}^N} |f|^p |DU|^{p\wedge 2} d\mu \le \frac{C}{\inf\limits_{\mathbb{R}^N \setminus B(r)} |DU|^{p\wedge 2}},$$

for any $f \in E$. Choosing r large enough, from (8.5.7) we get

$$\left(\int_{\mathbb{R}^N \setminus B(r)} |f|^p d\mu \right)^{\frac{1}{p}} \le \varepsilon, \qquad f \in E.$$

Therefore, extending the functions g_j ($j = 1, \ldots, n$) by zero to the whole $\mathbb{R}^N$ we obtain

$$E \subset \bigcup_{i=1}^{n} \{ f : \; \|f - g_i\|_p < 2\varepsilon \},$$

that is, E is totally bounded in L^p_μ.

The compactness of $T(t)$ in L^2_μ, for any $t > 0$, now follows from Proposition 8.1.29. ∎

The following corollary is an immediate consequence of Proposition 8.3.2 and Theorem 8.5.3.

Corollary 8.5.4 *Assume that the operator $\mathcal{A}$ in* (8.4.1) *satisfies both Hypotheses* 8.3.4 *and the assumptions of Theorem* 8.5.3 *for some $p > 1$. Then, $T(t)$ is compact in L^p_μ for any $t > 0$.*

However, in general the embedding $W^{1,p}_\mu \subset L^p_\mu$ is not compact. Here, we see a counterexample.

Example 8.5.5 Let $N = 1$ and let $U \in C^2(\mathbb{R})$ be any function such that $U(x) = x$ for any $x > 0$ and $U(x) = 0$ for any $x \le -1$. Moreover, let $\mu(dx) = e^{-U(x)}dx$. Consider the sequence of functions

$$f_n(x) = \chi_{(0,+\infty)}x^n, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Then, we have

$$\|f_n\|_1 = \int_0^{+\infty} x^n e^{-x}dx = n!$$

and

$$\|Df_n\|_1 = n\int_0^{+\infty} x^{n-1}e^{-x}dx = n!.$$

Thus the sequence of functions $g_n = (n!)^{-1}f_n$ ($n \in \mathbb{N}$) is bounded in $W^{1,1}_\mu$, but it is not relatively compact in L^1_μ because $\lim_{n\to+\infty} g_n(x) = 0$ for any $x \in \mathbb{R}$, whereas $\|g_n\|_{L^1_\mu} = 1$ for any $n \in \mathbb{N}$.

In this example Hypotheses 8.1.25 and the conditions (8.5.1) and (8.5.3) are satisfied, so that the estimates (8.5.2) and (8.5.4) hold, but the condition (8.5.7) fails.

To conclude this section, we observe that, taking Lemma 8.5.2 into account, we can improve the results in Theorem 8.4.2.

Proposition 8.5.6 *Let the function U satisfy Hypothesis* 8.4.1 *and the condition* (8.5.1). *Then, $D(L) = W^{2,2}_\mu$ and the graph-norm of $D(L)$ is equivalent to the $W^{2,2}_\mu$-norm.*

8.6 The Poincaré inequality and the spectral gap

In this section we study the Poincaré inequality in L^p_μ. We say that the Poincaré inequality holds if there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |f - \overline{f}|^p d\mu \le C \int_{\mathbb{R}^N} |Df|^p d\mu, \qquad f \in W^{1,p}_\mu, \tag{8.6.1}$$

where $\overline{f} = \int_{\mathbb{R}^N} f d\mu$. The following proposition gives a first sufficient condition for the Poincaré inequality to hold.

Proposition 8.6.1 *Let $p \in [1,+\infty)$. If the embedding $W^{1,p}_\mu \subset L^p_\mu$ is compact, then the Poincaré inequality* (8.6.1) *holds.*

Proof. Fix $p \in [1,+\infty)$ and suppose, by contradiction, that (8.6.1) does not hold. Then, there exists a sequence $\{f_n\} \subset W^{1,p}_\mu$ such that

$$\overline{f_n} = 0, \qquad \|f_n\|_p = 1, \qquad \|Df_n\|_p \le \frac{1}{n}, \tag{8.6.2}$$

for any $n \in \mathbb{N}$. Since $W^{1,p}_\mu$ is compactly embedded in L^p_μ, then there exists a subsequence $\{f_{n_k}\}$ converging in L^p_μ to a function f^*, which satisfies

$$\overline{f^*} = 0, \qquad \|f^*\|_p = 1. \tag{8.6.3}$$

Now fix $r > 0$; according to Proposition 8.1.5, $L^p(B(r),\mu) \subset L^p(B(r)) := L^p(B(r),dx)$. Therefore,

$$\lim_{k\to+\infty} \|f_{n_k} - f^*\|_{L^1(B(r))} = 0 \tag{8.6.4}$$

and, moreover, from (8.6.2) it follows that

$$\|Df_{n_k}\|_{L^1(B(r))} \le \frac{C}{n_k}, \qquad k \in \mathbb{N}, \tag{8.6.5}$$

for some $C > 0$. Now, from the Poincaré inequality in $W^{1,1}(B(r))$ (see e.g., [54, Theorem 4.5.2]) we have

$$\|f_{n_k} - (f_{n_k})_r\|_{L^1(B(r))} \le C'\|Df_{n_k}\|_{L^1(B(r))}, \tag{8.6.6}$$

where

$$(f_{n_k})_r = \frac{1}{\omega_N r^N}\int_{B(r)} f_{n_k}\,dx,$$

and ω_N is the Lebesgue measure of the ball $B(1) \subset \mathbb{R}^N$. Letting k tend to $+\infty$ in (8.6.6), taking (8.6.4) and (8.6.5) into account, it follows that

$$\|f^* - (f^*)_r\|_{L^1(B(r))} = 0.$$

Therefore, f^* is almost everywhere constant in $B(r)$. Since $r > 0$ is arbitrary, f^* is almost everywhere constant in $\mathbb{R}^N$. But this is in contradiction with (8.6.3). ■

As an immediate consequence of the previous proposition and the results of Section 8.5, we get the following result.

Theorem 8.6.2 *Under the assumptions of Theorem* 8.5.3 *the Poincaré inequality* (8.6.1) *holds for any* $p \in [1,+\infty)$.

Now, we provide another proof of the Poincaré inequality, based on the pointwise gradient estimate (7.3.4).

Theorem 8.6.3 *Assume that Hypotheses* 8.3.4 *hold for some* $d_0 < 0$. *Then, the Poincaré inequality* (8.6.1) *holds for* $p = 2$ *and with* $C = -1/d_0$.

Proof. Observe that (8.3.2) implies that $D(L)$ is dense in $W^{1,2}_\mu$. Indeed, it contains $C_c^\infty(\mathbb{R}^N)$ which is dense in $W^{1,p}_\mu$; see the last part of the proof of Proposition 8.1.27. Hence, it is sufficient to consider the case when $f \in D(L)$. Then, by (8.3.9) we have

$$\frac{d}{dt}\int_{\mathbb{R}^N} |T(t)f|^2 d\mu = 2\int_{\mathbb{R}^N} T(t)f\, LT(t)f\, d\mu = -2\int_{\mathbb{R}^N} |DT(t)f|^2 d\mu,$$

for any $t > 0$. Using the pointwise estimate (7.3.4), with $p = 2$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^N} |T(t)f|^2 d\mu \geq -2e^{2d_0 t}\int_{\mathbb{R}^N} T(t)\big(|Df|^2\big) d\mu = -2e^{2d_0 t}\int_{\mathbb{R}^N} |Df|^2 d\mu.$$

Now, integrating with respect to t in $(0,+\infty)$ and using Theorem 8.1.16, we get

$$\overline{f}^2 - \int_{\mathbb{R}^N} f^2 d\mu \geq \frac{1}{d_0}\int_{\mathbb{R}^N} |Df|^2 d\mu,$$

that is (8.6.1) for $p = 2$ and with $C = -1/d_0$. ∎

We now show that the Poincaré inequality implies that $T(t)f$ converges exponentially to $\overline{f}$ in L^2_μ, as t tends to $+\infty$. This improves Theorem 8.1.16 and should be compared with Theorem 8.1.24. In the proof of Proposition 8.6.4 we will need that $D(L) \subset W^{1,2}_\mu$ and that

$$\int_{\mathbb{R}^N} fLf\, d\mu = -\int_{\mathbb{R}^N} |Df|^2 d\mu, \qquad f \in D(L). \tag{8.6.7}$$

Both the two previous properties hold. Actually we have already proved them twice: first in the symmetric case, that is under Hypotheses 8.1.25, see Proposition 8.1.27. Secondly we have proved them in the nonsymmetric case in Propositions 8.3.3 and 8.3.6, under Hypotheses 8.3.4.

Proposition 8.6.4 *Assume Hypotheses* 8.1.25 *or Hypotheses* 8.3.4. *Moreover, assume that the Poincaré inequality* (8.6.1) *holds with* $p = 2$. *Then,*

$$\|T(t)f - \overline{f}\|_2 \leq e^{-\frac{t}{C}}\|f - \overline{f}\|_2, \qquad t > 0, \tag{8.6.8}$$

for any $f \in L^2_\mu$, *where* $C > 0$ *is the same constant as in* (8.6.1). *Further, we have a spectral gap for* L, *namely*

$$\sigma(L)\setminus\{0\} \subset \{\lambda \in \mathbb{C} :\ \mathrm{Re}\,\lambda \leq -1/C\}.$$

Proof. First we prove (8.6.8). By density, it is sufficient to consider the case when $f \in D(L)$. Moreover, we can assume that $\overline{f} = 0$, since the general case then follows by considering the function $f - \overline{f}$. Note that $\overline{f} = 0$ implies that $\overline{T(t)f} = 0$ for any $t > 0$.

Let us observe that the function $t \mapsto \|T(t)f\|_2^2$ is differentiable in $[0, +\infty)$, and, by the relation (8.6.7), we have

$$\frac{d}{dt}\|T(t)f\|_2^2 = 2\int_{\mathbb{R}^N} T(t)f\, LT(t)f\, d\mu = -2\int_{\mathbb{R}^N} |DT(t)f|^2 d\mu.$$

Now, the Poincaré inequality (8.6.1) yields

$$\frac{d}{dt}\|T(t)f\|_2^2 \le -\frac{2}{C}\int_{\mathbb{R}^N} |T(t)f - \overline{T(t)f}|^2 d\mu = -\frac{2}{C}\int_{\mathbb{R}^N} |T(t)f|^2 d\mu.$$

Then, the estimate (8.6.8) follows from the Gronwall Lemma.

To prove the spectral gap property, it suffices to argue as in the proof of Theorem 8.1.24, observing that the estimate (8.6.8) implies that

$$\|T(t)\|_{L(H)} \le e^{-\frac{t}{C}}, \qquad t > 0, \tag{8.6.9}$$

where, as in the proof of the quoted theorem, we have set $H = \{f \in L^2_\mu : \overline{f} = 0\}$. In particular, (8.6.9) implies that

$$\sigma(L_{|H}) \subset \left\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \le -\frac{1}{2}\right\}.$$

See the beginning of Section B.1. ∎

8.7 The logarithmic Sobolev inequality and hypercontractivity

In this section we deal with the logarithmic Sobolev inequality (in short LSI). Fix $p \in [1, +\infty)$; the logarithmic Sobolev inequality reads as follows:

$$\int_{\mathbb{R}^N} f^p \log f d\mu \le \|f\|_p^p \log \|f\|_p + \frac{p}{\lambda}\int_{\mathbb{R}^N} f^{p-2}|Df|^2 d\mu, \tag{8.7.1}$$

where $\lambda > 0$ and $f \in L^p_\mu$ is a regular positive function. In this subsection we see some conditions under which the LSI holds and its main consequences.

Roughly speaking the LSI is, for invariant measures, the counterpart of the Sobolev embedding theorems which hold when the underlining measure is the Lebesgue one. Indeed, it is well known that, if both f and its gradient belong to some L^p-space related to the Lebesgue measure, then $|f|^q$ belongs to $L^1(\mathbb{R}^N)$ for some $q>p$. In general, this is no more the case for a general measure (hence, in particular, for invariant measure). We can see it by a simple counterexample.

Example 8.7.1 Let μ be the Gaussian measure, i.e., $\mu(dx)=\pi^{-N/2}e^{-|x|^2}$. Moreover, for any $p>1$ and any $\varepsilon>0$, let $u:\mathbb{R}^N\to\mathbb{R}$ be defined by

$$u(x)=\exp\left(\frac{2}{2p+\varepsilon}|x|^2\right),\qquad x\in\mathbb{R}^N.$$

Then, u belongs to L^p_μ, is continuously differentiable in $\mathbb{R}^N$ and

$$D^\alpha u(x)=P_{|\alpha|}(x)u(x),\qquad x\in\mathbb{R}^N,$$

where $P_{|\alpha|}$ is a real polynomial with degree $|\alpha|$. Since

$$\int_{\mathbb{R}^N}|D^\alpha u(x)|^p d\mu=\frac{1}{\pi^{N/2}}\int_{\mathbb{R}^N}|P_{|\alpha|}(x)|^p\exp\left(-\frac{\varepsilon}{2p+\varepsilon}|x|^2\right)dx<+\infty,$$

then $u\in W^{k,p}_\mu$ for any $k\in\mathbb{N}$.

However, u does not belong to $L^{p+\varepsilon}_\mu$. Indeed,

$$\int_{\mathbb{R}^N}|u(x)|^{p+\varepsilon}d\mu=\frac{1}{\pi^{N/2}}\int_{\mathbb{R}^N}\exp\left(\frac{\varepsilon}{2p+\varepsilon}|x|^2\right)dx=+\infty.$$

In the case when $p\ge 2$, what one can only expect is that $f^p\log(f)\in L^1_\mu$, if f is positive. Indeed, by the Hölder inequality we have

$$\int_{\mathbb{R}^N}|f|^{p-2}|Df|^2d\mu\le\left(\int_{\mathbb{R}^N}|f|^pd\mu\right)^{\frac{p-2}{p}}\left(\int_{\mathbb{R}^N}|Df|^pd\mu\right)^{\frac{2}{p}}.$$

This result is very sharp, as Example 8.7.6 shows.

Theorem 8.7.2 *Assume that Hypotheses* 8.3.4 *hold with* $d_0<0$*. Then, the LSI* (8.7.1) *holds, with* $\lambda=2d_0$*, for any* $p\in[1,+\infty)$ *and any nonnegative* $f\in D(\widehat{A})$*.*

Proof. We split the proof into two steps. First in Step 1 we prove the assertion for the functions $f\in D(\widehat{A})$ (see Section 2.3) such that $f\ge\delta$ for some positive constant δ. Then, in Step 2, we extend the result to all the nonnegative functions $f\in D(\widehat{A})$,

Step 1. Let $f\in D(\widehat{A})$ be strictly greater than δ for some $\delta>0$. We first prove the LSI with $p=1$. By Proposition 8.3.3, $T(t)f\in W^{1,2}_\mu$ for any $t>0$.

Moreover, by Remark 2.2.3 and Theorem 2.2.5, $T(t)f \geq T(t)\delta \equiv \delta$ for any $t \geq 0$. Therefore, we can define the function

$$\psi(t) = e^{-2d_0 t} \int_{\mathbb{R}^N} \frac{1}{T(t)f} |DT(t)f|^2 d\mu, \qquad t \geq 0.$$

The pointwise estimate (7.3.4) with $p = 2$ implies that

$$\psi(t) \leq \int_{\mathbb{R}^N} \frac{1}{T(t)f} |T(t)(|Df|)|^2 d\mu, \qquad t > 0.$$

Moreover, applying the Hölder inequality to the representation formula (2.2.8) we get

$$|T(t)(|Df|)|^2 = \left(T(t)\left(\sqrt{f}\frac{1}{\sqrt{f}}|Df|\right)\right)^2 \leq (T(t)f)T(t)\left(\frac{1}{f}|Df|^2\right);$$

therefore,

$$\psi(t) \leq \int_{\mathbb{R}^N} T(t)\left(\frac{1}{f}|Df|^2\right) d\mu = \int_{\mathbb{R}^N} \frac{1}{f}|Df|^2 d\mu = \psi(0). \tag{8.7.2}$$

Consider now the function $F : [0, +\infty) \to \mathbb{R}$ defined by

$$F(t) = \int_{\mathbb{R}^N} T(t)f \log(T(t)f)\, d\mu, \qquad t \geq 0.$$

By Theorem 8.1.16, $T(t)f$ tends to $\overline{f}$ in L^2_μ as t tends to $+\infty$; besides, since $T(t)f \geq \delta > 0$ for any $t > 0$, and since the function $x \mapsto \log x$ is Lipschitz continuous in $[\delta, +\infty)$, it follows that $\log T(t)f$ tends to $\log \overline{f}$ in L^2_μ as t tends to $+\infty$. This implies that

$$\lim_{t \to +\infty} F(t) = \overline{f} \log \overline{f}. \tag{8.7.3}$$

Moreover, $F \in C^1([0, +\infty))$ and a straightforward computation yields

$$F'(t) = \int_{\mathbb{R}^N} (\mathcal{A}T(t)f) \log T(t)f\, d\mu, \qquad t \geq 0, \tag{8.7.4}$$

since, by (8.3.8), $\int_{\mathbb{R}^N} \mathcal{A}T(t)f d\mu = \int_{\mathbb{R}^N} LT(t)f d\mu = 0$ for any $t > 0$. Now, since

$$\mathcal{A}(\log u) = \frac{1}{u}\mathcal{A}u - \frac{1}{u^2}|Du|^2, \qquad u \in D(\widehat{A}),$$

for any positive function $u \in C^2(\mathbb{R}^N)$, it follows that $\log T(t)f \in D_{\max}(\mathcal{A})$ for any $t > 0$ (see (2.0.1)). Since $\{T(t)\}$ is conservative (see Proposition 8.1.10), from Propositions 2.3.6, 4.1.10 and 8.1.9 we deduce that $\log T(t)f \in D(L)$ for any $t > 0$. Now, (8.3.8) and (8.7.4) yield

$$\begin{aligned} F'(t) &= -\int_{\mathbb{R}^N} T(t)fL(\log T(t)f)d\mu - \int_{\mathbb{R}^N} 2\langle DT(t)f, D\log T(t)f\rangle d\mu \\ &= -\int_{\mathbb{R}^N} \frac{1}{T(t)f}|DT(t)f|^2 d\mu = -e^{2d_0 t}\psi(t), \qquad t \geq 0. \end{aligned}$$

Integrating with respect to t in $(0,+\infty)$ and using (8.7.2) and (8.7.3), we get

$$\overline{f}\log\overline{f} - F(0) = -\int_0^{+\infty} e^{2d_0 t}\psi(t)dt \geq -\frac{1}{2d_0}\psi(0),$$

that is the LSI (8.7.1) with $p = 1$.

Finally, we consider the case when $p \in (1,+\infty)$. We have

$$\mathcal{A}(f^p) = pf^{p-1}\mathcal{A}f + p(p-1)f^{p-2}|Df|^2,$$

for any $f \in D(\widehat{A})$. According to Theorem 6.2.2, $D(\widehat{A}) = D_{\max}(\mathcal{A}) \subset C_b^1(\mathbb{R}^N)$, and thus $f^p \in D_{\max}(\mathcal{A})$. Moreover $f^p \geq \delta^p > 0$, and thus, by the first part of the proof, the function f^p satisfies (8.7.1) with $p = 1$. This yields the conclusion.

Step 2. We now consider the general case, first extending the validity of the LSI to all the nonnegative functions $f \in D(\widehat{A})$. For this purpose, we fix a nonnegative $f \in D(\widehat{A})$ and, for any $\delta > 0$, we define the function $f_\delta = f + \delta$. Since $\mathcal{A}f_\delta = \mathcal{A}f$ and $D_{\max}(\mathcal{A}) = D(\widehat{A})$, it is immediate to check that $f_\delta \in D(\widehat{A})$. Therefore, we can apply (8.7.1), with f_δ instead of f, getting

$$\int_{\mathbb{R}^N} f_\delta^p \log f_\delta d\mu \leq \|f_\delta\|_p^p \log\|f_\delta\|_p + \frac{p}{2d_0}\int_{\mathbb{R}^N} f_\delta^{p-2}|Df_\delta|^2 d\mu. \tag{8.7.5}$$

We are going to show that we can let δ go to 0^+ in the previous inequality, obtaining (8.7.1). As a first step we apply the Fatou lemma. From (8.7.5), we deduce that

$$\int_{\mathbb{R}^N} f^p \log f d\mu \leq \liminf_{\delta\to 0^+}\left(\|f_\delta\|_p^p \log\|f_\delta\|_p + \frac{p}{2d_0}\int_{\mathbb{R}^N} f_\delta^{p-2}|Df|^2 d\mu\right). \tag{8.7.6}$$

Now, since $f_\delta \leq f_1 \in C_b(\mathbb{R}^N) \subset L^p_\mu$ for any $\delta \in (0,1)$, from the dominated convergence theorem we immediately deduce that f_δ tends to f in L^p_μ as δ tends to 0. Therefore, $\|f_\delta\|_p^p \log\|f_\delta\|_p$ tends to $\|f\|_p^p\log\|f\|_p$ as δ vanishes. So, we just need to consider the last term in the right-hand side of (8.7.6). We observe that, if $p \leq 2$, then $f_\delta^{p-2} \leq f^{p-2}$ and, therefore, we are done. On the other hand, in the case when $p \geq 2$, by the Hölder inequality, it follows that the function $f_\delta^{p-2}|Df|^2$ is in L^1_μ and, as above, $f_\delta^{p-2}|Df|^2 \leq f_1^{p-1}|Df|^2$. Hence, the dominated convergence theorem implies that the last integral in (8.7.5) converges to the corresponding one, where we replace f_δ with f. Therefore, the estimate (8.7.1) follows also in this case. ∎

In order to extend the LSI (8.7.1) to all the positive functions $f \in W^{1,p}_\mu$, we prove the following proposition.

Proposition 8.7.3 *Let μ be the invariant measure of $\{T(t)\}$ and denote by ρ its density with respect to the Lebesgue measure. Suppose that ρ is locally bounded in $\mathbb{R}^N$. Then, $C^\infty_c(\mathbb{R}^N)$ is dense in $W^{1,p}_\mu$.*

Proof. The proof is similar to the classical proof of the density of $C^\infty_c(\mathbb{R}^N)$ in the Sobolev space $W^{1,p}(\mathbb{R}^N)$ associated with the Lebesgue measure. For the reader's convenience, we go into details. Let $f \in W^{1,p}_\mu(\mathbb{R}^N)$. We approximate f by a sequence of compactly supported functions $f_n \in W^{1,p}_\mu$, by setting $f_n = f\xi_n$, where $\xi_n(x) = \xi(|x|/n)$, for any $x \in \mathbb{R}^N$ and any $n \in \mathbb{N}$, $\xi \in C^\infty([0,+\infty))$ being any smooth function such that $\xi = 1$ in $[0,1/2]$ and $\xi = 0$ in $[1,+\infty)$. Each function f_n belongs to $W^{1,p}_\mu$ and the dominated convergence theorem implies that f_n converges to f in $W^{1,p}_\mu$. Therefore, we can limit ourselves to proving the assertion in the case when f is compactly supported. Let $\varphi \in C^\infty_c(\mathbb{R}^N)$ be a smooth function, compactly supported in $B(1)$ such that $\varphi \equiv 1$ in $B(1/2)$, $0 \le \varphi \le 1$ in $\mathbb{R}^N$ and $\|\varphi\|_{L^1(\mathbb{R}^N)} = 1$. We set $\varphi_n(x) = n^N\varphi(x/n)$ for any $x \in \mathbb{R}^N$ and any $n \in \mathbb{N}$, and, then, for any f compactly supported in $\mathbb{R}^N$, we set

$$f_n(x) = (f \star \varphi_n)(x) = \int_{\mathbb{R}^N} \varphi_n(x-y)f(y)dy, \qquad x \in \mathbb{R}^N, \ n \in \mathbb{N}.$$

It is immediate to check that all the functions f_n belong to $C^\infty_c(\mathbb{R}^N)$ and $\mathrm{Supp}\, f_n \subset B(1) + \mathrm{supp}\, f$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^N} |f_n - f|^p d\mu &= \int_{B(K)} |f_n - f|^p d\mu \\ &\le \sup_{x\in B(K)} \rho(x) \int_{B(K)} |f_n - f|^p dx \\ &\le \sup_{x\in B(K)} \rho(x) \int_{\mathbb{R}^N} |f_n - f|^p dx,\end{aligned}$$

where K is such that $B(1)+\mathrm{supp}(f) \subset B(K)$. Since, it is well known that f_n tends to f if $L^p(\mathbb{R}^N)$, from the previous chain of inequalities we deduce that f_n tends to f in L^p_μ as well. The same argument can be applied to check the convergence of Df_n to Df. Indeed, $Df_n = Df \star \varphi_n$ for any $n \in \mathbb{N}$. ∎

Remark 8.7.4 The assumptions of Proposition 8.7.3 are satisfied, for instance, under Hypothesis 8.2.1(i).

Theorem 8.7.5 *Under the assumptions of Proposition* 8.7.3*, the logarithmic Sobolev inequality* (8.7.1) *holds for any nonnegative function* $f\in W^{1,p}_\mu$ *and any* $p\geq 2$*.*

Proof. Fix a nonnegative function $f\in W^{1,p}_\mu$ and let $\{f_n\}\in C^\infty_c(\mathbb{R}^N)$ be a sequence of smooth functions converging to f in $W^{1,p}_\mu$ as n tends to $+\infty$. As the proof of Proposition 8.7.3 shows, we can assume that f_n is nonnegative for any $n\in\mathbb{N}$. Since $C^\infty_c(\mathbb{R}^N)\subset D(\widehat{A})$ (see Proposition 2.3.6), we can write the estimate (8.7.1) with f replaced with f_n ($n\in\mathbb{N}$) obtaining

$$\int_{\mathbb{R}^N} f_n^p\log f_n d\mu\leq \|f_n\|_p^p\log\|f_n\|_p+\frac{p}{2d_0}\int_{\mathbb{R}^N} f_n^{p-2}|Df_n|^2d\mu.$$

Without loss of generality, we can also assume that f_n converges to f pointwise μ-almost everywhere in $\mathbb{R}^N$. Therefore, the Fatou lemma implies that

$$\int_{\mathbb{R}^N} f^p\log f d\mu\leq \|f\|_p^p\log\|f\|_p+\frac{p}{2d_0}\liminf_{n\to+\infty}\int_{\mathbb{R}^N} f_n^{p-2}|Df_n|^2d\mu.$$

So, to conclude the proof, we just need to show that

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^N} f_n^{p-2}|Df_n|^2d\mu=\int_{\mathbb{R}^N} f^{p-2}|Df|^2d\mu. \tag{8.7.7}$$

For this purpose, we observe that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} f_n^{p-2}|Df_n|^2d\mu-\int_{\mathbb{R}^N} f^{p-2}|Df|^2d\mu\right|\\ &\leq \int_{\mathbb{R}^N} f_n^{p-2}||Df_n|^2-|Df|^2|d\mu+\int_{\mathbb{R}^N}|f_n^{p-2}-f^{p-2}||Df|^2d\mu\\ &:= I_1+I_2.\end{aligned}$$

As far as I_1 is concerned, we observe that the Hölder inequality implies that

$$\begin{aligned}I_1&\leq \|f_n\|_p^{p-2}\left(\int_{\mathbb{R}^N}|\,|Df_n|^2-|Df|^2\,|^{\frac{p}{2}}\right)^{\frac{2}{p}}\\ &\leq \|f_n\|_p^{p-2}(\|Df_n\|_p+\|Df\|_p)\|Df_n-Df\|_p,\end{aligned}$$

so that it vanishes as n tends to $+\infty$. As far as I_2 is concerned, we observe that if $p\in[2,3]$, then $|f_n^{p-2}-f^{p-2}|\leq|f_n-f|^{p-2}$ for any $n\in\mathbb{N}$. Therefore, the Hölder inequality yields

$$I_2\leq\int_{\mathbb{R}^N}|f_n-f|^{p-2}|Df|^2d\mu\leq\|f_n-f\|_p^{p-2}\|Df\|_p^2. \tag{8.7.8}$$

On the other hand, if $p>3$, then

$$\begin{aligned}|f_n^{p-2}-f_n^{p-2}|&\leq(p-2)|f_n^{p-3}+f^{p-3}||f_n-f|\\ &\leq(p-2)(2^{4-p}\vee 1)|f_n+f|^{p-3}|f_n-f|,\end{aligned}$$

for any $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} I_2 &\le (2^{4-p} \vee 1)(p-2) \int_{\mathbb{R}^N} |f_n + f|^{p-3}|f_n - f||Df|^2 d\mu \\ &\le (2^{4-p} \vee 1)(p-2) \left(\int_{\mathbb{R}^N} |f_n - f|^{\frac{p}{p-2}} |f_n + f|^{\frac{p(p-3)}{p-2}} d\mu \right)^{\frac{p-2}{p}} \|Df\|_p^2 \\ &\le (2^{4-p} \vee 1)(p-2)\|f_n - f\|_p (\|f_n\|_p + \|f\|_p)^{\frac{p-3}{p}} \|Df\|_p^2. \end{aligned} \tag{8.7.9}$$

From (8.7.8) and (8.7.9) we deduce that, both in the cases when $p \in [2,3]$ and $p > 3$, I_2 tends to 0 as n tends to $+\infty$. Summing up, (8.7.7) follows and this completes the proof. ∎

Example 8.7.6 Let $\mu(dx) = (2\pi)^{-1/2} \exp(-x^2/2)dx$ be the one-dimensional Gaussian measure. Further, let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{e^{\frac{x^2}{4}}}{\{(x^2+2)\log(x^2+2)\}^{\frac{3}{4}}} := e^{\frac{x^2}{4}} h(x),$$

for any $x \ge M$, where M is such that $f(x) \ge 1$ for any $x \in (-\infty, M) \cup (M, +\infty)$. As it is immediately seen, f belongs to L^2_μ, since at $\pm\infty$ h^2 is asymptotic to the function $x \mapsto |x|^{-3/2}(\log(|x|)^{-3/4}$ which is integrable in a neighborhood of $\pm\infty$. Moreover, f is continuously differentiable in $\mathbb{R}$ and

$$f'(x) = \frac{x}{2} \frac{(x^2-1)\log(x^2+2) - 3}{((x^2+2)\log(x^2+2))^{\frac{7}{4}}} e^{\frac{x^2}{4}},$$

for any $x \in (-\infty, -M) \cup (M, +\infty)$. Therefore, $x \mapsto (f'(x))^2 \exp(-x^2/2)$ is asymptotic to the function $x \mapsto |x|^{-1}(\log(|x|)^{-3/2}$ at $\pm\infty$, and this latter function is integrable in a neighborhood of $\pm\infty$.

On the other hand, the function

$$x \mapsto (f(x))^2 \log(f(x)) \log(\log(f(x))) \exp(-x^2/2)$$

is asymptotic to the function $x \mapsto (2|x|^2 \log(|x|))^{-1/2}$ at $\pm\infty$, and this latter function is integrable neither in a neighborhood of $+\infty$ nor in a neighborhood of $-\infty$.

This example, due to L. Gross (see [69, p. 1074]), shows that the LSI (8.7.1) is sharp.

Remark 8.7.7 As a consequence of Proposition 8.7.3, Remark 8.7.4 and Theorem 8.7.5, we immediately deduce that, if $\mathcal{A}u = \Delta u - \langle DU, Du\rangle$, with U satisfying Hypotheses 8.1.25 as well as the dissipative type condition

$$\langle DU(x) - DU(y), y - x\rangle \le d_0|x-y|^2, \qquad x, y \in \mathbb{R}^N, \tag{8.7.10}$$

for some $d_0\in\mathbb{R}$, then the LSI may be written for any nonnegative $f\in W^{1,p}_\mu(\mathbb{R}^N)$ ($p\geq 2$). Since $D(L_p)\subset W^{1,p}_\mu$ (see Proposition 8.3.3), if $p\geq 2$, we can write the LSI for any $f\in D(L_p)$.

The following proposition shows that actually the LSI can be written for any nonnegative $f\in D(L_p)$ also in the case when $p\in(1,2)$. For this purpose, we show that integrating by parts we can write the second integral term in the right-hand side of (8.7.1) in a more suitable form.

Proposition 8.7.8 *Assume Hypotheses* 8.1.25 *with the function U wherein defined satisfying* (8.7.10). *Then, for any $p\in(1,+\infty)$ and any positive function $f\in D(L_p)$, we have*

$$\int_{\mathbb{R}^N} f^{p-2}|Df|^2 d\mu=-\frac{1}{(p-1)}\int_{\mathbb{R}^N} f^{p-1}L_pf\,d\mu. \tag{8.7.11}$$

In particular the LSI (8.7.1) *holds true for any nonnegative $f\in D(L_p)$ ($p\in(1,+\infty)$).*

Proof. Let us prove (8.7.11). First, we suppose that $p\geq 2$ and fix $f\in D(L_p)$. Then, by Proposition 8.3.3, $f\in W^{1,p}_\mu$ and, therefore, $f^{p-1}\in W^{1,p/(p-1)}_\mu$; besides $D(f^{p-1})=(p-1)f^{p-2}Df$. Thus, (8.7.11) follows from the formula (8.3.11).

Now let $p\in(1,2)$ and let us fix $f\in D(L_p)$. For any $\varepsilon>0$ we consider the function $g_\varepsilon=(f^2+\varepsilon)^{(p-1)/2}$. As it is easily seen, g_ε belongs to $W^{1,p/(p-1)}_\mu$. Applying the previous argument with g_ε instead of f, we get

$$\int_{\mathbb{R}^N} f(f^2+\varepsilon)^{\frac{p-3}{2}}|Df|^2 d\mu=-\frac{1}{p-1}\int_{\mathbb{R}^N}(f^2+\varepsilon)^{\frac{p-1}{2}}L_pf\,d\mu.$$

Letting ε go to 0^+, by monotone and dominated convergence, we get (8.7.11).

To conclude the proof, let us show that the LSI (8.7.1) can be extended to all the positive functions $f\in D(L_p)$ for any $p\in(1,+\infty)$. Thanks to Remark 8.7.7, we can limit ourselves to considering the case when $p\in(1,2)$. For this purpose, we fix $f\in D(L_p)$ for some $p\in(1,2)$. Since $D(\widehat{A})$ is a core of L_p (see Proposition 8.1.9), we can determine a sequence $\{f_n\}\subset D(\widehat{A})$ converging to f, in L^p_μ, as n tends to $+\infty$, and such that L_pf_n tends to L_pf in L^p_μ, as well. Arguing as in the proof of Theorem 8.7.5, we can easily show that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} f_n^{p-1}L_pf_n d\mu=\int_{\mathbb{R}^N} f^{p-1}L_pf d\mu.$$

Now, using twice (8.7.11) gives

$$\begin{aligned}\lim_{n\to+\infty}\int_{\mathbb{R}^N} f_n^{p-2}|Df_n|^2 d\mu &= -\frac{1}{p-1}\lim_{n\to+\infty}\int_{\mathbb{R}^N} f_n^{p-1}L_p f_n d\mu\\ &= -\frac{1}{p-1}\int_{\mathbb{R}^N} f^{p-1}L_p f d\mu\\ &= \int_{\mathbb{R}^N} f^{p-2}|Df|^2 d\mu.\end{aligned}$$

Therefore, writing the LSI with f_n instead of f and, then, letting n go to $+\infty$, we can show the assertion. We refer the reader to the proof of Theorem 8.7.2 for further details. ■

The following result, due to L. Gross (see [69, 70]), states that the LSI implies that the semigroup $\{T(t)\}$ is hypercontractive, that is for any $t > 0$ the operator $T(t)$ is a contraction from L^2_μ to $L^{q(t)}_\mu$, where $q(t) = 1+e^{\lambda t}$. Actually, the hypercontractivity is equivalent to the logarithmic Sobolev inequality (see again [69, 70]). As the LSI, also the hypercontractivity is a very sharp result: in general $T(t)$ is not bounded from L^2_μ to L^p_μ when $p > q(t)$.

Theorem 8.7.9 *Assume Hypotheses* 8.1.25 *and* (8.7.10) *with* $d_0 < 0$. *Then,*

$$\|T(t)f\|_{q(t)} \le \|f\|_2, \qquad q(t) = 1+e^{\lambda t}, \tag{8.7.12}$$

for any $t > 0$ *and any* $f \in L^2_\mu$, *where* λ *is the constant appearing in* (8.7.1).

Proof. First, we assume that $f \in D(\widehat{A})$ and $f \ge \delta > 0$. Consider the function

$$g(t,x) = ((T(t)f)(x))^{q(t)}, \qquad t \ge 0, \quad x \in \mathbb{R}^N.$$

Since $f \in D(\widehat{A})$, then, for any fixed $x \in \mathbb{R}^N$, $g(\cdot,x) \in C^1([0,+\infty))$ (see Proposition 2.3.5) and

$$\begin{aligned}\frac{\partial g}{\partial t}(t,x) = q(t)((T(t)f)(x))^{q(t)-1}&(\widehat{A}T(t)f)(x)\\ &+((T(t)f)(x))^{q(t)}q'(t)\log((T(t)f)(x)),\end{aligned}$$

for any $t \ge 0$. Since $\widehat{A}T(t)f = T(t)\widehat{A}f$ and $0 < \delta \le T(t)f \le \|f\|_\infty$ for any $t > 0$ (see Lemma 2.3.3), then, for any fixed $T > 0$, there exists a constant $C_T > 0$ such that

$$\left|\frac{\partial g}{\partial t}(t,x)\right| \le C_T, \qquad t \ge 0, \quad x \in \mathbb{R}^N.$$

It follows that the function $G : [0,+\infty) \to \mathbb{R}$ defined by

$$G(t) = \int_{\mathbb{R}^N} g(t,\cdot)d\mu, \qquad t \ge 0,$$

belongs to $C^1([0,+\infty))$ and

$$G'(t)=\int_{\mathbb{R}^N}\frac{\partial g}{\partial t}(t,\cdot)d\mu,\qquad t\ge 0.$$

Now, consider the function $z:[0+\infty)\to\mathbb{R}$ defined by

$$z(t)=\log\|T(t)f\|_{q(t)}=\frac{1}{q(t)}\log G(t),\qquad t\ge 0.$$

It belongs to $C^1([0,+\infty))$ and

$$\begin{aligned}
z'(t)&=-\frac{q'(t)}{(q(t))^2}\log G(t)+\frac{G'(t)}{q(t)G(t)}\\
&=-\frac{q'(t)}{q(t)}\log\|T(t)f\|_{q(t)}+\frac{1}{\|T(t)f\|_{q(t)}^{q(t)}}\int_{\mathbb{R}^N}(T(t)f)^{q(t)-1}\widehat{A}T(t)fd\mu\\
&\quad+\frac{q'(t)}{q(t)\|T(t)f\|_{q(t)}^{q(t)}}\int_{\mathbb{R}^N}(T(t)f)^{q(t)}\log T(t)fd\mu,
\end{aligned}\tag{8.7.13}$$

for any $t\ge 0$. Observing that $q'(t)=\lambda(q(t)-1)$ and that $\widehat{A}=L$ on $D(\widehat{A})$ (see Proposition 8.1.9), and using the formula (8.7.11), we get

$$\int_{\mathbb{R}^N}(T(t)f)^{q(t)-1}\widehat{A}T(t)fd\mu=-\frac{q'(t)}{\lambda}\int_{\mathbb{R}^N}(T(t)f)^{q(t)-2}|DT(t)f|^2d\mu.\tag{8.7.14}$$

Replacing (8.7.14) into (8.7.13) we get

$$\begin{aligned}
\frac{q(t)}{q'(t)}\|T(t)f\|_{q(t)}^{q(t)}z'(t)&=-\|T(t)f\|_{q(t)}^{q(t)}\log\|T(t)f\|_{q(t)}\\
&\quad-\frac{q(t)}{\lambda}\int_{\mathbb{R}^N}(T(t)f)^{q(t)-2}|DT(t)f|^2d\mu\\
&\quad+\int_{\mathbb{R}^N}(T(t)f)^{q(t)}\log T(t)fd\mu.
\end{aligned}\tag{8.7.15}$$

Now, according to Theorem 8.7.2, the function $T(t)f$ satisfies the LSI (8.7.1) for any $p\in[1,+\infty)$ and hence, in particular, for $p=q(t)$. This implies that the right-hand side of (8.7.15) is negative. Consequently, $z(t)\le z(0)$, which yields (8.7.12).

Next, we consider the case when $f\in D(\widehat{A})$ is nonnegative. In this situation (8.7.12) follows from the previous step, by approximating f with $f+\varepsilon$ ($\varepsilon>0$). In particular, (8.7.12) holds for any nonnegative $f\in C_c^\infty(\mathbb{R}^N)$.

Finally, let $f\in L^2_\mu$. Consider a sequence of functions $\{f_n\}\subset C_c^\infty(\mathbb{R}^N)$ converging to $|f|$ in L^2_μ. Then $T(t)f_n$ converges to $T(t)|f|$ in L^2_μ as n tends to $+\infty$. By (8.7.12), applied to $T(t)(f_n-f_m)$, we see that $T(t)f_n$ converges to

$T(t)|f|$ in $L^{q(t)}_\mu$ as well, and that $|f|$ satisfies (8.7.12). Since $|T(t)f| \le T(t)|f|$ (recall that $\{T(t)\}$ is a positive semigroup, see Remark 2.2.3 and Theorem 2.2.5), f satisfies (8.7.12) as well. ■

The assumptions under which we proved the LSI in Theorem 8.7.2 imply also the Poincaré inequality (by Theorem 8.6.3). But one can also prove directly that the LSI implies the Poincaré inequality. We show here the proof in [134].

Proposition 8.7.10 *If the LSI* (8.7.1) *holds for any* $f \in D(\widehat{A})$ *such that* $f \ge \delta > 0$*, then the Poincaré inequality* (8.6.1) *holds with* $p = 2$ *and* $C = 2/\lambda$.

Proof. Without loss of generality, we can assume that $\overline{f} = 0$, since the general case then follows by considering the function $f - \overline{f}$. Moreover, to fix ideas, we assume that $\delta = 1/2$. Let $\varepsilon > 0$ be such that

$$\varepsilon \|f\|_\infty < \frac{1}{2} \tag{8.7.16}$$

and define the function

$$g = 1 + \varepsilon f \in D(\widehat{A}).$$

Then $g \ge 1/2$, and from the LSI (8.7.1) with $p = 2$, it follows that

$$\int_{\mathbb{R}^N} (1+\varepsilon f)^2 \log(1+\varepsilon f) d\mu \le \|g\|_2^2 \log \|g\|_2 + \frac{2\varepsilon^2}{\lambda} \int_{\mathbb{R}^N} |Df|^2 d\mu. \tag{8.7.17}$$

Now, we use the Taylor expansion of the function

$$\varphi(s) = (1+s)^2 \log(1+s), \qquad s > -1,$$

at $s = 0$. We have

$$\varphi(s) = s + \frac{3}{2}s^2 + r(s), \qquad r(s) = \frac{\varphi'''(\xi_s)}{6} s^3, \qquad s > -1, \tag{8.7.18}$$

where ξ_s is a suitable point in the interval joining 0 and s. Substituting the expansion (8.7.18) in (8.7.17), and dividing by ε^2, we get

$$\frac{3}{2}\|f\|_2^2 + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N} r(\varepsilon f) d\mu \le \frac{1}{2\varepsilon^2}(1+\varepsilon^2\|f\|_2^2)\log(1+\varepsilon^2\|f\|_2^2) + \frac{2}{\lambda}\int_{\mathbb{R}^N} |Df|^2 d\mu. \tag{8.7.19}$$

Since $\varphi'''(s) = 2/(1+s)$, we have

$$|r(s)| \le \frac{2}{3}|s|^3, \qquad s > -\frac{1}{2}. \tag{8.7.20}$$

Now, from (8.7.16) and (8.7.20) it follows that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N} r(\varepsilon f) d\mu = 0.$$

Moreover,

$$\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon^2}(1+\varepsilon^2\|f\|_2^2)\log(1+\varepsilon^2\|f\|_2^2) = \frac{1}{2}\|f\|_2^2.$$

Therefore, letting ε tend to 0^+ in (8.7.19), we obtain the Poincaré inequality (8.6.1) with $C = 2/\lambda$. ■

Remark 8.7.11 For higher order logarithmic Sobolev inequalities we refer the reader to the papers [55, 56], where the author proves that, if μ is a Gaussian measure and $f \in W^{2,2}_\mu$, then $f^2(\log(f))^2$ is in L^1_μ.

Chapter 9

The Ornstein-Uhlenbeck operator

9.0 Introduction

In this chapter we present some results concerned with the Ornstein-Uhlenbeck operator, which is the prototype of an elliptic operator with unbounded coefficients. Such an operator is defined on smooth functions φ by

$$(\mathcal{A}\varphi)(x) = \frac{1}{2}\mathrm{Tr}\,(QD^2\varphi(x)) + \langle Bx, D\varphi(x)\rangle, \qquad x \in \mathbb{R}^N, \tag{9.0.1}$$

where Q and B are $N \times N$ constant matrices, with Q (strictly) positive definite and $B \neq 0$. Here, if not otherwise specified, we consider only the case when Q is strictly positive. In any case, some of the results that we present hold also when Q is a degenerate positive definite matrix and the operator $\mathcal{A}$ in (9.0.1) is hypoelliptic, i.e., when the matrix

$$\int_0^t e^{sB}Qe^{sB^*}\,ds$$

is strictly positive definite for any $t > 0$. Such a condition can be expressed also by saying that the kernel of Q does not contain any invariant subspace of B^* (see, e.g., [98]).

Firstly, in Section 9.1, we show that an explicit formula for the Ornstein-Uhlenbeck semigroup is available both in the nondegenerate and in the degenerate case. Having such a formula simplifies the study of the main properties of the semigroup. For instance, one can prove uniform estimates for the space derivatives of any order of the function $T(t)f$ when $f \in C_b(\mathbb{R}^N)$ just differentiating under the integral sign. We do this in Section 9.2 in the nondegenerate case. The case of the degenerate Ornstein-Uhlenbeck operator is much more involved. It has been studied by A. Lunardi in [107]. Here, we limit ourselves to state the main results of [107].

As it has been claimed several times, the Ornstein-Uhlenbeck semigroup is neither analytic nor strong continuous in $C_b(\mathbb{R}^N)$. In particular, $T(t)f$ tends to f in $C_b(\mathbb{R}^N)$ as t tends to 0^+, if and only if $f \in BUC(\mathbb{R}^N)$ and $f(e^{tB}\cdot)$ tends to f uniformly in $\mathbb{R}^N$.

In Section 9.3, we deal with the invariant measure of $\{T(t)\}$. We show that when the spectrum of the matrix B is contained in the left halfplane, the

Ornstein-Uhlenbeck semigroup admits the Gaussian measure

$$\mu(dx) = \frac{1}{\sqrt{(2\pi)^N \det Q_\infty}} e^{-\frac{1}{2}\langle Q_\infty^{-1}x,x\rangle} dx$$

as the (unique) invariant measure, both in the nondegenerate and in the degenerate case. Here,

$$Q_\infty = \int_0^{+\infty} e^{sB} Q e^{sB^*} ds. \tag{9.0.2}$$

The assumptions on the location of the spectrum of the matrix B turns out to be also necessary to guarantee the existence of the invariant measure of $\{T(t)\}$.

From the results in Chapter 8, we know that the extension of the Ornstein-Uhlenbeck semigroup to the L^p-spaces associated with the invariant measure μ (in short L^p_μ) gives rise to a strongly continuous semigroup for any $p \in [1, +\infty)$. Moreover, for any $f \in L^p_\mu$, $T(t)f$ is still given by the same formula as in the case when $f \in C_b(\mathbb{R}^N)$.

Actually, in the nondegenerate case, $\{T(t)\}$ is also analytic for any $p \in (1, +\infty)$. Also in this situation, having an explicit representation formula for $T(t)f$ is of much help. Indeed, we can quite easily show that $T(t)$ maps L^p_μ into $W^{k,p}_\mu$ (for any $k \in \mathbb{N}$) and we can also give precise estimates on the behaviour of the space derivatives of $T(t)f$ in L^p_μ when t approaches 0^+.

An important feature of the Ornstein-Uhlenbeck semigroup in L^p_μ is that a complete characterization of its infinitesimal generator L_p is available. More precisely, we show that

$$D(L_p) = W^{2,p}_\mu, \qquad p \in (1, +\infty)$$

and that the graph norm is equivalent to the Euclidean norm of $W^{2,p}_\mu$. Such a result was firstly proved by A. Lunardi and G. Da Prato in the Hilbert case, and then it has been proved by G. Metafune, D. Pallara, A. Rhandi and R. Schnaubelt for a general p.

Since the Ornstein-Uhlenbeck semigroup in L^p_μ is compact for any $p \in (1, +\infty)$, the spectrum of L_p is a discrete set. It has been completely characterized by G. Metafune, D. Pallara and E. Priola in terms of the eigenvalues $\lambda_1, \ldots, \lambda_N$ of the matrix B. More precisely, they show that

$$\sigma(L_p) = \left\{\lambda = \sum_{i=1}^r n_i\lambda_i : n_i \in \mathbb{N} \cup \{0\},\ i = 1, \ldots, r\right\}.$$

In particular, $\sigma(L_p)$ is independent of p and the eigenfunctions of L_p are polynomials with degree at most $(\mathrm{Re}\lambda)/s(B)$. Moreover, the eigenvalues of L_p are all semisimple if and only if B is diagonalizable in $\mathbb{C}$.

The picture changes drastically in the case when $p = 1$. Indeed, in such a situation $\sigma(L_1)$ is the set of all the $\lambda \in \mathbb{C}$ with nonpositive real part. In particular, any λ with negative real part is an eigenvalue of L_1.

Since the Ornstein-Uhlenbeck semigroup is analytic in L^p_μ, the function $z \mapsto T(z)$ is analytic in a sector centered at the origin. A recent result by R. Chill, E. Fasangova, G. Metafune and D. Pallara explicitly characterizes the amplitude of such a sector.

Next, we exploit some relations between the Ornstein-Uhlenbeck semigroup and the Hermite polynomials.

Finally, in Section 9.4, we consider the Ornstein-Uhlenbeck semigroup in the L^p-spaces associated with the Lebesgue measure, for any $p \in [1, +\infty)$. In such a setting $\{T(t)\}$ is strongly continuous but it is not analytic. The domain of its infinitesimal generator A_p is the set of all the functions $u \in W^{2,p}(\mathbb{R}^N)$ such that $\mathcal{A}u \in L^p(\mathbb{R}^N)$. Moreover, the spectrum of A_p consists of all the complex numbers with real parts not greater than $-(\mathrm{Tr}(B))/p$. In particular, differently from what happens when the underlining measure is the invariant measure, the spectrum depends explicitly on p.

9.1 The formula for $T(t)f$

One of the best features of the Ornstein-Uhlenbeck operator is that an explicit formula for the associated semigroup is available both in the nondegenerate and in the degenerate case. Such a formula is due to Kolmogorov (see [85]).

Let us introduce some notation. For any $t > 0$, we denote by $Q_t \in L(\mathbb{R}^N)$ the matrix defined by

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds. \tag{9.1.1}$$

It is readily seen that Q_t $(t > 0)$ is a positive definite matrix, and it is strictly positive definite whenever Q is.
Moreover, for any strictly positive definite matrix K and any $a \in \mathbb{R}^N$, we denote by $\mathcal{N}(a, K)$ the Gaussian measure defined by

$$\mathcal{N}(a,K)(dx) = \frac{1}{\sqrt{(2\pi)^N \det K}} e^{-\frac{1}{2}\langle K^{-1}(x-a), x-a\rangle} dx. \tag{9.1.2}$$

We can now prove the following theorem.

Theorem 9.1.1 *For any $f \in C_b(\mathbb{R}^N)$, the Cauchy problem*

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{9.1.3}$$

admits a unique classical solution u which is bounded in $[0,T] \times \mathbb{R}^N$ for any $T > 0$. In fact, such a solution is bounded in $[0,+\infty) \times \mathbb{R}^N$ and

$$\sup_{t>0} \|u(t,\cdot)\|_{C_b(\mathbb{R}^N)} \le \|f\|_\infty. \tag{9.1.4}$$

Moreover, u is given by

$$u(t,x) = (T(t)f)(x) := \frac{1}{\sqrt{(2\pi)^N \det Q_t}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Q_t^{-1}(e^{tB}x-y), e^{tB}x-y\rangle} f(y)dy, \tag{9.1.5}$$

for any $t > 0$ and any $x \in \mathbb{R}^N$. The semigroup $\{T(t)\}$ is called the Ornstein-Uhlenbeck semigroup associated with the operator $\mathcal{A}$ in $C_b(\mathbb{R}^N)$.

Proof. First of all, we observe that the existence of the classical solution to the problem (9.1.3) as well as the estimate (9.1.4) are guaranteed by Theorem 2.2.1, whereas its uniqueness is an immediate consequence of Theorem 4.1.3, where we can choose as a Lyapunov function, the function $\varphi : \mathbb{R}^N \to \mathbb{R}$ defined by $\varphi(x) = |x|^2$ for any $x \in \mathbb{R}^N$. Indeed, an explicit computation shows that

$$(\mathcal{A}\varphi)(x) = \operatorname{Tr} Q + 2\langle Bx, x\rangle \leq \operatorname{Tr} Q + 2\|B\|_\infty |x|^2, \qquad x \in \mathbb{R}^N.$$

Hence, it suffices to take $\lambda = 2\|B\|_\infty$ to get $\sup_{\mathbb{R}^N}(\mathcal{A}\varphi - \lambda\varphi) < +\infty$ and, by Remark 4.0.3, this is enough to apply Theorem 4.1.3.

So, to complete the proof we just need to check the formula (9.1.5). For this purpose, we use stochastic calculus. Indeed, the stochastic differential equation associated with the operator $\mathcal{A}$ is the following one:

$$d\xi_t^x = B\xi_t^x dt + \sigma dW_t, \qquad \xi_0^x \equiv x,$$

where $\sigma = Q^{1/2}$, W_t is a standard N-dimensional Brownian motion and $x \in \mathbb{R}^N$. It is easily verified that the solution of this equation is the process

$$\xi_t^x = e^{tB}x + \int_0^t \sigma e^{(t-s)B} dW_s, \qquad t > 0.$$

This is a Gaussian process. For any fixed $t > 0$ and any $x \in \mathbb{R}^N$ the mean value of ξ_t^x is $e^{tB}x$ and the covariance matrix is the matrix Q_t defined in (9.1.1). Therefore, the distribution of ξ_t^x is the normal distribution $\mathcal{N}(e^{tB}x, Q_t)(dx)$. This implies that the semigroup $\{T(t)\}$ is given by the formula

$$(T(t)f)(x) = \mathbb{E}\, f(\xi_t^x) = \int_{\mathbb{R}^N} f(y)\mathcal{N}(e^{tB}x, Q_t)(dy), \qquad x \in \mathbb{R}^N,$$

for any $f \in B_b(\mathbb{R}^N)$, and (9.1.5) follows.

Let us give also an analytical proof of the formula (9.1.5). For this purpose, we look for a solution u of (9.1.3) in the form

$$u(t,x) = v(t, M(t)x), \qquad t > 0,\ x \in \mathbb{R}^N, \tag{9.1.6}$$

where the matrix-valued function $M : (0,+\infty) \to L(\mathbb{R}^N)$, defined by $t \mapsto M(t)$, is to be properly chosen.

A straightforward computation shows that the function v solves the Cauchy problem

$$\begin{cases} D_t v(t,x) = \mathcal{A}(t)v(t,x), & t>0,\ x\in\mathbb{R}^N, \\ v(0,x) = f(M(0)x), & x\in\mathbb{R}^N, \end{cases} \tag{9.1.7}$$

where $\mathcal{A}(t)$ is given by

$$\mathcal{A}(t) = \frac{1}{2}\mathrm{Tr}(\Lambda(t)D^2v(t,x)) + \langle (M(t)B - M'(t))(M(t))^{-1}x, Dv(t,x)\rangle, \tag{9.1.8}$$

with $\Lambda(t) = M(t)Q(M(t))^*$, for any $t>0$.

Now, we choose $M(t)$ in order to eliminate the drift term in (9.1.8). This is possible if we choose $M(\cdot)$ satisfying the differential equation $M'(\cdot) = M(\cdot)B$. For instance, we can take $M(t) = e^{tB}$ for any $t>0$. With this choice of $M(\cdot)$ the elliptic equation in (9.1.7) reduces to

$$\begin{cases} D_t v(t,x) = \dfrac{1}{2}\mathrm{Tr}(e^{tB}Qe^{tB^*}D^2v(t,x)), & t>0,\ x\in\mathbb{R}^N, \\ u(0,x) = f(x), & x\in\mathbb{R}^N. \end{cases} \tag{9.1.9}$$

Taking the Fourier transform of both the sides of the differential equation in (9.1.9) gives the ordinary differential equation

$$D_t\hat{v}(t,\xi) = -\frac{1}{2}\langle\Lambda(t)\xi,\xi\rangle\,\hat{v}(t,\xi), \qquad t>0,\ \xi\in\mathbb{R}^N.$$

Hence,

$$\hat{v}(t,\xi) = \exp\left(-\frac{1}{2}\langle Q_t\xi,\xi\rangle\right)\hat{f}(\xi), \qquad t>0,\ \xi\in\mathbb{R}^N, \tag{9.1.10}$$

where Q_t is given by (9.1.1). Taking the inverse Fourier transform gives

$$v(t,x) = \frac{1}{\sqrt{(2\pi)^N\det Q_t}}\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Q_t^{-1}(x-y),x-y\rangle}f(y)dy. \tag{9.1.11}$$

Now, the formulas (9.1.6) and (9.1.11) lead us to (9.1.5). ∎

Remark 9.1.2 The same results as in Theorem 9.1.1 hold true also in the case when Q is singular (but still positive definite) provided that the matrix Q_t in (9.1.1) is nonsingular for any $t>0$. The semigroup defined by the formula (9.1.5) is still called the (degenerate) Ornstein-Uhlenbeck semigroup associated with the operator $\mathcal{A}$ in (9.0.1). As it has been stressed in the introduction to this chapter, the condition $\det Q_t > 0$, for any $t>0$, is equivalent to the hypoellipticity of the operator $\mathcal{A}$ in the sense of Hörmander (see [73]).

9.2 Properties of $\{T(t)\}$ in $C_b(\mathbb{R}^N)$

In this section we study some properties of $\{T(t)\}$ in the space $C_b(\mathbb{R}^N)$. To begin with, let us consider the following proposition.

Proposition 9.2.1 *The following properties are met:*

(i) $\{T(t)\}$ *is conservative in* $C_b(\mathbb{R}^N)$*;*

(ii) $C_0(\mathbb{R}^N)$ *is invariant for* $\{T(t)\}$*;*

(iii) for any $t > 0$, $T(t)$ *is not compact in* $C_b(\mathbb{R}^N)$*.*

Proof. The property (i) follows immediately from Proposition 4.1.10 since $T(t)\mathbb{1} = \mathbb{1}$ for any $t > 0$.

The property (ii) follows immediately from Example 5.3.3.

Finally, the property (iii) is now a straightforward consequence of the properties (i) and (ii) and Proposition 5.3.4. ■

The estimates for the derivatives of $\{T(t)f\}$ can be easily obtained as a consequence of the results in Chapters 6 and 7. They can also be deduced differentiating under the integral sign the representation formula (9.1.5). This is the approach used in [39] that allows the authors to prove uniform (and pointwise) estimates for the space derivatives of $T(t)f$ of any order.

Proposition 9.2.2 *For any* $t > 0$ *and any* $f \in C_b(\mathbb{R}^N)$*, the function* $T(t)f$ *belongs to* $C_b^k(\mathbb{R}^N)$ *for any* $k \in \mathbb{N}$*. In particular, if* $f \in C_b^1(\mathbb{R}^N)$*,*

$$D_iT(t)f = \sum_{j=1}^{N}(e^{tB^*})_{ij}T(t)D_jf, \qquad t > 0,\ i,j = 1,\dots,N. \tag{9.2.1}$$

Moreover, for any $\varepsilon > 0$ *and any* $h \in \mathbb{N}$ *with* $h \le k$*, there exists a positive constant* $C = C_\varepsilon$ *such that*

$$\|T(t)f\|_{C_b^k(\mathbb{R}^N)} \le C\frac{e^{k(s(B)+\varepsilon)t}}{1\wedge t^{\frac{k-h}{2}}}\|f\|_{C_b^h(\mathbb{R}^N)}, \qquad t > 0, \tag{9.2.2}$$

for any $f \in C_b^h(\mathbb{R}^N)$*, where* $s(B)$ *is the spectral bound of the matrix* B*.*

Proof. The estimate (9.2.2) can be proved by induction on k. Let us begin by proving it in the case when $h = 0$. For $k = 1$ the estimate (9.2.2), as well as the formula (9.2.1), follows easily differentiating the formula (9.1.5) under the integral sign and observing that, for any $\delta > 0$, there exists a constant $\widetilde{C}_1 = \widetilde{C}_1(\delta)$ such that

$$\|e^{tB}\|_\infty \le \widetilde{C}_1 e^{(s(B)+\delta)t}, \qquad t > 0. \tag{9.2.3}$$

Let us suppose that $T(t)f$ belongs to $C_b^{m-1}(\mathbb{R}^N)$ for any $t>0$ and (9.2.2) holds for all the $(m-1)$-th order derivatives of $T(t)f$. We are going to prove that the m-th order derivative $D_{i_1,\ldots,i_m}T(t)f$ exists in $C_b(\mathbb{R}^N)$ and it satisfies (9.2.2). For this purpose, we observe that, using (9.2.1), we get

$$\begin{aligned}
&D_{i_2,\ldots,i_m}T(t)f\\
&= D_{i_2,\ldots,i_m}T(t/n)\cdot\ldots\cdot T(t/n)f\\
&= D_{i_2,\ldots,i_{m-1}}\left\{T((1-1/n)t)\sum_{j_m=1}^N (e^{\frac{n-1}{n}B^*t})_{i_mj_m}D_{j_m}T(t/n)f\right\}\\
&= T((1-(m-1)/n)t)\sum_{j_2,\ldots,j_m=1}^N (e^{\frac{n-m+1}{n}tB^*})_{i_2j_2}\cdot\ldots\cdot(e^{\frac{n-1}{n}tB^*})_{i_mj_m}\\
&\qquad\qquad\times D_{j_2}T(t/n)\circ\ldots\circ D_{j_m}T(t/n)f,
\end{aligned}\tag{9.2.4}$$

for any $n>m$ and any $t>0$. Similarly,

$$\begin{aligned}
&D_{j_2}T(t/n)\circ\ldots\circ D_{j_m}T(t/n)f\\
&= T((m/n-2/n)t)\sum_{l_2,\ldots,l_{m-1}=1}^N (e^{\frac{1}{n}tB^*})_{j_{m-1}l_{m-1}}\cdot\ldots\cdot(e^{\frac{m-2}{n}tB^*})_{j_2l_2}\\
&\qquad\qquad\times D_{l_2,\ldots,l_{m-1},j_m}T(t/n)f,
\end{aligned}\tag{9.2.5}$$

for any $t>0$. From (9.2.4), (9.2.5) and (9.2.2) (with $k=1$) we easily deduce that $D_{i_2,\ldots,i_m}T(t)\in C_b^1(\mathbb{R}^N)$. Moreover,

$$\begin{aligned}
&D_{i_1,\ldots,i_m}T(t)f\\
&= \sum_{j_2,\ldots,j_m=1}^N (e^{\frac{n-m+1}{n}tB^*})_{i_2j_2}\cdot\ldots\cdot(e^{\frac{n-1}{n}tB^*})_{i_mj_m}\\
&\qquad\times\sum_{l_1,\ldots,l_{m-1}=1}^N (e^{\frac{1}{n}tB^*})_{j_{m-1}l_{m-1}}\cdot\ldots\cdot(e^{\frac{m-2}{n}tB^*})_{j_2l_2}(e^{\frac{n-2}{n}tB^*})_{i_1l_1}\\
&\qquad\qquad\times T((1-2/n)t)D_{l_1}T(t/n)D_{l_2,\ldots,l_{m-1},j_m}T(t/n)f.
\end{aligned}\tag{9.2.6}$$

Therefore, taking $\delta=1/n$ in (9.2.3) and using (9.2.2) with $k=1$, $k=m-1$ and $\varepsilon=1/n$, we get

$$\|D_{i_1,\ldots,i_m}T(t)f\|_\infty\le C_n\exp\left\{\left(m-\frac{1}{n}\right)\left(s(B)+\frac{1}{n}\right)\right\}\frac{1}{1\wedge t^{m/2}}\|f\|_\infty,$$

for any $t>0$ and some positive constant C_n. The estimate (9.2.2) with $k=m$ follows, taking n sufficiently large.

The estimate (9.2.2) with $h = k$ follows immediately from (9.2.3). Indeed, a straightforward computation shows that

$$D_{i_1,\ldots,i_k}T(t)f = \sum_{j_1,\ldots,j_k=1}^{N} (e^{tB^*})_{i_1j_1}\cdot\ldots\cdot(e^{tB^*})_{i_kj_k}T(t)D_{j_1,\ldots,j_k}f, \qquad (9.2.7)$$

for any $i_1,\ldots,i_k \in \{1,\ldots,N\}$, any $k \in \mathbb{N}$ and any $t > 0$.

Finally, in the case when $0 < h < k$, the estimate (9.2.2) follows from (9.2.6) (with $m = k$) observing that, thanks to (9.2.7), we have

$$\begin{aligned}
&D_{l_1}T(t/n)D_{l_2,\cdots,l_{k-1},j_k}T(t/n)f \\
&= \sum_{r_1,\ldots,r_{h+1}=1}^{N} (e^{\frac{1}{n}tB^*})_{l_1r_1}(e^{\frac{1}{n}tB^*})_{l_{k-h+1}r_2}\cdot\ldots\cdot(e^{\frac{1}{n}tB^*})_{l_{k-1}r_h}(e^{\frac{1}{n}tB^*})_{j_kr_{h+1}} \\
&\qquad\qquad \times T(t/n)(D_{r_1,l_2,\ldots,l_{k-h}}T(t/n)D_{r_2,\ldots,r_{h+1}}f),
\end{aligned}$$

for any $t > 0$. ■

Remark 9.2.3 (i) From the proof of Proposition 9.2.2, it is clear that if (9.2.3) holds with $\delta = 0$, then the estimate (9.2.2) can be written also with $\varepsilon = 0$.

(ii) In view of Remark 7.2.4, when $k = 1, 2, 3$, the uniform estimate (9.2.2) is useful only in the particular case when $s(B) < 0$.

Remark 9.2.4 In the case when the matrix Q is singular and the operator $\mathcal{A}$ is hypoelliptic, uniform estimates of the Ornstein-Uhlenbeck semigroup are still available, but they differ from those in Proposition 9.2.2 since, as we will see in Chapter 14, the behaviour of the space derivatives of $T(t)f$ near $t = 0$ is anisotropic. Such estimates, which have been proved in [107, Proposition 3.2], can be written in an elegant way if the Ornstein-Uhlenbeck is written not in the usual Euclidean basis of $\mathbb{R}^N$ but in a suitable orthonormal basis $\{e'_i : i = 1,\ldots,N\}$ associated with the matrix Q.

To determine such a basis, we begin by observing that the operator $\mathcal{A}$ is hypoelliptic if and only if there exists $m \le N-1$ such that $\operatorname{rank} H_m = N$, where the matrix $H_m \in L(\mathbb{R}^{mN},\mathbb{R}^N)$ is given by $H_m = [Q^{1/2}, BQ^{1/2},\ldots,B^mQ^{1/2}]$. Such a condition is usually called the Kalman rank condition and its equivalence with the hypoellipticity condition is proved in [149, Chapter 1]. We denote by r the smallest integer such that $\operatorname{rank} H_r = N$, and by V_i $(i = 0,\ldots,r-1)$ the image of $\mathbb{R}^N$ through the matrix H_i $(0 = 1,\ldots,r)$. It is now easy to check that $\mathbb{R}^N = \bigoplus_{k=0}^{r} E_k(\mathbb{R}^N)$ where E_0 is the orthogonal projection of $\mathbb{R}^N$ in V_0 and E_i $(j = i,\ldots,r)$ is the orthogonal projection of $\mathbb{R}^N$ on $V_{i-1}^{\perp}$ in V_i, $i = 1,\ldots,r$. Then, we define the orthonormal basis $\{e'_i : i = 1,\ldots,N\}$ of $\mathbb{R}^N$ choosing e'_j in $E_i(\mathbb{R}^N)$ for any $j \in \{a_i+1,\ldots,a_{i+1}\} := I_i$ $(i = 0,\ldots,r)$, where $a_0 = 0$, $a_{r+1} = N$. In such a basis it can be shown that, for any $h \in \mathbb{N}$, any multi-index α with $|\alpha| = h$, and any $\varepsilon > 0$, there exists a constant $C_h = C_h(\varepsilon)$ such that

$$\|D^\alpha T(t)f\|_\infty \le C_h e^{(s(B)+\varepsilon)t} t^{-\frac{h}{2}-m_\alpha} \|f\|_\infty, \qquad t > 0, \tag{9.2.8}$$

where, $m_\alpha = \sum_{i=0}^{r} \sum_{j=a_i+1}^{a_{i+1}} \tau(\alpha_j)$. Here, $\tau(\alpha_j) = k$, where k is such that $j \in I_k$.

As we have already several times recalled, both the Ornstein-Uhlenbeck semigroup and the degenerate Ornstein-Uhlenbeck semigroup are neither analytic nor strongly continuous in $C_b(\mathbb{R}^N)$ and in $BUC(\mathbb{R}^N)$. Now we provide a proof of these properties. To begin with we show that $\{T(t)\}$ is not analytic, adapting the proof of [39, Lemma 3.3]. This result holds also in the case when Q is singular.

Proposition 9.2.5 *The Ornstein-Uhlenbeck semigroup is analytic neither in $C_b(\mathbb{R}^N)$ nor in $BUC(\mathbb{R}^N)$.*

Proof. We begin the proof considering the case when Q is not singular. Of course, we can limit ourselves to proving that $\{T(t)\}$ is not analytic in $BUC(\mathbb{R}^N)$. For this purpose, we prove that there exists $h \in \mathbb{R}^N$ such that

$$\sup_{h\in\mathbb{N}} \|\mathcal{A}T(1)f_h\|_\infty = +\infty, \tag{9.2.9}$$

where $f_h(x) = \sin(\langle h, x\rangle)$ for any $x \in \mathbb{R}^N$. Indeed, we have

$$(T(1)f_h)(x) = e^{-\frac{1}{2}\langle Q_1 h, h\rangle} \sin(\langle e^B x, h\rangle), \qquad x, h \in \mathbb{R}^N \tag{9.2.10}$$

(see also (9.3.6)). From (9.2.10) it follows that the function $\mathrm{Tr}\,(QD^2T(1)f_h)$ is bounded in $\mathbb{R}^N$ for any $h \in \mathbb{R}^N$. Therefore, to prove (9.2.9) we need to show that the function $x \mapsto \langle Bx, (DT(1)f_h)(x)\rangle$ is unbounded in $\mathbb{R}^N$ for some $h \in \mathbb{R}^N$. As it is easily seen

$$(B^*(DT(1)f_h)(x))_i = (B^* e^{B^*} h)_i e^{-\frac{1}{2}\langle Q_1 h, h\rangle} \cos(\langle e^B x, h\rangle),$$

for any $x \in \mathbb{R}^N$ and any $i = 1, \ldots, N$. Since $B \neq 0$, there exist $h_0 \in \mathbb{R}^N$ and $j \in \{1, \ldots, N\}$ such that $(B^* e^{B^*} h_0)_j \neq 0$. As a consequence, the function

$$r \mapsto \langle rBe_j, (DT(1)f_{h_0})(re_j)\rangle = r(B^* e^{B^*} h_0)_j e^{-\frac{1}{2}\langle Q_1 h_0, h_0\rangle} \cos(r(e^{B^*} h_0)_j)$$

is unbounded in $\mathbb{R}$ and, therefore, $\langle B\cdot, DT(1)f_{h_0}\rangle$ is unbounded in $\mathbb{R}^N$. This completes the proof in this case.

Now we assume that Q is nonnegative definite and singular, and we denote by $\tilde{\mathcal{A}}$ the operator $\mathcal{A}$ in the orthonormal basis $\{e_i' :\ i = 1, \ldots, N\}$ defined in Remark 9.2.4. Observe that $\mathcal{A} = M\tilde{\mathcal{A}}M^*$ for some orthogonal matrix M, where

$$\tilde{\mathcal{A}}\varphi = \sum_{i,j=1}^{a_1} q'_{ij}D_{ij}\varphi + \langle B\cdot, D\varphi\rangle,$$

for any smooth function φ, $Q' = (q'_{ij})$ and B' being suitable matrices, with Q' strictly positive definite, and a_1 being as in Remark 9.2.4. Then, $T(t) = M\tilde{T}(t)M^*$ for any $t > 0$, where $\{\tilde{T}(t)\}$ denotes the Ornstein-Uhlenbeck semigroup associated with the operator $\tilde{\mathcal{A}}$. Therefore, we can limit ourselves to showing that $\{\tilde{T}(t)\}$ is not analytic in $BUC(\mathbb{R}^N)$. For this purpose, we can repeat step by step the proof given in the nondegenerate case. ∎

The next proposition describes the set of strong continuity of $\{T(t)\}$. It holds for both nondegenerate and degenerate Ornstein-Uhlenbeck semigroups.

Proposition 9.2.6 *Let $f \in C_b(\mathbb{R}^N)$. Then, $\|T(t)f - f\|_\infty$ tends to 0 as t tends to 0 if and only if $f \in BUC(\mathbb{R}^N)$ and*

$$\lim_{t\to 0^+} (f(e^{tB}x) - f(x)) = 0,$$

uniformly with respect to $x \in \mathbb{R}^N$.

Proof. First of all let us notice that, if $T(t)f$ tends to f uniformly in $\mathbb{R}^N$, then $f \in BUC(\mathbb{R}^N)$. Indeed, Proposition 9.2.2 ensures that, for any $f \in C_b(\mathbb{R}^N)$, the function $T(t)f$ belongs to $BUC(\mathbb{R}^N)$. Therefore, in the rest of the proof we assume that $f \in BUC(\mathbb{R}^N)$.

For any $t > 0$, let G_t be the linear operator in $BUC(\mathbb{R}^N)$ defined by

$$(G_tf)(x) = \frac{1}{\sqrt{(2\pi)^N \det Q_t}} \int_{\mathbb{R}^N} f(x-y)e^{-\frac{1}{2}\langle Q_t^{-1}y,y\rangle}dy, \qquad x \in \mathbb{R}^N.$$

Then, we have $(T(t)f)(x) = (G_tf)(e^{tB}x)$ for any $t > 0$ and any $x \in \mathbb{R}^N$. Let us observe that G_tf tends to f uniformly in $\mathbb{R}^N$ as n tends to $+\infty$. Indeed, for any $n \in \mathbb{N}$,

$$\begin{aligned}
|(G_tf)(x) - f(x)| &= \left|\frac{1}{\sqrt{(2\pi)^N}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}|y|^2}\big(f(x+\sqrt{Q_t}y) - f(x)\big)dy\right| \\
&\le \frac{1}{\sqrt{(2\pi)^N}} \int_{B(n)} e^{-\frac{1}{2}|y|^2}|f(x+\sqrt{Q_t}y) - f(x)|dy \\
&\quad + \frac{2}{\sqrt{(2\pi)^N}}\|f\|_\infty \int_{\mathbb{R}^N\setminus B(n)} e^{-\frac{1}{2}|y|^2}dy, \qquad t > 0,\ x \in \mathbb{R}^N.
\end{aligned} \tag{9.2.11}$$

From (9.2.11) it is now clear that for any $\varepsilon > 0$ we can find first n sufficiently large and, then, t sufficiently close to 0 such that both the two last integrals are less then $\varepsilon/2$, uniformly with respect to $x \in \mathbb{R}^N$.

Now, the conclusion follows writing

$$(T(t)f)(x) - f(x) = (G_t f)(e^{tB}x) - f(e^{tB}x) + f(e^{tB}x) - f(x),$$

for any $t > 0$ and any $x \in \mathbb{R}^N$. ■

To conclude this section we characterize the spectrum of the weak generator $\widehat{A}$ of the Ornstein-Uhlenbeck operator in $C_b(\mathbb{R}^N;\mathbb{C})$ when Q is not singular. For the proof we refer the reader to [112, Corollary 6.3], where the characterization of the spectrum of the restriction of $\widehat{A}$ to $BUC(\mathbb{R}^N;\mathbb{C})$ is given. But the spectra of $\widehat{A}$ and its restriction to $BUC(\mathbb{R}^N;\mathbb{C})$ actually coincide. Indeed, of course, $\sigma(\widehat{A}_{|BUC(\mathbb{R}^N;\mathbb{C})}) \subset \sigma(\widehat{A})$. Moreover, since $D(\widehat{A}) \subset BUC(\mathbb{R}^N;\mathbb{C})$ the point spectra of $\widehat{A}$ and $\widehat{A}_{|BUC(\mathbb{R}^N;\mathbb{C})}$ coincide. Further, if $\mathrm{Re}\,\lambda > 0$, then $\lambda \in \rho(\widehat{A})$. Indeed, fix $f \in C_b(\mathbb{R}^N;\mathbb{C})$ and consider a bounded sequence $\{f_n\} \subset C_c^\infty(\mathbb{R}^N;\mathbb{C})$ converging locally uniformly in $\mathbb{R}^N$ to f. Taking the estimates in Proposition 9.2.2 into account it is easy to check that

$$(R(\lambda, \widehat{A}_{|BUC(\mathbb{R}^N;\mathbb{C})})f_n)(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f_n)(x)dt, \tag{9.2.12}$$

for any $x \in \mathbb{R}^N$ and any $n \in \mathbb{N}$, where we still denote by $\{T(t)\}$ the extension of the Ornstein-Uhlenbeck semigroup to complex-valued functions. From formula (9.2.12) and Proposition 2.2.9, it is immediate to check that $u_n = R(\lambda, \widehat{A}_{|BUC(\mathbb{R}^N;\mathbb{C})})f_n$ converges locally uniformly to a function $u \in C_b(\mathbb{R}^N;\mathbb{C})$. Since $\widehat{A}u_n = \lambda u_n - f_n$ and

$$\|R(\lambda, \widehat{A}_{|BUC(\mathbb{R}^N;\mathbb{C})})f_n\|_\infty \le K\|f_n\|_\infty, \qquad n \in \mathbb{N}, \tag{9.2.13}$$

for some positive constant K, independent of n, it follows that $\{\widehat{A}u_n\}$ is bounded in $C_b(\mathbb{R}^N;\mathbb{C})$ and it converges locally uniformly in $\mathbb{R}^N$. Therefore, according to Proposition 2.3.5, $u \in D(\widehat{A})$ and $\lambda u - \widehat{A}u = f$. Moreover, by (9.2.13) we get $\|u\|_\infty \le K\|f\|_\infty$ since we can assume that $\|f_n\|_\infty \le \|f\|_\infty$ for any $n \in \mathbb{N}$.

Theorem 9.2.7 *If $\sigma(B) \cap i\mathbb{R} = \varnothing$ then $\sigma(\widehat{A}) = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \le 0\}$.*

9.3 The invariant measure μ and the semigroup in L^p_μ

In this section we assume that

$$\sigma(B) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}. \tag{9.3.1}$$

As a straightforward consequence, the matrix Q_t (see (9.1.1)) can be defined also for $t = +\infty$. Q_∞ is a positive definite matrix and it is strictly positive definite if Q_t is strictly positive definite for some $t > 0$.

We set

$$\mu(dx) := \mathcal{N}(0, Q_\infty)(dx) = \frac{1}{\sqrt{(2\pi)^N \det Q_\infty}} e^{-\frac{1}{2}\langle Q_\infty^{-1}x, x\rangle} dx := \rho(x)dx. \quad (9.3.2)$$

As the following theorem shows, μ is the invariant measure of both the nondegenerate and the degenerate Ornstein-Uhlenbeck semigroups.

Proposition 9.3.1 *If* (9.3.1) *holds, then the Gaussian measure* μ *is the unique invariant measure of the semigroup* $\{T(t)\}$.

Proof. In the case of the nondegenerate Ornstein-Uhlenbeck semigroup, the uniqueness of the invariant measure of $\{T(t)\}$ follows immediately from Theorem 8.1.15.

Actually, Theorem 8.1.15 holds also in the case when the diffusion matrix Q is singular. Indeed, to make the proof of that proposition work, one needs to be able to extend the semigroup with continuity to a strong Feller and irreducible semigroup in to $B_b(\mathbb{R}^N)$ and to prove the convergence of the sequence $(T(t)f_n)(x)$ to $(T(t)f)(x)$, as n tends to $+\infty$, for any $t > 0$ and any $x \in \mathbb{R}^N$, whenever the sequence $\{f_n\} \subset B_b(\mathbb{R}^N)$ converges in a dominated way to f. Of course, these two properties are satisfied also by the degenerate Ornstein-Uhlenbeck semigroup, as it can be easily checked using the representation formula (9.1.5).

So, let us prove that, in both the nondegenerate and the degenerate case $\{T(t)\}$, the Gaussian measure μ is the invariant measure of $\{T(t)\}$. Throughout the proof, we extend $\{T(t)\}$ to the space of bounded and continuous complex-valued functions in the natural way, and we still denote by $\{T(t)\}$ the so extended semigroup.

Denote by $\mathcal{E}$ the linear span of the set $\{f_h : h \in \mathbb{R}^N\}$ where $f_h : \mathbb{R}^N \to \mathbb{C}$ is defined by $f_h(x) = e^{i\langle x,h\rangle}$ for any $x, h \in \mathbb{R}^N$. We observe that $\mathcal{E}$ is dense in L^1_μ. To check this property, it suffices to show that, if $T \in (L^1_\mu)'$ vanishes on $\mathcal{E}$, then $T = 0$. For this purpose, we recall that $(L^1_\mu)' = L^\infty_\mu \subset L^\infty(\mathbb{R}^N, dx)$ (see [135, Theorem 6.16]). Fix $T \in (L^1_\mu)'$ vanishing on $\mathcal{E}$ and let $g \in L^\infty(\mathbb{R}^N, dx)$ be such that $Tf = \int_{\mathbb{R}^N} fgd\mu$ for any $f \in L^1_\mu$. Then

$$\int_{\mathbb{R}^N} g(x)e^{i\langle x,h\rangle} e^{-\frac{1}{2}\langle Q_\infty x,x\rangle} dx = 0,$$

for any $h \in \mathbb{R}^N$. It follows that the Fourier transform of the function $x \mapsto g(x)e^{-\langle Q_\infty x,x\rangle/2}$ identically vanishes in $\mathbb{R}^N$. By the uniqueness of the Fourier transform, we deduce that $g \equiv 0$.

Hence, to prove that μ is the invariant measure of the Ornstein-Uhlenbeck semigroup, it suffices to show that

$$\int_{\mathbb{R}^N} T(t)f_h d\mu = \int_{\mathbb{R}^N} f_h d\mu, \qquad t>0, \;\; h \in \mathbb{R}^N. \tag{9.3.3}$$

For this purpose, fix $a \in \mathbb{R}^N$ and a strictly positive definite matrix K, and let us compute the Fourier transform of the probability measure $\mathcal{N}(a,K)(dx)$ (see (9.1.2)).
Let M be an orthogonal matrix such that $MKM^* = \operatorname{diag}(\lambda_1,\ldots,\lambda_N)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^N} f_h \mathcal{N}(a,K)(dy) &= \frac{e^{i\langle a,h\rangle}}{\sqrt{(2\pi)^N \det K}} \int_{\mathbb{R}^N} e^{i\langle M^* y,h\rangle} \exp\left(-\sum_{i=1}^N \frac{y_i^2}{2\lambda_i}\right) dy \\
&= \frac{e^{i\langle a,h\rangle}}{\sqrt{(2\pi)^N \lambda_1\cdot\ldots\cdot\lambda_N}} \prod_{k=1}^N \int_{\mathbb{R}} e^{iy_k(Mh)_k} e^{-\frac{y_k^2}{2\lambda_k}} dy_k \\
&= e^{i\langle a,h\rangle} \prod_{k=1}^N e^{-\frac{1}{2}\lambda_k (Mh)_k^2} \\
&= e^{i\langle a,h\rangle - \frac{1}{2}\langle Kh,h\rangle},
\end{aligned} \tag{9.3.4}$$

for any $h \in \mathbb{R}^N$. Therefore,

$$\int_{\mathbb{R}^N} f_h d\mu = \int_{\mathbb{R}^N} e^{i\langle h,y\rangle} \mathcal{N}(0,Q_\infty)(dy) = e^{-\frac{1}{2}\langle Q_\infty h,h\rangle}, \qquad h \in \mathbb{R}^N. \tag{9.3.5}$$

On the other hand, we have

$$(T(t)f_h)(x) = \int_{\mathbb{R}^N} e^{i\langle h,y\rangle} \mathcal{N}(e^{tB}x, Q_t)(dy) = e^{i\langle e^{tB}x,h\rangle - \frac{1}{2}\langle Q_t h,h\rangle}, \tag{9.3.6}$$

for any $t>0$, any $x,h \in \mathbb{R}^N$, and then

$$\begin{aligned}
\int_{\mathbb{R}^N} T(t)f_h d\mu &= e^{-\frac{1}{2}\langle Q_t h,h\rangle} \int_{\mathbb{R}^N} e^{i\langle e^{tB^*}h,x\rangle} \mathcal{N}(0,Q_\infty)(dx) \\
&= e^{-\frac{1}{2}\langle (Q_t + e^{tB}Q_\infty e^{tB^*})h,h\rangle}.
\end{aligned} \tag{9.3.7}$$

Now, observing that

$$\begin{aligned}
Q_t + e^{tB}Q_\infty e^{tB^*} &= \int_0^t e^{sB}Qe^{sB^*} ds + \int_0^{+\infty} e^{(t+s)B}Qe^{(t+s)B^*} ds \\
&= \int_0^t e^{sB}Qe^{sB^*} ds + \int_t^{+\infty} e^{sB}Qe^{sB^*} ds = Q_\infty,
\end{aligned} \tag{9.3.8}$$

for any $t>0$, from (9.3.5) and (9.3.7) we get (9.3.3). ∎

Remark 9.3.2 As it is shown in [43, Section 11.2.3], (9.3.1) is also a necessary condition to guarantee the existence of an invariant measure of $\{T(t)\}$.

In view of Remark 9.3.2, throughout the rest of this section we assume that the condition (9.3.1) is satisfied.

According to Proposition 8.1.8, the semigroup $\{T(t)\}$ can be extended to a strongly continuous semigroup in $L^p_\mu(\mathbb{R}^N)$. As in Chapter 8, we simply write L^p_μ for $L^p(\mathbb{R}^N,\mu)$ and denote by $\|\cdot\|_p$ its norm. Moreover, we denote $W^{k,p}_\mu$ the Sobolev space of the functions in L^p_μ with weak derivatives up to k-th-order in L^p_μ. Finally we denote by $L_p : D(L_p) \subset L^p_\mu \to L^p_\mu$ ($p \in [1,+\infty)$) the infinitesimal generator of $\{T(t)\}$ in L^p_μ and we simply write L for L_2.

To prove the main result of this first part of the section (i.e., the analyticity of $\{T(t)\}$ in L^p_μ for any $p \in (1,+\infty)$), we need some preliminary results.

Lemma 9.3.3 *For any $p \in [1,+\infty)$, $C^\infty_c(\mathbb{R}^N)$ is dense in L^p_μ and in $W^{k,p}_\mu$ for any $k \in \mathbb{N}$.*

Proof. The proof can be obtained using the same truncation argument as in the proof of Proposition 8.7.3. ∎

Lemma 9.3.4 *Let $p \in [1,+\infty)$. Then, for any $u \in W^{1,p}_\mu$ and any $j = 1,\ldots,N$, the function $x \mapsto x_j u(x)$ belongs to L^p_μ. Moreover, there exists a positive constant C_j such that*

$$\left(\int_{\mathbb{R}^N} |x_j|^p |u(x)|^p d\mu\right)^{\frac{1}{p}} \le C_j \left(\int_{\mathbb{R}^N} (|u(x)|^p + |Du(x)|^p) d\mu\right)^{\frac{1}{p}}.$$

Proof. Without loss of generality we can limit ourselves to proving the assertion in the case when $j = 1$. Moreover, we can assume that $Q_\infty = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$. Indeed, we can always reduce to this situation by means of a suitable linear change of variables as in the proof of Proposition 9.3.1. From the formula (9.1.5) we immediately deduce that

$$\rho(x) = \frac{1}{\sqrt{(2\pi)^N \lambda_1 \cdot \ldots \cdot \lambda_n}} \exp\left(-\sum_{i=1}^N \frac{x_j^2}{2\lambda_j}\right), \qquad x \in \mathbb{R}^N.$$

Therefore,

$$x_1 \rho(x) = -\lambda_1 D_1 \rho(x), \qquad x \in \mathbb{R}^N. \tag{9.3.9}$$

Let us now fix $u \in C^\infty_c(\mathbb{R}^N)$ and assume that $p > 1$ (the case $p = 1$ is similar and much easier). Taking (9.3.9) into account and integrating by parts, we deduce the following chain of inequalities

$$\int_{\mathbb{R}^N} |x_1 u(x)|^p d\mu$$
$$\le -\lambda_1 \int_{\mathbb{R}^N} x_1 |x_1|^{p-2} |u(x)|^p D_1 \rho(x) dx$$

$$
\begin{aligned}
&= \lambda_1 p \int_{\mathbb{R}^N} x_1 u(x)|x_1 u(x)|^{p-2} D_1 u(x) d\mu + \lambda_1(p-1)\int_{\mathbb{R}^N} |x_1|^{p-2}|u(x)|^p d\mu \\
&= I_1 + I_2. \qquad (9.3.10)
\end{aligned}
$$

Using the Hölder and Young inequalities we immediately see that

$$
\begin{aligned}
I_1 &\le \lambda_1 p \left(\int_{\mathbb{R}^N} |x_1 u(x)|^p d\mu\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |D_1 u|^p d\mu\right)^{\frac{1}{p}} \\
&\le \frac{1}{4}\int_{\mathbb{R}^N} |x_1 u(x)|^p d\mu + K_1 \int_{\mathbb{R}^N} |D_1 u|^p d\mu, \qquad (9.3.11)
\end{aligned}
$$

for any $\varepsilon > 0$ and some positive constant $K_1 = K_1(\varepsilon)$.

As far as I_2 is concerned, we observe that, if $p \ge 2$, by the Young inequality, we can determine a positive constant K_2 such that

$$
|x_1|^{p-2} \le \frac{1}{4\lambda_1(p-1)}|x_1|^p + K_2, \qquad x_1 \in \mathbb{R}.
$$

Therefore,

$$
I_2 \le \frac{1}{4}\int_{\mathbb{R}^N} |x_1|^p |u(x)|^p d\mu + K_3 \int_{\mathbb{R}^N} |u(x)|^p d\mu, \qquad (9.3.12)
$$

for some positive constant K_3.
If $p < 2$ we have to apply a different argument to estimate the term I_2. For notational convenience we split $x \in \mathbb{R}^N$ as $x = (x_1, z)$. Moreover, we denote by $d\mu_1$ and $d\mu_2$ the measures in $\mathbb{R}$ and $\mathbb{R}^{N-1}$ whose densities with respect to the Lebesgue measure are

$$
\rho_1(x_1) = \frac{1}{\sqrt{(2\pi)\lambda_1}} \exp\left(-\frac{x_1^2}{2\lambda_1}\right), \qquad x_1 \in \mathbb{R}
$$

and

$$
\rho_2(z) = \frac{1}{\sqrt{(2\pi)^{N-1}\lambda_2 \cdot \ldots \cdot \lambda_N}} \exp\left(-\sum_{j=1}^{N-1} \frac{z_j^2}{2\lambda_{j+1}}\right), \qquad z \in \mathbb{R}^{N-1},
$$

respectively. Applying the Fubini-Tonelli theorem we deduce that

$$
\begin{aligned}
I_2 &\le 2\lambda_1 \int_{\mathbb{R}^{N-1}} d\mu_2 \int_{\mathbb{R}} |x_1|^{p-2}|u(x_1,z)|^p d\mu_1 \\
&= 2\lambda_1 \int_{\mathbb{R}^{N-1}} d\mu_2 \int_{|x_1|\ge\sqrt{8\lambda_1}} |x_1|^{p-2}|u(x_1,z)|^p d\mu_1 \\
&\quad +2\lambda_1 \int_{\mathbb{R}^{N-1}} d\mu_2 \int_{-\sqrt{8\lambda_1}}^{\sqrt{8\lambda_1}} |x_1|^{p-2}|u(x_1,z)|^p d\mu_1
\end{aligned}
$$

$$\leq \frac{1}{4}\int_{\mathbb{R}^{N-1}} d\mu_2 \int_{\mathbb{R}} |x_1|^p |u(x_1,z)|^p d\mu_1$$
$$+ \int_{\mathbb{R}^{N-1}} d\mu_2 \int_{-\sqrt{8\lambda_1}}^{\sqrt{8\lambda_1}} |x_1|^{p-2}|u(x_1,z)|^p d\mu_1. \qquad (9.3.13)$$

Let us estimate the second integral in the last side of (9.3.13). For this purpose we observe that since $W^{1,p}(-\sqrt{8\lambda_1},\sqrt{8\lambda_1})$ is continuously embedded in $L^\infty(-\sqrt{8\lambda_1},\sqrt{8\lambda_1})$ (see [2, Theorem 5.4]), then

$$\int_{-\sqrt{8\lambda_1}}^{\sqrt{8\lambda_1}} |x_1|^{p-2}|u(x_1,z)|^p d\mu_1$$
$$\leq \left(\sup_{|x_1|\leq\sqrt{8\lambda_1}} |u(x_1,z)|\right)^p \int_{-\sqrt{8\lambda_1}}^{\sqrt{8\lambda_1}} |x_1|^{p-2} dx$$
$$\leq K_4 \int_{-\sqrt{8\lambda_1}}^{\sqrt{8\lambda_1}} (|u(x_1,z)|^p + |D_1 u(x_1,z)|^p) dx_1$$
$$\leq K_4 \sup_{|x_1|\leq\sqrt{8\lambda_1}} |\rho_1(x_1)|^{-1} \int_{-\sqrt{8\lambda_1}}^{\sqrt{8\lambda_1}} (|u(x_1,z)|^p + |D_1 u(x_1,z)|^p) d\mu_1,$$

for some positive constant K_4. Integrating in $\mathbb{R}^{N-1}$ with respect to the measure μ_2 gives

$$\int_{\mathbb{R}^{N-1}} d\mu_2 \int_{-\sqrt{8\lambda_1}}^{\sqrt{8\lambda_1}} |x_1|^{p-2}|u(x_1,z)|^p d\mu_1 \leq K_5 \|u\|_{1,p}^p,$$

for some constant $K_5 > 0$. Therefore,

$$I_2 \leq \frac{1}{4}\int_{\mathbb{R}^{N-1}} d\mu_2 \int_{\mathbb{R}} |x_1|^p |u(x_1,z)|^p d\mu_1 + K_5\|u\|_{1,p}^p. \qquad (9.3.14)$$

Summing up, from (9.3.10)-(9.3.12) and (9.3.14), we deduce that there exists a positive constant K_6, independent of u, such that

$$\int_{\mathbb{R}^N} |x_1 u(x)|^p d\mu \leq K_6 \|u\|_{1,p}^p. \qquad (9.3.15)$$

Since by Lemma 9.3.3 $C_c^\infty(\mathbb{R}^N)$ is dense in $W_\mu^{1,p}$, the estimate (9.3.15) yields the assertion. ■

The following corollary is now a straightforward consequence of Lemma 9.3.4.

Corollary 9.3.5 *The realization of the Ornstein-Uhlenbeck operator $\mathcal{A}$ in L_μ^p with domain $W_\mu^{2,p}$ is a bounded linear operator.*

Lemma 9.3.6 *For any $f \in L^p_\mu$ ($p \in [1,+\infty)$) and any $t > 0$, $T(t)f$ is given by the formula* (9.1.5).

Proof. In order to avoid misunderstanding, throughout the proof we denote by $\{T(t)\}$ the Ornstein-Uhlenbeck semigroup in $C_b(\mathbb{R}^N)$ and by $\{T_p(t)\}$ its extension to L^p_μ.

To begin with, let us observe that since, for any $t > 0$, Q_t and Q_∞ differ in a (strictly) positive definite matrix, then $Q_t < Q_\infty$ in the sense of the positive matrices (i.e., $\langle Q_t h, h\rangle < \langle Q_\infty h, h\rangle$ for any $h \in \mathbb{R}^N$). Hence, $Q_t Q_\infty^{-1} < Q_\infty Q_\infty^{-1} = I$ or, equivalently, $Q_\infty^{-1} = Q_t^{-1} Q_t Q_\infty^{-1} < Q_t^{-1}$. As a consequence,

$$\rho_t(x) := \frac{1}{\sqrt{(2\pi)^N \det Q_t}} e^{-\frac{1}{2}\langle Q_t^{-1}x,x\rangle} \le C_t \rho(x), \qquad x \in \mathbb{R}^N,$$

for some positive constant C_t, and this implies that $L^1_\mu \subset L^1_{\mu_t}$ where $\mu_t(dx) = \rho_t(x)dx$. Therefore, if $\{f_n\}$ is a sequence of smooth functions converging to f in L^p_μ as n tends to $+\infty$, then, for any $t > 0$ and any $x \in \mathbb{R}^N$, $f_n(e^{tB}x + \cdot)$ converges to $f(e^{tB}x + \cdot)$ in L^1_μ and, hence,

$$\lim_{n\to+\infty} (T(t)f_n)(x) = \lim_{n\to+\infty} \int_{\mathbb{R}^N} f_n(e^{tB}x + y)\mu_t(dy) = \int_{\mathbb{R}^N} f(e^{tB}x + y)\mu_t(dy).$$

Now, the assertion follows if we recall that, by definition, $T(t)f_n$ converges to $T_p(t)f$ in L^p_μ (see Proposition 8.1.8). ■

Remark 9.3.7 Note that the results in Corollary 9.3.5 and Lemma 9.3.6 hold true also in the case when the matrix Q is singular.

Arguing as in the proof of Proposition 9.2.2, we can prove the following estimates for the derivatives of the function $T(t)f$ in the L^p_μ-norm.

Proposition 9.3.8 *For any $p \in (1,+\infty)$, any $k \in \mathbb{N}$, any $t > 0$ and any $f \in L^p_\mu$ the function $T(t)f$ belongs to $W^{k,p}_\mu$. Moreover, for any multi-index α and any $\varepsilon > 0$, there exists a positive constant $C = C(\varepsilon, |\alpha|, p)$ such that*

$$\|D^\alpha T(t)f\|_p \le Ce^{|\alpha|(s(B)+\varepsilon)t} t^{-\frac{|\alpha|}{2}} \|f\|_p, \qquad t > 0, \tag{9.3.16}$$

for any $p \in (1,+\infty)$, and

$$\|D^\alpha T(t)f\|_p \le Ce^{|\alpha|(s(B)+\varepsilon)t} \sum_{|\beta|=|\alpha|} \|D^\beta f\|_p, \qquad t > 0, \tag{9.3.17}$$

for any $p \in [1,+\infty)$. Further, the function $t \mapsto D^\alpha T(t)f$ is continuous in $(0,+\infty)$ with values in L^p_μ for any $f \in L^p_\mu$ ($p \in (1,+\infty)$).

Proof. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{k,p}_\mu$ (see Lemma 9.3.3), it suffices to prove the assertion in the case when $f \in C_c^\infty(\mathbb{R}^N)$.

To prove the estimate (9.3.16) it suffices to argue by induction on the length of $|\alpha|$ as in the proof of Proposition 9.2.2. So, we can limit ourselves to checking (9.3.16) in the case when $|\alpha| = 1$. For this purpose, we fix $f \in C_c^\infty(\mathbb{R}^N)$, $i \in \{1, \ldots, N\}$, $p \in (1, +\infty)$ and observe that

$$(D_i T(t)f)(x) = -\frac{1}{\sqrt{(2\pi)^N \det Q_t}} \int_{\mathbb{R}^N} (e^{tB^*} Q_t^{-1} y)_i e^{-\frac{1}{2}\langle Q_t^{-1} y, y\rangle} f(e^{tB}x - y)dy,$$

for any $t > 0$ and any $x \in \mathbb{R}^N$. Since

$$|(e^{tB^*} Q_t^{-1} y)_i| = |\langle Q_t^{-1/2} y, Q_t^{-1/2} e^{tB} e_i\rangle| \le |Q_t^{-1/2} y||(Q_t^{-1/2} e^{tB})_i|,$$

for any $t > 0$, any $y \in \mathbb{R}^N$ and any $i = 1, \ldots, N$, using the Hölder inequality and taking (9.2.3) into account, we easily see that

$$\begin{aligned} |(D_i T(t)f)(x)|^p &\le \left(\int_{\mathbb{R}^N} |(e^{tB^*} Q_t^{-1} y)_i|^{\frac{p}{p-1}} e^{-\frac{1}{2}\langle Q_t^{-1} y, y\rangle} dy \right)^{p-1} (T(t)(|f|^p))(x) \\ &\le \|(Q_t^{-1/2} e^{tB})_i\|^{\frac{p}{p-1}} (T(t)(|f|^p))(x) \\ &\quad \times \left(\frac{1}{\sqrt{(2\pi)^N \det Q_t}} \int_{\mathbb{R}^N} |Q_t^{-1/2} y|^{\frac{p}{p-1}} e^{-\frac{1}{2}\langle Q_t^{-1} y, y\rangle} dy \right)^{p-1} \\ &\le C_p t^{-\frac{1}{2}} e^{(s(B)+\varepsilon)t} (T(t)(|f|^p))(x), \end{aligned} \tag{9.3.18}$$

for any $t > 0$, any $x \in \mathbb{R}^N$ and some positive constant C_p. Hence integrating the first and the last term in (9.3.18) in $\mathbb{R}^N$, with respect to the measure μ, we get (9.3.16).

The estimate (9.3.17) is easier to be proved: indeed, it suffices to iterate the formula (9.2.1).

Finally, let us prove that for any multi-index α, the function $D^\alpha T(\cdot)f$ is continuous in $(0, +\infty)$ with values in L^p_μ. To see it, we observe that the estimate (9.3.16) implies that

$$\|D^\alpha T(t)f - D^\alpha T(s)f\|_p \le C e^{|\alpha|(s(B)+\varepsilon)t} s^{-\frac{|\alpha|}{2}} \|T(t-s)f - f\|_p,$$

for any $0 < s < t$ and any $p \in (1, +\infty)$. Now, the strong continuity of $\{T(t)\}$ in L^p_μ yields the assertion. ∎

Now, we can show that the Ornstein-Uhlenbeck semigroup is analytic in L^p_μ for any $p \in (1, +\infty)$. Such a result has been proved in [39] and in [61, 67] also in infinite dimensional settings. Here, we present a simplified proof taken from [106].

Theorem 9.3.9 *For any $p \in (1, +\infty)$, the Ornstein-Uhlenbeck semigroup is analytic in L^p_μ.*

Proof. Fix $p \in (1, +\infty)$. According to Theorem B.2.8 to prove the assertion we can limit ourselves to showing that the function $T(\cdot)f$ is differentiable in $(0, +\infty)$, for any $f \in L^p_\mu$, and there exists a positive constant C, independent of f, such that

$$\|tD_tT(t)f\|_p \le C\|f\|_p, \qquad t \in (0,1). \tag{9.3.19}$$

Suppose first that $f \in C^\infty_c(\mathbb{R}^N)$. In such a case, taking Propositions 2.3.5, 2.3.6 and 4.1.10 into account, it is easy to check that the function $T(\cdot)f$ is differentiable in $[0, +\infty)$ with values in L^p_μ and $D_tT(t)f = \mathcal{A}T(t)f$ for any $t \ge 0$. Moreover, taking Corollary 9.3.5 and the estimate (9.3.16) into account, it is immediate to check that

$$t\|\mathcal{A}T(t)f\|_p \le C_1\|f\|_p, \qquad t > 0, \tag{9.3.20}$$

for some positive constant C_1, independent of f, and (9.3.19) follows.

Now, suppose that $f \in L^p_\mu$ and let $\{f_n\} \in C^\infty_c(\mathbb{R}^N)$ be a sequence of smooth functions converging to f in L^p_μ (see Lemma 9.3.3). Since

$$\frac{T(t)f_n - T(t_0)f_n}{t - t_0} - \mathcal{A}T(t_0)f_n = \frac{1}{t - t_0}\int_{t_0}^t (\mathcal{A}T(s)f_n - \mathcal{A}T(t_0)f_n)ds, \tag{9.3.21}$$

for any $t_0, t \in (0, +\infty)$ and any $n \in \mathbb{N}$ (where the integral is meant in L^p_μ), letting n go to $+\infty$, and observing that (9.3.20) is satisfied by f_n ($n \in \mathbb{N}$) and f, with the same constant C_1, we easily see that (9.3.21) holds true also with f instead of f_n. Now using the continuity of the function $t \mapsto \mathcal{A}T(t)f$ in $(0, +\infty)$ with values in L^p_μ (see Corollary 9.3.5 and Proposition 9.3.8), we can conclude that $T(\cdot)f$ is differentiable in $(0, +\infty)$ with values in L^p_μ and $D_tT(t)f = \mathcal{A}T(t)f$ for any $t > 0$. Hence, the estimate (9.3.19) follows and we are done. ∎

The next proposition characterizes the symmetric Ornstein-Uhlenbeck semigroups, i.e., the case when $T(t)$ coincides with its adjoint operator for any $t > 0$. The characterization of the symmetric Ornstein-Uhlenbeck semigroups has been studied in [32] and [33] also in the infinite dimensional setting.

Proposition 9.3.10 *The Ornstein-Uhlenbeck semigroup is symmetric in L^2_μ if and only if $BQ = QB^*$. In such a case L is self-adjoint. Further, if $B = -Q_\infty^{-1}$, then*

$$\int_{\mathbb{R}^N} Lf\, g d\mu = -\int_{\mathbb{R}^N} \langle QDf, Dg\rangle d\mu, \tag{9.3.22}$$

for any $f, g \in D(L)$.

Proof. As a first step we show that, for any $t > 0$, the adjoint operator of $T(t)$ is given by

$$(T(t)^*f)(x) = \int_{\mathbb{R}^N} f(Q_\infty^{1/2}e^{tB_\infty^*}Q_\infty^{-1/2}x + Q_\infty^{1/2}(I - e^{tB_\infty^*}e^{tB_\infty})^{1/2}Q_\infty^{-1/2}y)d\mu, \tag{9.3.23}$$

for any $t>0$ and any $f\in\mathcal{D}:=\{g_h^{(j)}:\ h\in\mathbb{R}^N,\ j=1,2\}$, where $g_h^{(1)}(x)=\cos(\langle h,x\rangle)$ and $g_h^{(2)}(x)=\sin(\langle h,x\rangle)$ for any x and any h in $\mathbb{R}^N$. Here, $B_\infty=Q_\infty^{-1/2}BQ_\infty^{1/2}$. Note that the matrix $(I-e^{tB_\infty^*}e^{tB_\infty})^{1/2}$ is well defined. Indeed, using the formula (9.3.8), it is easy to check that

$$Q_t=Q_\infty^{1/2}(I-e^{tB_\infty}e^{tB_\infty^*})Q_\infty^{1/2},\qquad t>0. \tag{9.3.24}$$

As a consequence, the matrix $I-e^{tB_\infty}e^{tB_\infty^*}$ is strictly definite positive and this implies that $\|e^{tB_\infty}\|_{L(L^2_\mu)}=\|e^{tB_\infty^*}\|_{L(L^2_\mu)}<1$ for any $t>0$. Therefore, $I-e^{tB_\infty^*}e^{tB_\infty}$ is a strictly positive definite matrix.

Let us fix $t>0$ and $g\in\mathcal{D}$, and denote by $R(t)$ the operator defined by the right-hand side of (9.3.23). Since $\mathcal{D}$ is dense in L^2_μ (see the proof of Proposition (9.3.1)), to prove that $T(t)^*g=R(t)g$ it suffices to show that

$$\int_{\mathbb{R}^N}gT(t)f\,d\mu=\int_{\mathbb{R}^N}fR(t)g\,d\mu, \tag{9.3.25}$$

for any $f\in\mathcal{D}$. For this purpose, we begin by observing that, from the formula (9.3.6), we get

$$T(t)g_h^{(j)}=e^{-\frac12\langle Q_th,h\rangle}g^{(j)}_{e^{tB^*}h},\qquad j=1,2. \tag{9.3.26}$$

Similarly, setting $P_{1,t}=Q_\infty^{1/2}(I-e^{tB_\infty^*}e^{tB_\infty})^{1/2}Q_\infty^{-1/2}$, $P_{2,t}=Q_\infty^{1/2}e^{tB_\infty^*}Q_\infty^{-1/2}$, we obtain that

$$\begin{aligned}(R(t)g_h^{(j)})(x)&=\int_{\mathbb{R}^N}g_h^{(j)}(P_{2,t}x+P_{1,t}y)\mathcal{N}(0,Q_\infty)(dy)\\&=\int_{\mathbb{R}^N}g_h^{(j)}(P_{1,t}y)\mathcal{N}(P_{1,t}^{-1}P_{2,t}x,Q_\infty)(dy)\\&=\int_{\mathbb{R}^N}g_h^{(j)}(y)\mathcal{N}(P_{2,t}x,P_{1,t}Q_\infty P_{1,t}^*)(dy)\\&=e^{-\frac12\langle Q_\infty^{1/2}(I-e^{tB_\infty^*}e^{tB_\infty})Q_\infty^{1/2}h,h\rangle}g^{(j)}_{Q_\infty^{-1/2}e^{tB_\infty}Q_\infty^{1/2}h},\end{aligned} \tag{9.3.27}$$

for any $x\in\mathbb{R}^N$ and $j=1,2$, where we took advantage of (9.3.4) in the last side of (9.3.27), and $\mathcal{N}(a,K)$ is defined in (9.1.2) for any $a\in\mathbb{R}^N$ and any positive definite matrix K.

Since the density of μ (with respect to the Lebesgue measure) is even with respect to each variable, from (9.3.26) and (9.3.27) we easily deduce that we can limit ourselves to checking (9.3.25) in the case when $(f,g)=(g_h^{(j)},g_k^{(j)})$, for $j=1,2$. This can easily be done, observing that $2g_h^{(j)}g_k^{(j)}=g_{h-k}^{(j)}+(-1)^{j-1}g_{h+k}^{(j)}$ for any $h,k\in\mathbb{R}^N$ and $j=1,2$. Indeed, we get

$$\begin{aligned}\int_{\mathbb{R}^N}g_k^{(j)}T(t)g_h^{(j)}d\mu=\frac12e^{-\frac12\langle Q_th,h\rangle}\Big(&e^{-\frac12\langle Q_\infty(e^{tB^*}h-k),e^{tB^*}h-k\rangle}\\&+(-1)^{j-1}e^{-\frac12\langle Q_\infty(e^{tB^*}h+k),e^{tB^*}h+k\rangle}\Big),\end{aligned}$$

whereas

$$\begin{aligned}
&\int_{\mathbb{R}^N} g_h^{(j)} R(t) g_k^{(j)}\, d\mu \\
&= \frac{1}{2} e^{-\frac{1}{2}\langle Q_\infty^{1/2}(I-e^{tB_\infty^*}e^{tB_\infty})Q_\infty^{1/2}k,k\rangle} \\
&\qquad \times \Big(e^{\frac{1}{2}\langle Q_\infty(h-Q_\infty^{-1/2}e^{tB_\infty}Q_\infty^{1/2}k),h-Q_\infty^{-1/2}e^{tB_\infty}Q_\infty^{1/2}k\rangle} \\
&\qquad\qquad + (-1)^{j-1} e^{\frac{1}{2}\langle Q_\infty(h+Q_\infty^{-1/2}e^{tB_\infty}Q_\infty^{1/2}k),h+Q_\infty^{-1/2}e^{tB_\infty}Q_\infty^{1/2}k\rangle} \Big),
\end{aligned}$$

from which the equality (9.3.26) immediately follows observing that

$$e^{tB}Q_\infty e^{tB^*} + Q_t = Q_\infty, \qquad t > 0.$$

Now, the formulas (9.3.26) and (9.3.27) imply that $T(t)$ is symmetric for any $t > 0$ if and only

$$(i)\ Q_\infty^{-1/2} e^{tB_\infty} Q_\infty^{1/2} = e^{tB^*}, \qquad (ii)\ Q_\infty^{1/2}(I - e^{tB_\infty^*}e^{tB_\infty})Q_\infty^{1/2} = Q_t. \quad (9.3.28)$$

Differentiating both the members of (9.3.28)(i) at $t = 0$, we see that such a condition holds if and only if $B_\infty Q_\infty^{1/2} = Q_\infty^{1/2} B^*$ or, equivalently, if and only if B_∞ is symmetric. Taking (9.3.24) into account, the condition (9.3.28)(ii) is now an immediate consequence of (9.3.28)(i). Summing up, we have shown that $T(t)$ is symmetric in L^2_μ for any $t > 0$ if and only if the matrix B_∞ is symmetric or, equivalently, if and only if $BQ_\infty = Q_\infty B^*$. In such a case, according to [123, Chapter 1, Corollary 10.6], the operator L is self-adjoint in L^2_μ.

To conclude the first part of the proof, let us show that $BQ_\infty = Q_\infty B^*$ if and only if $BQ = QB^*$. Showing that $BQ = QB^*$ implies that $BQ_\infty = Q_\infty B^*$ follows immediately from (9.0.2). Conversely, let us assume that $BQ_\infty = Q_\infty B^*$. Using the formula (9.3.8) it is immediate to check that

$$Q_t B^* = BQ_t, \qquad t > 0. \tag{9.3.29}$$

Differentiating (9.3.29) at $t = 0$ gives immediately $BQ = QB^*$.

We now assume that $B = -Q_\infty^{-1}$ and we prove the formula (9.3.22). For this purpose, since

$$D_i\rho(x) = -\frac{1}{2}\rho(x) D_i(\langle Q_\infty^{-1}x, x\rangle) = -\rho(x)(Q_\infty^{-1}x)_i, \qquad i = 1, \dots, N,$$

an integration by parts gives

$$\begin{aligned}
&\int_{\mathbb{R}^N} \langle QDf(x), Dg(x)\rangle d\mu \\
&= \sum_{i,j=1}^N \int_{\mathbb{R}^N} q_{ij} D_i f(x) D_j g(x)\, \rho(x) dx \\
&= -\sum_{i,j=1}^N \int_{\mathbb{R}^N} q_{ij} g(x) \left(D_{ij} f(x)\rho(x) + D_i f(x) D_j \rho(x)\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} q_{ij}g(x)\left(D_{ij}f(x) - D_if(x)(Q_\infty^{-1}x)_j\right)\rho(x)dx \\
&= -\int_{\mathbb{R}^N} g(x)(\mathrm{Tr}\,(QD^2f(x)) - \langle QDf(x), Q_\infty^{-1}x\rangle)d\mu,
\end{aligned}
$$

for any $f, g \in C_c^\infty(\mathbb{R}^N)$, that is (9.3.22). Since $C_c^\infty(\mathbb{R}^N)$ is a core of L (see the forthcoming Lemma 9.3.13), the formula (9.3.22) follows for any $f, g \in D(L)$. ■

To conclude the first part of this section, let us show that the Ornstein-Uhlenbeck semigroup satisfies the Poincaré inequality.

Theorem 9.3.11 *The Ornstein-Uhlenbeck semigroup satisfies the Poincaré inequality* (8.6.1) *with* $p = 2$ *and* $C_\varepsilon = -(s(B) + \varepsilon)^{-1}$ *for any* $\varepsilon < -s(B)$.

Proof. It can be obtained repeating step by step the proof of Theorem 8.6.3, replacing everywhere the constant d_0 defined in (8.3.6) with $s(B) + \varepsilon$, where ε is as in the assertion of the theorem. ■

Remark 9.3.12 Note that our assumptions on the matrix B do not imply, in general, that the constant d_0 in (8.3.6) is negative. This is the reason why we cannot apply immediately Theorem 8.6.3.

Suppose, for instance, that

$$
B = \begin{pmatrix} -1 & 1 \\ -6 & -4 \end{pmatrix}.
$$

Then, B has two eigenvalues with negative real parts. Moreover,

$$
\langle B\xi, \xi\rangle = -\xi_1^2 - 5\xi_1\xi_2 - 4\xi_2^2, \qquad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,
$$

which is not strictly less than 0 in $\mathbb{R}^2$.

9.3.1 The domain of the realization of $\{T(t)\}$ in $L^p_\mu(\mathbb{R}^N)$

In this subsection we give a complete characterization of the domain of L_p for any $p \in [1, +\infty)$. The results that we present are taken from [118]. We notice that in the case when $p = 2$ the characterization of the domain of L_2 has been proved in [106] and in [36] (also in the infinite dimensional case) and, then, it has been generalized to a more general context first in [41] and, then, in [40]. Here, we consider the case when Q is not singular.

The following lemma will be very useful to characterize the domain of L_p.

Lemma 9.3.13 *For any* $p \in [1, +\infty)$, $C_c^\infty(\mathbb{R}^N)$ *is a core of* L_p.

Proof. We first prove that the Schwarz space $\mathcal{S}$, of all the smooth functions $f:\mathbb{R}^N\to\mathbb{R}$ such that $\lim_{|x|\to+\infty}|x|^kD^\alpha f(x)=0$ for any $k\in\mathbb{N}$ and any $\alpha\in\mathbb{N}^N$, is a core of L_p. For this purpose, let us notice that the same arguments as in the first part of the proof of Theorem 9.3.9 show that $\mathcal{S}$ is contained in $D(L_p)$ for any $p\in[1,+\infty)$. Moreover, a straightforward computation shows that $T(t)$ maps $\mathcal{S}$ into itself for any $t>0$. Therefore, according to Lemma 9.3.3 and Proposition B.1.10, $\mathcal{S}$ is a core of $\{T(t)\}$.

Now to prove that $C_c^\infty(\mathbb{R}^N)$ is a core of L_p, it suffices to show that, for any $f\in\mathcal{S}$, there exists a sequence $\{f_n\}\subset C_c^\infty(\mathbb{R}^N)$ such that f_n and $\mathcal{A}f_n$ converge, respectively, to f and $\mathcal{A}f$ in L^p_μ, as n tends to $+\infty$. So, let $f\in\mathcal{S}$ and, for any $x\in\mathbb{R}^N$ and any $n\in\mathbb{N}$, let $\vartheta_n(x)=\vartheta(x/n)$ where $\vartheta\in C_c^\infty(\mathbb{R}^N)$ satisfies $\chi_{B(1/2)}\le\vartheta\le\chi_{B(1)}$. Then, the function $f_n=f\vartheta_n$ belongs to $C_c^\infty(\mathbb{R}^N)$ for any $n\in\mathbb{N}$ and it converges to f in $W^{2,p}_\mu$ as n tends to $+\infty$. Moreover, since

$$\langle x,Df_n(x)\rangle=\vartheta_n(x)\langle x,Df(x)\rangle+\frac{1}{n}f(x)\langle x,D\vartheta(n^{-1}x)\rangle,\qquad x\in\mathbb{R}^N,$$

$\mathcal{A}f_n$ converges to $\mathcal{A}f$ in L^p_μ. ■

Now, we characterize the domain of L_p for any $p\in(1,+\infty)$, when the Ornstein-Uhlenbeck operator $\mathcal{A}$ is given by

$$(\mathcal{A}u)(x)=\frac{1}{2}\Delta u(x)-\sum_{i=1}^N\frac{x_i}{2\lambda_i}D_iu(x),\qquad x\in\mathbb{R}^N,\tag{9.3.30}$$

where λ_i $(i=1,\ldots,N)$ are suitable positive constants. This a crucial step in order to characterize the domain of the more general Ornstein-Uhlenbeck operator. In the case when $\mathcal{A}$ is given by (9.3.30), $\{T(t)\}$ is symmetric (see Proposition 9.3.10) and it is given by

$$(T(t)f)(x)=\frac{1}{\sqrt{(2\pi)^N\prod_{i=1}^N\lambda_i(1-e^{-\frac{t}{\lambda_i}})}}\times\int_{\mathbb{R}^N}\exp\left(-\sum_{i=1}^N\frac{(e^{-\frac{t}{2\lambda_i}}x_i-y_i)^2}{2\lambda_i(1-e^{-\frac{t}{\lambda_i}})}\right)f(y)dy,\tag{9.3.31}$$

for any $t>0$, any $x\in\mathbb{R}^N$ and any $f\in L^p_\mu$. Moreover, the density ρ of the invariant measure of the semigroup $\{T(t)\}$, with respect to the Lebesgue measure, is given by

$$\rho(x)=\frac{1}{\sqrt{(2\pi)^N\lambda_1\cdot\ldots\cdot\lambda_N}}\exp\left(-\sum_{i=1}^N\frac{x_i^2}{2\lambda_i}\right),\qquad x\in\mathbb{R}^N.$$

Proposition 9.3.14 *Suppose that the operator $\mathcal{A}$ is given by* (9.3.30). *Then, for any $p\in(1,+\infty)$, $D(L_p)=W^{2,p}_\mu$.*

Proof. Since the proof is rather long, we divide it into three steps.

Step 1. Let us introduce the semigroup $\{T_i(t)\}$ $(i = 1, \ldots, N)$ defined in $L^p_\mu(\mathbb{R}^N)$ $(p \in (1, +\infty)$ by

$$(T_i(t)f)(x) = \frac{1}{\sqrt{2\pi\lambda_i(1 - e^{-\frac{t}{\lambda_i}})}} \int_{\mathbb{R}} \exp\left(-\frac{y^2}{2\lambda_i(1 - e^{-\frac{t}{\lambda_i}})}\right) f(E_i(t, x, y))dy, \tag{9.3.32}$$

for any $t > 0$ and any $x \in \mathbb{R}^N$, where $E_i(t, x, y) \in \mathbb{R}^N$ is defined by

$$\langle E_i(t, x, y), e_j\rangle = \begin{cases} x_j, & \text{if } i \neq j; \\ e^{-t/(2\lambda_i)}x_i - y, & \text{if } i = j. \end{cases}$$

Arguing as in the proof of Proposition 8.1.8 and Theorem 9.3.9, it can be shown that $\{T_i(t)\}$ is an analytic strongly continuous semigroup of positive contractions in L^p_μ. We denote by $L_{p,i}$ its infinitesimal generator and we prove that it coincides with the operator $A_{p,i}$ defined by $A_{p,i}u(x) = \frac{1}{2}D_{ii}u(x) - (2\lambda_i)^{-1}x_iD_iu(x)$ for any $u \in D(A_{p,i}) = \{u \in L^p_\mu : D_iu,\ D_{ii}u \in L^p_\mu\}$. For this purpose, we observe that, arguing as in the proof of Lemma 9.3.13, we can easily show that $C^\infty_c(\mathbb{R}^N)$ is a core of $L_{p,i}$ and that $L_{p,i} = A_{p,i}$ on $C^\infty_c(\mathbb{R}^N)$. Since, as the proof of Lemma 9.3.4 shows, $A_{p,i}$ is a bounded operator in $D(A_{p,i})$ endowed with the norm $\|u\|_{i,p} = \|u\|_p + \|D_iu\|_p + \|D_{ii}u\|_p$, then $L_{p,i}$ is the closure of the operator $A_{p,i}$. Therefore, to prove the assertion it suffices to show that $A_{p,i}$ is closed in L^p_μ. This will follow immediately if we prove that there exists a positive constant C_1 such that

$$\|D_iu\|_p + \|D_{ii}u\|_p \le C_1\left(\|u\|_p + \|A_{i,p}u\|_p\right), \qquad u \in C^\infty_c(\mathbb{R}^N), \tag{9.3.33}$$

for any $i = 1, \ldots, N$. Equivalently, we can prove that

$$\int_{\mathbb{R}} |u'|^p d\mu_i + \int_{\mathbb{R}} |u''|^p d\mu_i \le C_2 \int_{\mathbb{R}} (|u|^p + |\tilde{L}_iu|^p) d\mu_i, \qquad i = 1, \ldots, N, \tag{9.3.34}$$

for any $u \in C^\infty_c(\mathbb{R})$ and some positive constant C_2, where

$$(\tilde{L}_iu)(x) = \frac{1}{2}u''(x) - \frac{1}{2\lambda_i}xu'(x), \qquad x \in \mathbb{R},$$

and $d\mu_i(x) = (\sqrt{2\pi\lambda_i})^{-1}\exp(-x^2/(2\lambda_i))dx$.

Let us observe that we can limit ourselves to showing that there exists $C_3 > 0$ such that

$$\int_{\mathbb{R}} |xu'(x)|^p d\mu_i \le C_3 \int_{\mathbb{R}} (|u|^p + |\tilde{L}_iu|^p) d\mu_i, \qquad u \in C^\infty_c(\mathbb{R}). \tag{9.3.35}$$

Indeed, once (9.3.35) is checked, it will be immediate to show that the $L^p_{\mu_i}$-norm of u'' satisfies (9.3.34). Concerning the $L^p_{\mu_i}$-norm of u', we observe that

$$\begin{aligned}
\frac{1}{\sqrt{2\pi\lambda_i}}|u'(x)|^p e^{-\frac{x^2}{2\lambda_i}} &= \frac{1}{\sqrt{2\pi\lambda_i}}\int_{-\infty}^x \frac{d}{dt}\left(|u'(t)|^p e^{-\frac{t^2}{2\lambda_i}}\right) dt \\
&\le p\int_{\mathbb{R}} |u'(x)|^{p-1}|u''(x)|d\mu_i + \frac{1}{\lambda_i}\int_{\mathbb{R}} |x||u'(x)|^p d\mu_i \\
&\le p\left(\int_{\mathbb{R}} |u'(x)|^p d\mu_i\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}} |u''(x)|^p d\mu_i\right)^{\frac{1}{p}} \\
&\quad +\frac{1}{4}\int_{\mathbb{R}} |u'(x)|^p d\mu_i + C_4\int_{\mathbb{R}} |x|^p|u'(x)|^p d\mu_i \\
&\le \frac{1}{2}\int_{\mathbb{R}} |u'(x)|^p d\mu_i + C_4\int_{\mathbb{R}} |x|^p|u'(x)|^p d\mu_i \\
&\quad +C_5\int_{\mathbb{R}} |u''(x)|^p d\mu_i, \qquad (9.3.36)
\end{aligned}$$

for any $x \in \mathbb{R}$ and some positive constants C_4 and C_5. Here, in the last two inequalities we took advantage of the Young inequality. Integrating (9.3.36) in $\mathbb{R}$ with respect to the Lebesgue measure gives

$$\int_{\mathbb{R}} |u'(x)|^p d\mu_i \le 2C_4\int_{\mathbb{R}} |x|^p|u'(x)|^p d\mu_i + 2C_5\int_{\mathbb{R}} |u''(x)|^p d\mu_i$$

and the estimate for u' in (9.3.34) follows.

So, let us prove (9.3.35). For this purpose, we fix $u \in C_c^\infty(\mathbb{R})$, we multiply both the sides of the equation $u-\tilde{L}_1 u =: f$ by $e^{-x^2/2\lambda_i}$ and, then, we integrate in $(-\infty, x)$ and in $(x, +\infty)$, with respect to the measure $d\mu_i$. We get

$$u'(x) = 2\int_x^{\pm\infty} (f(y)-u(y))e^{\frac{x^2-y^2}{2\lambda_i}} dy, \qquad x \in \mathbb{R}^N.$$

Therefore,

$$u'(x)e^{-\frac{x^2}{2p\lambda_i}} = 2\int_x^{\pm\infty} h(y)e^{\frac{x^2-y^2}{2p'\lambda_i}} dy, \qquad x \in \mathbb{R}^N,$$

where $1/p+1/p' = 1$ and we have set $h(y) = e^{-y^2/(2p\lambda_i)}(f(y)-u(y))$ for any $y \in \mathbb{R}$. Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}} |xu'(x)|^p d\mu_i \\
&= 2\int_0^{+\infty}\left|x\int_x^{+\infty} h(y)e^{\frac{x^2-y^2}{2p'\lambda_i}} dy\right|^p d\mu_i + 2\int_{-\infty}^0\left|x\int_{-\infty}^x h(y)e^{\frac{x^2-y^2}{2p'\lambda_i}} dy\right|^p d\mu_i \\
&\le 4\int_0^{+\infty}\left|x\int_x^{+\infty} (|h(y)|+|h(-y)|)e^{\frac{x^2-y^2}{2p'\lambda_i}} dy\right|^p d\mu_i.
\end{aligned}$$

Now, observing that $e^{(x^2-y^2)/(2p'\lambda_i)} \le e^{x(x-y)/(p'\lambda_i)}$ for any $0 \le x \le y$, and using the Hölder inequality and then the Fubini-Tonelli theorem, we deduce that

$$\begin{aligned}\int_{\mathbb{R}} |xu'(x)|^p d\mu_i &\le 4\int_0^{+\infty} dx \left(x\int_x^{+\infty} (|h(y)|+|h(-y)|)^p e^{\frac{x(x-y)}{p'\lambda_i}} dy\right) \\ &\qquad\qquad\qquad \times \left(\int_x^{+\infty} xe^{\frac{x(x-y)}{p'\lambda_i}} dy\right)^{p-1} \\ &= 4(p'\lambda_i)^{p-1}\int_0^{+\infty} (|h(y)|+|h(-y)|)^p dy \int_0^y xe^{\frac{x(x-y)}{p'\lambda_i}} dx.\end{aligned} \tag{9.3.37}$$

Since

$$\begin{aligned}\int_0^y xe^{\frac{x(x-y)}{p'\lambda_i}} dx &\le y^2\int_0^1 e^{-\frac{y^2t(1-t)}{p'\lambda_i}} dt \\ &= y^2\int_0^{\frac12} e^{-\frac{y^2t(1-t)}{p'\lambda_i}} dt + y^2\int_{\frac12}^1 e^{-\frac{y^2t(1-t)}{p'\lambda_i}} dt \\ &= 2y^2\int_0^{\frac12} e^{-\frac{y^2t(1-t)}{p'\lambda_i}} dt \\ &\le 2y^2\int_0^{\frac12} e^{-\frac{y^2t}{2p'\lambda_i}} dt \le 4p'\lambda_i,\end{aligned}$$

from (9.3.37) we easily deduce that

$$\begin{aligned}\int_{\mathbb{R}} |xu'(x)|^p d\mu_i &\le 2^{p+3}(p'\lambda_i)^p\|h\|_{L^p(\mathbb{R})} \\ &\le 2^{p+3}(p'\lambda_i)^p(\|u\|_p + \|\tilde{L}_i u\|_p),\end{aligned}$$

and (9.3.35) follows.

Step 2. We now prove that

$$D(L_p) = \{u \in L^p_\mu : D_i u,\ D_{ii}u \in L^p_\mu,\ i = 1,\dots,N\}. \tag{9.3.38}$$

By virtue of Step 1, we can assume that $N > 1$. A straightforward computation shows that

$$(T(t)f = (T_1(t)\circ\ldots\circ T_N(t))\,f, \qquad t > 0,\ \ f \in L^p_\mu.$$

Since for any $i, j \in \{1,\dots,N\}$ the operators $L_{p,i}$ and $L_{p,j}$ commute in the resolvent sense, then a general theorem for commuting analytic semigroups implies that L_p is the closure of the operator $\sum_{i=1}^N L_{p,i}$ with domain given by the right-hand side of (9.3.38) (see [38, Theorem 3.3]). Hence, to prove (9.3.38) it suffices to show that the operator $\sum_{i=1}^N L_{p,i}$ is closed. For this purpose, we use Theorem B.1.15. According to Proposition B.1.13(iii), the operators $I - L_{i,p}$ ($i = 1, \ldots, N$) admit bounded imaginary powers in L^p_μ with power angles $\theta_{i,p} := \theta_{I-L_{i,p}}$ less or equal to $\pi/2$. Now, since $I - L_{i,2}$ is self-adjoint in L^2_μ (see Proposition 9.3.10), from Proposition B.1.13(v) we deduce that $\theta_{i,2} = 0$, and the Riesz-Thorin interpolation theorem (see Theorem A.4.9) implies that $\theta_{i,p} < \pi/2$ for any $p \in (1, +\infty)$. Hence, Theorem B.1.15 applies and yields the closedness of the sum $\sum_{i=1}^N (I - L_{i,p})$ (equivalently, the closedness of the operator $\sum_{i=1}^N L_{i,p}$) in $\{u \in L^p_\mu : D_i u,\ D_{ii}u \in L^p_\mu,\ i = 1, \ldots, N\}$.

Step 3. To conclude the proof, we need to show that $W^{2,p}_\mu = \{u \in L^p_\mu : D_i u,\ D_{ii} u \in L^p_\mu,\ i = 1, \ldots, N\}$. For this purpose, we fix $u \in L^p_\mu$ such that $D_i u, D_{ii} u \in L^p_\mu$, for any $i = 1, \ldots, N$, and introduce the function $v \in L^p(\mathbb{R}^N)$ defined by

$$v(x) = u(x) \exp\left(-\sum_{i=1}^N \frac{x_i^2}{2p\lambda_i}\right), \qquad x \in \mathbb{R}^N,$$

and we prove that $\Delta v \in L^p(\mathbb{R}^N)$. Since

$$\begin{aligned} D_{ii} v(x) = &\left(D_{ii} u(x) - \frac{x_i}{p\lambda_i} D_j u(x) - \frac{1}{p\lambda_i} u(x) + \frac{x_i^2}{(p\lambda_i)^2} u(x)\right) \\ &\times \exp\left(-\sum_{i=1}^N \frac{x_i^2}{2p\lambda_i}\right), \end{aligned}$$

for any $x \in \mathbb{R}^N$ and any $i = 1, \ldots, N$, according to the proof of Lemma 9.3.4, the function $x \mapsto x_i D_i u$ belongs to $L^p_\mu(\mathbb{R}^N)$. Repeating the same arguments as in the proof of the quoted lemma, we can easily show that the function $x \mapsto x_i^2 u(x)$ belongs to L^p_μ as well. Therefore, $D_{ii} v \in L^p(\mathbb{R}^N)$ and, consequently, $\Delta v \in L^p(\mathbb{R}^N)$. By classical results for elliptic equations (see Theorem C.1.3(iii)), $v \in W^{2,p}(\mathbb{R}^N)$.

As a consequence of the previous results, we also deduce that the functions $x \mapsto x_i x_j v(x)$ and $x \mapsto x_i D_i v(x)$ belong to $L^p(\mathbb{R}^N)$ for any $i, j \in \{1, \ldots, N\}$. Since

$$\begin{aligned} D_{ij} u(x) = &\left(D_{ij} v(x) + \frac{x_i}{p\lambda_i} D_j v(x) + \frac{x_j}{p\lambda_j} D_i v(x) + \frac{x_i x_j}{p^2 \lambda_i \lambda_j} v(x)\right) \\ &\times \exp\left(\sum_{i=1}^N \frac{x_i^2}{2p\lambda_i}\right), \end{aligned}$$

for any $x \in \mathbb{R}^N$ and any $i, j \in \{1, \ldots, N\}$, it follows that $D_{ij} u \in L^p_\mu$. ∎

In order to characterize the domain of the more general Ornstein-Uhlenbeck operator $\mathcal{A}$, let us prove the following proposition which allows us to transform the operator $\mathcal{A}$ in a perturbation of the operator defined in (9.3.30). For this purpose, with any nonsingular matrix M we associate the operator Φ_M defined by $(\Phi_M u)(x) = u(Mx)$, for any $x \in \mathbb{R}^N$ and any $u\,\mathbb{R}^N \to \mathbb{R}$. It is clearly seen that Φ_M is an isometry between $L^p_{\tilde{\mu}}$ and L^p_{μ} $(p \in [1, +\infty))$, where

$$\tilde{\mu}(dx) = \frac{1}{|\det M|\sqrt{(2\pi)^N \det Q_\infty}} e^{-\frac{1}{2}\langle (M^*)^{-1}Q_\infty^{-1}M^{-1}x, x\rangle} dx := \tilde{\rho}(x)dx. \tag{9.3.39}$$

Lemma 9.3.15 *There exists a nonsingular matrix $M \in L(\mathbb{R}^N)$ such that the operator $\tilde{\mathcal{A}} = \Phi_{M^{-1}}\mathcal{A}\Phi_M$ is the Ornstein-Uhlenbeck operator defined, on smooth functions, by*

$$(\tilde{\mathcal{A}}u)(x) = \mathcal{A}^0 u(x) + \langle B^0 x, Du(x)\rangle, \qquad x \in \mathbb{R}^N, \tag{9.3.40}$$

where

$$(\mathcal{A}^0 u)(x) = \frac{1}{2}\Delta u(x) - \sum_{i=1}^N \frac{x_i}{2\lambda_i} D_i u(x), \qquad x \in \mathbb{R}^N \tag{9.3.41}$$

and

$$B^0 = MBM^{-1} + \frac{1}{2}\mathrm{diag}\,(\lambda_1^{-1}, \ldots, \lambda_N^{-1}). \tag{9.3.42}$$

The measure

$$\tilde{\mu}(dx) = \frac{1}{\sqrt{(2\pi)^N \lambda_1 \cdot \ldots \cdot \lambda_N}} \exp\left(-\sum_{i=1}^N \frac{x_i^2}{2\lambda_i}\right) dx \tag{9.3.43}$$

is the invariant measure of both the semigroup $\{\tilde{T}(t)\}$ associated with the operator $\tilde{\mathcal{A}}$ and the semigroup $\{T^0(t)\}$ associated with the operator $\mathcal{A}^0$. Finally, denoting by $\tilde{L}_p$ the generator of $\{\tilde{T}(t)\}$ in $L^p_{\tilde{\mu}}$ $(p \in (1, +\infty))$, it holds that $D(L_p) = \Phi_M(D(\tilde{L}_p))$ and $\tilde{L}_p := \Phi_{M^{-1}} L_p \Phi_M$.

Proof. Since Q is strictly positive definite, then the matrix $Q^{-1/2}$ is well defined. Now, let M_1 be an orthogonal matrix such that

$$M_1 Q^{-1/2} Q_\infty (Q^{-1/2}) M_1^* = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_N).$$

A straightforward computation shows that, setting $M = M_1 Q^{-1/2}$, we have

$$(\tilde{\mathcal{A}}u)(x) = \frac{1}{2}\Delta u(x) + \langle MBM^{-1}x, Du(x)\rangle, \qquad x \in \mathbb{R}^N. \tag{9.3.44}$$

Therefore, $\tilde{\mathcal{A}}$ is still an Ornstein-Uhlenbeck operator. The decomposition in (9.3.40)-(9.3.42) now easily follows.

The associated Ornstein-Uhlenbeck semigroup is given by (9.1.5) where now the matrix Q_t is replaced with the matrix

$$\begin{aligned}\tilde{Q}_t &= \int_0^t e^{sMBM^{-1}} e^{s(M^{-1})^*B^*M^*} ds \\ &= \int_0^t Me^{sB}M^{-1}(M^{-1})^* e^{sB^*} M^* ds \\ &= \int_0^t Me^{sB}Qe^{sB^*}M^* ds = MQ_tM^*, \qquad t > 0. \qquad (9.3.45)\end{aligned}$$

In particular, $\tilde{Q}_\infty = MQ_\infty M^* = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Therefore, according to Proposition 9.3.1, the measure $\tilde{\mu}$ in (9.3.43) is the invariant measure of both the semigroups $\{\tilde{T}(t)\}$ and $\{T^0(t)\}$.

Let us now denote by $\tilde{L}_p$ the infinitesimal generator of $\{\tilde{T}(t)\}$ in $L^p_{\tilde{\mu}}$ ($p \in (1, +\infty)$). Since by Lemma 9.3.13, $C_c^\infty(\mathbb{R}^N)$ is a core of both L_p and $\tilde{L}_p$, and $\tilde{L}_p = \tilde{\mathcal{A}} = \Phi_{M^{-1}}\mathcal{A}\Phi_M = \Phi_{M^{-1}}L_p\Phi_M$ in $C_c^\infty(\mathbb{R}^N)$, we easily deduce that $D(\tilde{L}_p) = \Phi_{M^{-1}}(D(L_p))$ and $\tilde{L}_p = \Phi_{M^{-1}}L_p\Phi_M$. This completes the proof. ∎

Lemma 9.3.16 *Let $\tilde{\mu}$ and B^0 be the measure and the matrix defined, respectively, in* (9.3.39) *and* (9.3.42). *Then, for any $p \in [1, +\infty)$, the family of bounded operators $\{S(t) : t \geq 0\}$ in $L^p_{\tilde{\mu}}$, defined by $(S(t)f)(x) = f(e^{tB^0}x)$ for any $t \in [0, +\infty)$, any $x \in \mathbb{R}^N$ and any $f \in L^p_{\mu}$, is a semigroup of isometries. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core of the infinitesimal generator B^0_p and $(B^0_p u)(x) = \langle B^0x, Du(x)\rangle$ for any $u \in C_c^\infty(\mathbb{R}^N)$ and any $x \in \mathbb{R}^N$.*

Proof. To begin with, let us observe that a straightforward computation shows that

$$\begin{aligned}&\int_{\mathbb{R}^N} |S(t)f|^p d\tilde{\mu} \\ &= \frac{e^{-t\mathrm{Tr}\, B^0}}{\sqrt{(2\pi)^N(\lambda_1 \cdot \ldots \cdot \lambda_N)}} \int_{\mathbb{R}^N} |f(y)|^p \exp\left(-\frac{1}{2}\langle \tilde{Q}_\infty^{-1} e^{-tB^0}y, e^{-tB^0}y\rangle\right) dy,\end{aligned}$$

for any $t > 0$, where $\tilde{Q}_\infty = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_N)$. To prove that $\|S(t)f\|_p = \|f\|_p$, it suffices to show that $\mathrm{Tr}\, B^0 = 0$ and

$$\langle \tilde{Q}_\infty^{-1} e^{-tB^0}h, e^{-tB^0}h\rangle = \langle \tilde{Q}_\infty^{-1}h, h\rangle, \qquad t > 0, \;\; h \in \mathbb{R}^N. \qquad (9.3.46)$$

Applying (9.3.8) to the Ornstein-Uhlenbeck operator $\tilde{\mathcal{A}}$ (see (9.3.44)), we get

$$\tilde{Q}_t e^{-t(MBM^{-1})^*} + e^{tMBM^{-1}}\tilde{Q}_\infty = \tilde{Q}_\infty e^{-t(MBM^{-1})^*}, \qquad (9.3.47)$$

for any $t > 0$, where $\tilde{Q}_t$ is given by (9.3.45). Differentiating both the sides of (9.3.47) at $t = 0$ gives

$$I + MBM^{-1}\tilde{Q}_\infty = -\tilde{Q}_\infty(MBM^{-1})^*$$

or, equivalently,

$$\tilde{Q}_\infty^{-1}B^0\tilde{Q}_\infty = -(B^0)^*. \tag{9.3.48}$$

From (9.3.48) we get $\operatorname{Tr} B^0 = 0$ and

$$\tilde{Q}_\infty^{-1}e^{-tB^0} = e^{t(B^0)^*}\tilde{Q}_\infty^{-1}, \qquad t > 0,$$

which implies (9.3.46).

Let us now observe that $\{S(t)\}$ is a strongly continuous group. Indeed, for any $f \in C_c^\infty(\mathbb{R}^N)$, $f(e^{tB^0}\cdot)$ converges uniformly to f as t tends to 0. Therefore, $S(t)f$ tends to f as t tends to 0 in $L^p_{\tilde{\mu}}$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p_{\tilde{\mu}}$ (see Lemma 9.3.3) and $\{S(t)\}$ is a semigroup of isometries in $L^p_{\tilde{\mu}}$, using Proposition A.1.2, it is immediate to check that, for any $f \in L^p_{\tilde{\mu}}$, $S(t)f$ tends to f as t tends to 0.

Denote now by B_p^0 the infinitesimal generator of the semigroup $\{S(t)\}$ in L^p_μ. To prove that $C_c^\infty(\mathbb{R}^N)$ is a core of B_p^0, one can argue as in the proof of Lemma 9.3.13. Indeed, it is easy to verify that the Schwarz space $\mathcal{S}$ is contained in $D(B_p^0)$ and $(B_p^0 f)(x) = \langle B^0x, Df(x)\rangle$ for any $x \in \mathbb{R}^N$ and any $f \in \mathcal{S}$. Moreover, $S(t)$ maps $\mathcal{S}$ into itself for any $t > 0$. Therefore, by Proposition B.1.10, $\mathcal{S}$ is a core of $\mathcal{B}_p^0$. Then, the same approximation argument as in the proof of Lemma 9.3.13 shows that $C_c^\infty(\mathbb{R}^N)$ is a core of B_p^0 and $(B_p^0 f)(x) = \langle Bx, Du(x)\rangle$ for any $x \in \mathbb{R}^N$ and any $u \in W^{2,p}_{\tilde{\mu}}$. ∎

Theorem 9.3.17 *For any $p \in (1,+\infty)$, $D(L_p) = W^{2,p}_\mu$ and $L_p u = \mathcal{A}u$ for any $u \in W^{2,p}_\mu$.*

Proof. By virtue of the Lemma 9.3.15, to characterize $D(L_p)$, it suffices to characterize $D(\tilde{L}_p)$. In particular, since Φ_M is an isometry between $W^{2,p}_{\tilde{\mu}}$ and $W^{2,p}_\mu$, if we show that $D(\tilde{L}_p) = W^{2,p}_{\tilde{\mu}}$ ($p \in (1,+\infty)$), we will immediately deduce that $D(L_p) = W^{2,p}_\mu$.

We fix now, and for the rest of the proof, an arbitrary $p \in (1,+\infty)$. Moreover, we denote by L_0^p the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $\{T^0(t)\}$ in L^p_μ, associated with the operator $\mathcal{A}^0$ in (9.3.41), and, by B_p^0, the operator in Lemma 9.3.16. By Proposition 9.3.14, $D(L_p^0) = W^{2,p}_{\tilde{\mu}}$ and $L_p^0 u = \mathcal{A}^0 u$ for any $u \in W^{2,p}_{\tilde{\mu}}$. Since $C_c^\infty(\mathbb{R}^N)$ is a core of $\tilde{L}_p$ (see Lemma 9.3.13) and the operator $L_p^0 + B_p^0$ with domain $W^{2,p}_{\tilde{\mu}}$ coincides with the bounded operator $\tilde{\mathcal{A}}$ (see Corollary 9.3.5), then $\tilde{L}_p$ is the closure of the sum of the operators L_p^0 and B_p^0. To complete the proof it suffices to show that the operator $L_p^0 + B_p^0$, with domain $W^{2,p}_\mu$, is closed. For this purpose, we prove that all the assumptions of Theorem B.1.16 are satisfied.

As a first step we observe that, repeating the same arguments as in the proof of Proposition 9.3.14 and observing that the operator L_2^0 is self-adjoint in $L^2_{\tilde{\mu}}$ (see Proposition 9.3.10), one can easily check that, for any $\lambda > 0$, both the operators $I-L^0_p$ and $\lambda I-B^0_p$ admit bounded imaginary powers. Moreover, by Proposition B.1.13(iii) it follows that the power angle $\theta_{\lambda I-B^0_p}\le\pi/2$ of $\lambda I-B^0_p$ is not greater than $\pi/2$. On the other hand, since L^0_2 is self-adjoint, then, by Proposition B.1.13(v), $\theta_{I-L^0_2}=0$. Therefore, $\theta_{I-L^0_2}<\pi/2$ for any $p\in(1,+\infty)$.

We now show that if λ is sufficiently large, then the operators $I-L^0_p$ and $\lambda I-B^0_p$ satisfy the condition (B.1.6). For this purpose, for any $\lambda>0$, any $\mu\in\rho(B^0_p)$ and any $\nu\in L^0_p$, we define the bounded operator $E(\lambda,\mu,\nu)$: $C^4_b(\mathbb{R}^N)\to L^p_{\tilde\mu}$ by setting

$$E(\lambda,\mu,\nu)u=(I-L^0_p)R(\nu+1,L^0_p)[R(1,L^0_p),R(\lambda+\mu,B^0_p)]u,\qquad u\in W^{-2,p}_{\tilde\mu}.$$

An explicit computation and Proposition 9.2.2 show that both $\{S(t)\}$ and $\{T^0(t)\}$ map $C^4_b(\mathbb{R}^N)$ into itself for any $t>0$ and there exist two constants $M\ge 1$ and $\omega_0\ge 0$ such that

$$e^{-\omega_0 t}\|S(t)\|_{L(C^4_b(\mathbb{R}^N))}+\|T^0(t)\|_{L(C^4_b(\mathbb{R}^N))}\le M,\qquad t>0.$$

This implies that both the resolvent operators $R(\xi+\omega_0,B^0_p)$ and $R(\xi,L^0_p)$ map $C^4_b(\mathbb{R}^N)$ into itself for any $\xi\in\mathbb{C}$ with $\mathrm{Re}\,\xi>0$. Therefore, if $\lambda>\omega_0$,

$$\begin{aligned}&[R(1,L^0_p),R(\lambda+\mu,B^0_p)]\\&=-R(\lambda+\mu,B^0_p)R(1,L^0_p)[B^0_p,L^0_p]R(1,L^0_p)R(\lambda+\mu,B^0_p).\end{aligned}$$

Observing that

$$\begin{aligned}&(I-L^0_p)R(\nu+1,L^0_p)R(\lambda+\mu,B^0_p)R(1,L^0_p)u\\&=-(I-L^0_p)R(\nu+1,L^0_p)[R(1,L^0_p),R(\lambda+\mu,B^0_p)]u\\&\quad+(I-L^0_p)R(\nu+1,L^0_p)R(1,L^0_p)R(\lambda+\mu,B^0_p)u\\&=-E(\lambda,\mu,\nu)u+(I-L^0_p)R(1,L^0_p)R(\nu+1,L^0_p)R(\lambda+\mu,B^0_p)u\\&=-E(\lambda,\mu,\nu)u+R(\nu+1,L^0_p)R(\lambda+\mu,B^0_p)u,\end{aligned}$$

for any $u\in C^4_b(\mathbb{R}^N)$, we finally get

$$\begin{aligned}&E(\lambda,\mu,\nu)u\\&=\{E(\lambda,\mu,\nu)-R(\nu+1,L^0_p)R(\lambda+\mu,B^0_p)\}[B_p,L^0_p]R(1,L^0_p)R(\lambda+\mu,B^0_p)u,\end{aligned}\tag{9.3.49}$$

for any $u\in C^4_b(\mathbb{R}^N)$ and any λ sufficiently large.

Observe that the equality in (9.3.49) can be extended to all the functions $u\in L^p_\mu$. To see it, it suffices to show that the operator $[B^0_p,L^0_p]R(1,L_p)$: $C^4_b(\mathbb{R}^N)\to L^p_\mu$ extends to a bounded operators in L^p_μ, but this is immediate since an explicit computation and (9.3.48) show that

$$([B_p,L^0_p]u)(x)=-\mathrm{Tr}\,(B^0D^2u(x))+\frac{1}{2}\langle(B^0+(B^0)^*)\tilde Q_\infty^{-1}x,Du(x)\rangle,$$

for any $x \in \mathbb{R}^N$ and any $u \in C_b^4(\mathbb{R}^N)$. Lemma 9.3.4 and Proposition 9.3.14 then allow us to conclude.

Now, we observe that, according to Proposition B.1.13(ii), we can determine two angles φ_1 and φ_2 with $\varphi_1 > \max\{\theta_{B_p^0}, \pi/2\}$, $\varphi_2 \in (\theta_{L_p^0}, \pi/2)$ and $\varphi_1+\varphi_2 < \pi$, such that

$$\|R(\mu+\lambda, B_p^0)\|_{L(L^p_{\tilde{\mu}})} \le \frac{C}{|\mu+\lambda|}, \qquad \|R(\nu+1, L_p^0)\|_{L(L^p_{\tilde{\mu}})} \le \frac{C}{1+|\nu|}, \tag{9.3.50}$$

for any $|\arg\mu| < \pi-\varphi_1$, any $|\arg\nu| < \pi-\varphi_2$ and some positive constant C, independent of μ, ν. Note that we can take $|\nu|+1$ instead of $|\nu|$ in the second estimate in (9.3.50) since the operator $\nu \mapsto R(\nu+1, L_p^0)$ is bounded in a neighborhood of 0.

Now, from (9.3.49) and (9.3.50) we get

$$\|E(\lambda,\mu,\nu)\|_{L(L^p_{\tilde{\mu}})} \le \frac{C_1}{|\mu+\lambda|}\|E(\lambda,\mu,\nu)\|_{L(L^p_{\tilde{\mu}})} + \frac{C_2}{(1+|\nu|)|\mu+\lambda|^2}, \tag{9.3.51}$$

for some positive constants C_1 and C_2, independent of λ, μ, ν. Taking $\lambda > \max\{\omega_0, 2C_1\}$, we easily deduce that

$$\|E(\lambda,\mu,\nu)\|_{L(L^p_{\tilde{\mu}})} \le \frac{2C_2}{(1+|\nu|)|\mu+\lambda|^2} \le \frac{2C_2}{(1+|\nu|)|\mu|^2},$$

for any $\mu, \nu \in \mathbb{C}\setminus\{0\}$ such that $|\arg\mu| < \pi-\varphi_1$ and $|\arg\nu| < \pi-\varphi_2$. Theorem B.1.16 allows us to conclude that the operator $(\lambda - B_p^0) + (I - L_p^0)$ (or, equivalently, the operator $L_p^0 + B_p^0$) is closed in $W^{2,p}_{\tilde{\mu}}$. This concludes the proof. ∎

Remark 9.3.18 In the case when $p=1$ the characterization of $D(L_1)$ is still an open problem. From Lemma 9.3.13, it follows easily that $D(L_1)$ is the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the graph norm of the operator L_1.

On the other hand, the set equality $D(L_1) = W^{2,1}_\mu$ cannot hold, since, in particular, $D(L_1)$ is not continuously embedded in $W^{1,1}_\mu$. Indeed, suppose on the contrary that $D(L_1)$ is continuously embedded in $W^{1,1}_\mu$. Then, the operator $R(1, L_1) : W^{1,1}_\mu \to W^{1,1}_\mu$ should be compact. Therefore the spectrum of the part of L_1 in $W^{1,1}_\mu$ (i.e., the restriction of the operator L_1 to the set $\{u \in D(L_1) : L_1 u \in W^{1,1}_\mu\}$) should be a discrete set. But this cannot be the case, since, as it is easily seen, it coincides with $\sigma(L_1)$, which, according to the forthcoming Theorem 9.3.24, is a halfplane.

9.3.2 The spectrum of the Ornstein-Uhlenbeck operator in L^p_μ

In this subsection we explicitly characterize the spectrum of the operator L_p for any $p \in [1, +\infty)$. The results that we present here are taken from [113] and they hold also for degenerate Ornstein-Uhlenbeck operators.

First of all, we show that the operator $R(\lambda, L_p)$ is compact in L^p_μ for any $p \in (1, +\infty)$.

Theorem 9.3.19 *For any $p \in (1, +\infty)$, $W^{1,p}_\mu$ is compactly embedded in L^p_μ. As a consequence, for any p as above, $T(t)$ and $R(\lambda, L_p)$ are compact operators in L^p_μ, respectively, for any $t > 0$ and any $\lambda \in \rho(L_p)$.*

Proof. Of course, we can limit ourselves to showing that $W^{1,p}_\mu$ is compactly embedded in L^p_μ for any $p \in (1, +\infty)$. Indeed, since $T(t)$ maps L^p_μ in $W^{1,p}_\mu$ for any $t > 0$ (see Proposition 9.3.8), the compactness of $T(t)$ then easily follows. Finally, since $R(\lambda, L_p)$ is the Laplace transform of the semigroup, for any λ with positive real part (see the formula (B.1.3)), then, taking Proposition A.1.1 and the resolvent formula (A.3.2) into account, we will deduce that $R(\lambda, L_p)$ is compact as well.

To prove that $W^{1,p}_\mu$ is compactly embedded in L^p_μ, we adapt the technique in the proof of Theorem 8.5.3. For this purpose, it suffices to show that

$$\lim_{r \to +\infty} \int_{\mathbb{R}^N \setminus B(r)} |f|^p d\mu = 0, \tag{9.3.52}$$

uniformly with respect to $f \in W^{1,p}_\mu$ with $\|f\|_{W^{1,p}_\mu} \le 1$. Taking Lemma 9.3.4 into account, we can write

$$\int_{\mathbb{R}^N \setminus B(r)} |f|^p d\mu \le \frac{1}{r^p} \int_{\mathbb{R}^N} |xf(x)|^p d\mu \le \frac{C}{r^p} \|f\|_{W^{1,p}_\mu},$$

for any $f \in W^{1,p}_\mu$ and some positive constant C, independent of f. Now, (9.3.52) easily follows. ∎

As a consequence of Theorem 9.3.19, the spectrum of L_p consists of at most a sequence of eigenvalues for any $p \in (1, +\infty)$. In the remainder of this subsection, we characterize explicitly the spectrum of L_p. To begin with, we observe that 0 is an eigenvalue of the operator L_p for any $p \in (1, +\infty)$ and, according to the Liouville type theorem 8.1.17 and Proposition 9.3.8, the corresponding eigenspace is one-dimensional and consists of all the constant functions. As the following proposition shows, if u is a generalized eigenfunction corresponding to the eigenvalue $\lambda \in \mathbb{C}$, then u is a polynomial. Of course, if u is a polynomial it belongs to $W^{2,p}_\mu = D(L_p)$ for any $p \in (1, +\infty)$ (see Theorem 9.3.17).

Proposition 9.3.20 *Fix $p \in (1, +\infty)$ and let $u \in D(L^r_p)$ satisfy $(\lambda - L_p)^r u = 0$ for some $\lambda \in \mathbb{C}$ and some $r \in \mathbb{N}$. Then,* $\mathrm{Re}\,\lambda \le 0$ *and u is a polynomial with degree at most equal to* $\mathrm{Re}\,\lambda / s(B)$.

Proof. We prove the assertion by induction on r. Suppose that $r = 1$ and let $\lambda \in \mathbb{C}$ and $u \in D(L_p)$ be such that $\lambda u - L_p u = 0$. Then, as it is easily checked,

$T(t)u = e^{\lambda t}u$ for any $t \geq 0$ and, from Proposition 9.3.8, we get $u \in W^{k,p}_\mu$ for any $k \in \mathbb{N}$. From the formula (9.3.17), we easily deduce that, for any $\varepsilon > 0$ and any multi-index α, there exists a positive constant $C = C(\varepsilon, |\alpha|)$ such that

$$e^{\mathrm{Re}\lambda\, t}\|D^\alpha u\|_p \leq C e^{|\alpha|(s(B)+\varepsilon)t} \sum_{|\beta|=|\alpha|} \|D^\beta u\|_p, \qquad t > 0.$$

From such an inequality we obtain that if $\mathrm{Re}\lambda > |\alpha| s(B)$, then C should be zero. Therefore, the derivatives of u of order $k > \mathrm{Re}\lambda/s(B)$ should vanish and the assertion follows in this particular case.

Now, suppose that the assertion is true with $r = n > 1$ and let us prove it with $r = n+1$. For this purpose we observe that if $u \in D(L_p^{n+1})$ satisfies $(\lambda u - L_p u)^{n+1} = 0$, then the function $v = \lambda u - L_p u$ belongs to $D(L_p^n)$ and it satisfies the equation $(\lambda I - L_p)^r v = 0$. Therefore, v is a polynomial of degree at most $\mathrm{Re}\,\lambda/s(B)$. Moreover, since $(\lambda I - L_p)^s v = 0$ for any $s \geq n+1$, we have

$$T(t)u = e^{\lambda t}\sum_{i=0}^{n} \frac{t^i}{i!}(\lambda I - L_p)^i u = e^{\lambda t}u + e^{\lambda t}\sum_{i=0}^{n-1} \frac{t^{i+1}}{(i+1)!}(\lambda I - L_p)^i v, \quad (9.3.53)$$

for any $t > 0$. From the formula (9.3.53), we obtain that

$$D^\alpha T(t)u = e^{\lambda t}D^\alpha u + e^{\alpha t}\sum_{i=0}^{n-1} \frac{t^{i+1}}{(i+1)!}D^\alpha(\lambda v - L_p v)^i, \qquad t > 0,$$

for any multi-index α. Since v is a polynomial with degree at most $\mathrm{Re}\lambda/s(B)$, then

$$D^\alpha T(t)u = e^{\lambda t}D^\alpha u, \qquad t > 0,$$

for any multi-index α with $|\alpha| > \mathrm{Re}\,\lambda/s(B)$. Now the proof goes on as in the case when $r = 1$. ∎

We now show that the problem of determining the eigenvalues of the operator L_p can be stated in an equivalent (and simpler as far as computations are concerned) way. For this purpose, we introduce the operator $\mathcal{B}_p : W^{2,p}_\mu \to L^p_\mu$ defined by

$$\mathcal{B}_p u(x) = \langle Bx, Du(x)\rangle, \qquad x \in \mathbb{R}^N, \quad u \in W^{2,p}_\mu. \quad (9.3.54)$$

According to Lemma 9.3.4, $\mathcal{B}_p$ is well defined and it is bounded in $W^{2,p}_\mu$.

Proposition 9.3.21 *Let $p \in (1, +\infty)$. The following statements are equivalent:*

(i) $\lambda \in \mathbb{C}$ is an eigenvalue of the operator L_p;

(ii) *λ is an eigenvalue of the operator $\mathcal{B}_p$ and the corresponding eigenspace contains a nonzero homogeneous polynomial.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of the operator L_p. Further, let u be a corresponding eigenfunction so that $\lambda u - \mathcal{B}_p u = \sum_{i,j=1}^N q_{ij} D_{ij} u$. By Proposition 9.3.20, u is a polynomial. We denote by m its degree. If $m = 0, 1$ then it is immediate to check that λ is an eigenvalue of $\mathcal{B}_p$. So, let us assume that $m \geq 2$. Note that for any $n \in \mathbb{N}$, both $\mathcal{B}_p$ and L_p map the set $\mathcal{P}_n$ of all the polynomials with degree at most n into itself. Suppose that $\lambda I - \mathcal{B}_p$ is invertible in $\mathcal{P}_{m-2}$. Then, there exists a polynomial v with degree $m-2$ such that

$$\lambda v - \mathcal{B}_p v = \sum_{i,j=1}^N q_{ij} D_{ij} u.$$

The function $w = u - v$, which is a nontrivial polynomial with degree m, turns out to be an eigenfunction of the operator L_p corresponding to the eigenvalue λ. If the operator $\lambda I - \mathcal{B}_p$ is not invertible in $\mathcal{P}_{m-2}$, of course λ is an eigenvalue of $\mathcal{B}_p$ and we can take as a corresponding eigenfunction a polynomial w with degree $m-2$.

To complete the proof of the property (ii) we have to check that there exists a homogeneous polynomial among the eigenfunctions of the operator $\mathcal{B}_p$ corresponding to the eigenvalue λ. For this purpose, it suffices to notice that, for any $k \in \mathbb{N}$, $\mathcal{B}_p$ maps the space of homogeneous polynomials with degree k into itself. Therefore, the homogeneous part with maximum degree of the eigenfunction w is itself an eigenfunction.

Let us now assume that the property (ii) holds and let us prove that the property (i) holds as well. For this purpose, we assume that $\lambda \in \mathbb{C}$ is an eigenvalue of the operator $\mathcal{B}_p$ and that the homogeneous polynomial u is a corresponding eigenfunction. Of course,

$$\lambda u - L_p u = -\sum_{i,j=1}^N q_{ij} D_{ij} u.$$

Repeating the same arguments used above, now applied to the operator L_p, we can easily show that λ is an eigenvalue of L_p. This concludes the proof. ∎

By virtue of Proposition 9.3.21, to characterize the spectrum of the operator A_p we can limit ourselves to characterize the spectrum of the operator $\mathcal{B}_p$, which is easier to handle.

Theorem 9.3.22 *Denote by $\lambda_1, \ldots, \lambda_r$ the r distinct eigenvalues of the matrix B. Then, for any $p \in (1, +\infty)$,*

$$\sigma(L_p) = \left\{ \lambda = \sum_{i=1}^r n_i \lambda_i : n_i \in \mathbb{N} \cup \{0\},\ i = 1, \ldots, r \right\}. \tag{9.3.55}$$

In particular, $\sigma(L_p)$ is independent of p.

Proof. To begin with, we observe that a straightforward computation shows that $v : \mathbb{R}^N \to \mathbb{C}$ is an eigenfunction of the operator $\mathcal{B}_p$, corresponding to the eigenvalue $\lambda \in \mathbb{C}$, if and only if

$$v(e^{tB}x) = e^{\lambda t}v(x), \qquad t > 0, \ x \in \mathbb{R}^N. \tag{9.3.56}$$

According to Proposition 9.3.21, we can assume that v is a homogeneous polynomial. Therefore, the equality (9.3.56) can be extended to all $z \in \mathbb{C}$.

Denote by $\lambda_1, \ldots, \lambda_r$ the distinct eigenvalues of the matrix B. By a general result of linear algebra (see [100, Chapter 11, Section 6]), there exists a nonsingular $N \times N$ complex valued matrix M such that $B = M^{-1}CM$, with $C = \mathrm{diag}(C_1, \ldots, C_r)$, where each block C_i is a Jordan block, which can be split into the sum $C_i = \lambda_i I_{n_i} + N_{n_i}$, n_i being the algebraic multiplicity of the eigenvalue λ_i and N_{λ_i} being a nihilpotent matrix. It follows that $e^{tB} = M^{-1}e^{tC}M$ and, consequently, setting $w(z) = v(M^{-1}z)$ for any $z \in \mathbb{C}^N$, we easily see that v satisfies (9.3.56) if and only if

$$w(e^{tC}z) = e^{\lambda t}w(z), \qquad t > 0, \ z \in \mathbb{C}^N. \tag{9.3.57}$$

We can now prove the inclusion "$\subset$" in the formula (9.3.55). For this purpose, let $\lambda \in \sigma(L_p)$ and let v be a corresponding eigenfunction. Since we have assumed that v is a homogeneous polynomial, w is a homogeneous polynomial as well and we can write

$$w(z) = \sum_{|\alpha_1|+\ldots+|\alpha_n|=n} a_{\alpha_1,\ldots,\alpha_r} \prod_{j=1}^{r} z_j^{\alpha_j}, \qquad z \in \mathbb{C}^N,$$

for some $n \in \mathbb{N}$ and some $a_{\alpha_1,\ldots,\alpha_r} \in \mathbb{C}$ ($|\alpha_1| + \ldots + |\alpha_n| = n$), where $z = (z_1, \ldots, z_r)$ and $z_i \in \mathbb{C}^{n_i}$. Since $e^{tC}z = (e^{tC_1}z_1, \ldots, e^{tC_r}z_r)$ and N_i is nihilpotent, we have

$$e^{tC_i}y = e^{\lambda t}(P_{1,i}(t,y), \ldots, P_{r_i,i}(t,y)), \qquad t > 0, \ y \in \mathbb{C}^{n_i}, \qquad i = 1, \ldots, r,$$

where $P_{k,i}$ ($i = 1, \ldots, r$) is a polynomial in both its entries. It follows that

$$\begin{aligned} e^{\lambda t}w(z) &= w(e^{tC}z) \\ &= \sum_{|\alpha_1|+\ldots+|\alpha_n|=n} a_{\alpha_1,\ldots,\alpha_r} e^{(\lambda_1|\alpha_1|+\ldots+\lambda_n|\alpha_n|)t} P_{\alpha_1,\ldots,\alpha_r}(t,z), \end{aligned} \tag{9.3.58}$$

for any $t > 0$ and any $z \in \mathbb{C}^N$. Here, $P_{\alpha_1,\ldots,\alpha_r}$ is a polynomial. Fixing $z^* \in \mathbb{C}^N$ such that $w(z^*) \neq 0$ and letting t tend first to $+\infty$ and then to $-\infty$ in (9.3.58), we easily see that $\lambda = \lambda_1|\alpha_1| + \ldots + \lambda_n|\alpha_n|$ for some $\alpha_1, \ldots, \alpha_n$, and the inclusion "$\subset$" in (9.3.55) follows.

We now show the other inclusion in (9.3.55). For this purpose, let $\lambda = \sum_{i=1}^r n_i\lambda_i$ for some $n_i \in \mathbb{N}$ $(i = 1,\dots,r)$ and let v be the polynomial defined by

$$w(z) = \hat{z}_1^{n_1} \cdot \hat{z}_2^{n_2} \cdot \ldots \cdot \hat{z}_r^{n_r}, \qquad z \in \mathbb{C}^N,$$

where $\hat{z}_i$ is the last component of the vector z_i $(i = 1,\dots,r)$ defined above. Since the last row of the matrix N_i is 0 for any $i = 1,\dots,r$, then, for any $z \in \mathbb{C}^{n_i}$, any $k \in \mathbb{N}$ and any $i = 1,\dots,r$, the last entry of the vector $N_i^k h$ is 0. It follows that the last entry of the vector $e^{tC_i}z_i$ is $e^{\lambda t}\hat{z}_i$ for any $i = 1,\dots,r$ and, consequently,

$$w(e^{tC}z) = e^{\lambda t}w(z), \qquad t > 0,\ \ z \in \mathbb{C}^N.$$

Hence, λ is an eigenvalue of $\mathcal{B}_p$ and the (homogeneous) polynomial v defined by $v(x) = w(Mx)$ for any $x \in \mathbb{R}^N$ is a corresponding eigenfunction. Proposition 9.3.21 allows us to conclude that $\lambda \in \sigma(L_p)$. This completes the proof. ∎

Some more information is available on the eigenspaces corresponding to the eigenvalues of the operator L_p. In the following theorem we give a necessary and sufficient condition which guarantees that the eigenvalues of L_p are semisimple. For notational convenience, for any linear operator S with compact resolvent operator, we denote by $i_\lambda(S)$ the index of the eigenvalue λ (i.e., the smallest integer k such that $\mathrm{Ker}\,(\lambda I - S)^k = \mathrm{Ker}\,(\lambda I - S)^{k+1}$). According to Proposition A.3.7, $i_\lambda(S)$ is well defined for any $\lambda \in \sigma(S)$ and $R(\cdot, S)$ has a pole of order $i_\lambda(S)$ at λ.

Theorem 9.3.23 *The index $i_\lambda(L_p)$ is one for any eigenvalue λ of L_p (equivalently, all the eigenvalues of L_p are semisimple) if and only if the matrix B is diagonalizable in $\mathbb{C}$.*

Proof. To show the assertion we prove that for any eigenvalue λ of the operator A_p, it holds that

$$1 + \max\left\{\sum_{i=1}^r n_i(i_{\lambda_i}(B) - 1) : \sum_{i=1}^r n_i\lambda_i = \lambda\right\} \le i_\lambda(L_p) \le 1 + \frac{\mathrm{Re}\lambda}{s(B)}(i(B) - 1), \tag{9.3.59}$$

where $i(B)$ denotes the maximum of the indices $i_{\lambda_i}(B)$ $(i = 1,\dots,r)$.

To prove (9.3.59) we proceed into some steps. As in the proof of Theorem 9.3.22, we split any $z \in \mathbb{C}^N$ into blocks as $z = (z_1,\dots,z_r)$, where $z_i \in \mathbb{C}^{n_i}$ and n_i is the algebraic multiplicity of the eigenvalue λ_i of the matrix B. Moreover, we recall (see again the proof of Theorem 9.3.22) that we can determine a nonsingular matrix M such that $B = M^{-1}CM$, where $C = \mathrm{diag}\,(C_1,\dots,C_r)$ and each block C_i is a Jordan block, which can be split into the sum $C_i = \lambda_i I_{n_i} + N_{n_i}$ for some nihilpotent matrix N_{λ_i}. By well known results of linear algebra, $i_{\lambda_i}(B)$ coincides with the algebraic multiplicity of the eigenvalue λ, i.e., it coincides with n_i. As a consequence

$$e^{tC_i} = e^{t\lambda_i}p_i(t), \qquad t \in \mathbb{R}, \quad i = 1,\dots,N, \tag{9.3.60}$$

where p_i are suitable matrices whose entries are polynomials with degree at most $i_{\lambda_i}(B)-1$. Moreover, there exists $h\in\mathbb{R}^{n_i}$ such that at least one component of $p_i(t)x$ is a polynomial with degree $i_{\lambda_i}(B)-1$.

Step 1. Here, we prove that $i_\lambda(\mathcal{B}_p)\le i_\lambda(L_p)$. For this purpose, we show that, for any $\mu\in\rho(L_p)$, it holds that

$$R(\mu,\mathcal{B}_p)u=\sum_{i=1}^{+\infty}P_iR(\mu,L_p)P_iu, \tag{9.3.61}$$

for any polynomial u. Here, P_i ($i\in\mathbb{N}\cup\{0\}$) is the canonical projection of the space of polynomials into the space $\mathcal{H}_i$ of the homogeneous polynomials with degree i.

Once the formula (9.3.61) is checked, we can quite easily show that $R(\cdot,L_p)$ has a pole at λ, whose order is greater or equal to the order of the pole of $R(\cdot,\mathcal{B}_p)$ at λ. Indeed, using such a formula we deduce that for any polynomial u the function $\mu\mapsto(\mu-\lambda)^{i_\lambda(L_p)}R(\mu,\mathcal{B}_p)u$ can be analytically extended to $\lambda+B(r)\subset\mathbb{C}$ for some $r>0$, independent of u. By density, $\mu\mapsto(\mu-\lambda)^{i_\lambda(L_p)}R(\mu,\mathcal{B}_p)u$ is analytic in $\lambda+B(r)$ for any $u\in L^p_\mu$, and the Banach-Steinhaus theorem (see [147, Chapter 2, Section 4]) implies that $\|(\mu-\lambda)^{i_\lambda(L_p)}R(\mu,\mathcal{B}_p)\|_{L(L^p_\mu)}\le C$ for any $\mu\in\lambda+B(r)$ and some positive constant C, independent of λ. Of course, this implies that $R(\mu,\mathcal{B}_p)u$ has a pole of order $m\le i_\lambda(L_p)$ at λ. Therefore, $\lambda\in\sigma(\mathcal{B}_p)$. According to the formula (A.3.5), $(\lambda I-\mathcal{B}_p)^mP=0$ and $(\lambda I-\mathcal{B}_p)^{m-1}P\neq0$, where P is the spectral projection with λ, i.e.,

$$P=\frac{1}{2\pi i}\int_\gamma R(\xi,\mathcal{B}_p)d\xi,$$

γ being the boundary of a suitable ball centered at λ and oriented counterclockwise (see Proposition A.3.5). Therefore, $P(L^p_\mu)\subset\mathcal{E}_\lambda$, where by $\mathcal{E}_\lambda$ we have denoted the space of generalized eigenfunction of $\mathcal{B}_p$ corresponding to λ. But since the restriction of $\lambda I-\mathcal{B}_p$ to $(I-P)(L^p_\mu)$ is an invertible operator (see again Proposition A.3.5), then $P(L^p_\mu)=\mathcal{E}_\lambda$. Now, it is immediate to check that $i_\lambda(\mathcal{B}_p)$ coincides with m.

By linearity we can limit ourselves to checking (9.3.61) in the case when $u\in\mathcal{H}_i$ ($i\in\mathbb{N}$). Set $v=R(\mu,L_p)u$. Since $\mathcal{B}_p$ maps $\mathcal{H}_n$ into itself, we easily get

$$u=P_iu=P_i(\mu v-L_pv)=P_i(\mu v-\mathcal{B}_pv)=(\mu I-\mathcal{B}_p)P_iv.$$

The formula (9.3.61) now follows.

Step 2. We now prove that

$$i_\lambda(\mathcal{B}_p)=1+\max\left\{\sum_{i=1}^r n_i(i_{\lambda_i}(B)-1):\sum_{i=1}^r n_i\lambda_i=\lambda\right\}. \tag{9.3.62}$$

Let v be a generalized eigenfunction of $\mathcal{B}_p$ corresponding to the eigenvalue $\lambda = m_1\lambda_1+\ldots+m_r\lambda_r$ such that $(\lambda I-\mathcal{B}_p)^{i_\lambda(\mathcal{B}_p)-1}v\neq 0$ and $(\lambda I-B_p)^{i_\lambda(B_p)}v=0$. Let us show that v is a linear combination of polynomials $p_i\in\mathcal{H}_{m_i}$, $i=1,\ldots,r$, such that each p_i depends only on the block of variables z_i. For this purpose, we observe that $v(e^{tB}x)=(S(t)v)(x)$ for any $t\geq 0$ and any $x\in\mathbb{R}^N$, where $\{S(t)\}$ is the strongly continuous group generated by the closure of the operator $\mathcal{B}_p$ in L^p_μ (the strong continuity of $\{S(t)\}$ can be proved by arguing as in the proof of Lemma 9.3.16). Since v is a generalized eigenfunction of $\mathcal{B}_p$, then v is a polynomial. This can be easily checked arguing as in the proof of Proposition 9.3.20, observing that

$$\|D^\alpha S(t)f\|_p\leq Ce^{|\alpha|(s(B)+\varepsilon)t}\sum_{|\beta|=|\alpha|}\|D^\alpha f\|_p,\qquad t>0,$$

for any multi-index α, any $\varepsilon>0$, any $f\in W^{k,p}_\mu$ and some positive constant $C=C(|\alpha|,\varepsilon)$. Moreover, arguing as in the proof of (9.3.53), we can show that

$$(S(t)v)(x)=v(e^{tB}x)=e^{t\lambda}p(t,x),\qquad t>0,\ \ x\in\mathbb{R}^N,\tag{9.3.63}$$

where p is a polynomial in (t,x) whose degree in t is less or equal to $i_B(\lambda)-1$. Since v is a polynomial, we can extend (9.3.63) to any $z\in\mathbb{C}^N$. Moreover, we can determine some constants $c_{\alpha_1,\ldots,\alpha_r}$ such that

$$v(z)=\sum_{|\alpha_1|+\ldots+|\alpha_r|\leq n}c_{\alpha_1,\ldots,\alpha_r}\prod_{i=1}^r z_i^{\alpha_i},\qquad z\in\mathbb{C}^N,$$

and, hence,

$$v(e^{tB}z)=\sum_{|\alpha_1|+\ldots+|\alpha_r|\leq n}c_{\alpha_1,\ldots\alpha_r}e^{t(\lambda_1|\alpha_1|+\ldots+\lambda_r|\alpha_r|)}p_{\alpha_1,\ldots,\alpha_r}(t,z),\tag{9.3.64}$$

for any $t>0$ and any $z\in\mathbb{C}^N$, where $p_{\alpha_1,\ldots,\alpha_r}$ are polynomials in (t,x) with degree at most $\sum_{i=1}^r|\alpha_i|(i_{\lambda_i}(B)-1)$ in t. Comparing (9.3.63) and (9.3.64) for large t, we easily deduce that $c_{\alpha_1,\ldots,\alpha_r}=0$, if $|\alpha_1|\lambda_1+\ldots+|\alpha_r|\lambda_r\neq\lambda$. It also follows that each polynomial $p_{\alpha_1,\ldots,\alpha_r}$ depends only on the block of variables x_i $(i=1,\ldots,r)$ for which $\alpha_i\neq 0$.

From (9.3.63) and (9.3.64) we now get that

$$p(t,z)=q(t,z):=\sum_{|\alpha_1|\lambda_1+\ldots+|\alpha_r|\lambda_r=\lambda}c_{\alpha_1,\ldots,\alpha_r}p_{\alpha_1,\ldots,\alpha_r}(t,z),\qquad t>0,\ \ z\in\mathbb{C}^N.$$

Moreover, from the arguments before Step 1, we deduce that there exists $z\in\mathbb{C}^N$ such that the degree of $q(\cdot,z)$ is $\max\{\sum_{i=1}^r n_i(i_{\lambda_i}(B)-1):\sum_{i=1}^r n_i\lambda_i=\lambda\}$ and the degree of $p(\cdot,z)$ is $i_\lambda(\mathcal{B}_p)-1$. Now, (9.3.62) easily follows.

Step 3. We now show that

$$i_\lambda(L_p)\leq 1+\frac{\mathrm{Re}\lambda}{s(B)}(i(B)-1).\tag{9.3.65}$$

For this purpose, let $u = \sum_{|\alpha|\le m} c_\alpha x^\alpha$ be a polynomial (with degree $m \le \mathrm{Re}\,\lambda/s(B)$, see Proposition 9.3.20) such that $(\lambda I - L_p)^{i_\lambda(L_p)-1}u \neq 0$ and $(\lambda I - L_p)^{i_\lambda(L_p)}u = 0$. From the formula (9.3.53) with $n = i_\lambda(L_p) - 1$, we deduce that

$$(T(t)u)(x) = e^{\lambda t} p(t,x), \qquad t>0,\ x\in\mathbb{R}^N, \tag{9.3.66}$$

and there exists $x_0 \in \mathbb{R}^N$ such that the degree of $p(\cdot, x)$ is $i_\lambda(L_p) - 1$.

A long but straightforward computation shows that

$$\begin{aligned}(T(t)u)(x) &= \sum_{|\alpha|\le m} c_\alpha \sum_{h\le\alpha} \binom{\alpha}{h} (e^{tB}x)^h \\ &\qquad\qquad \times \frac{1}{\sqrt{(2\pi)^N \det Q_t}} \int_{\mathbb{R}^N} y^{\alpha-h} e^{-\frac{1}{2}\langle Q_t^{-1}y,y\rangle} dy \\ &= \sum_{|\alpha|\le m} c_\alpha \sum_{\substack{h\le\alpha \\ |\alpha-h|\ \mathrm{even}}} d_{\alpha,h} \binom{\alpha}{h} (e^{tB}x)^h D_w^{\alpha-h}\left((Q_t w, w)^{\frac{|\alpha-h|}{2}}\right)\Big|_{w=0},\end{aligned}$$

for any $t>0$ and any $x\in\mathbb{R}^N$, where $d_{\alpha,h}^{-1} = \left(\frac{2}{|\alpha-h|}\right)!$. Taking (9.3.60) into account, it easily follows that

$$(Q_t)_{ij} = \sum_{h=1}^{r} e^{\lambda_h t} p_{ij}^{(h)}(t) + c_{ij}^{(h)}, \qquad t>0,\ \ i,j = 1,\ldots,N,$$

where $p_{ij}^{(h)}$ are suitable polynomials with degree at most $2i(B)-2$, and $c_{ij}^{(h)}$ are complex constants. Therefore, $D_w^{\alpha-h}((Q_t w,w)^{|\alpha-h|/2})\big|_{w=0} = \sum_{i=1}^{l_1} e^{\tau_i t} q_i(t)$ for any $t>0$ and some $l_1 \ge m$, where τ_i are integer multiples of $\lambda_1,\ldots,\lambda_r$ and q_i are polynomials with degree at most $(i(B)-1)(|\alpha|-|h|)$. Summing up,

$$(T(t)u)(x) = \sum_{i=1}^{l_2} e^{\xi_i t} r_i(t,x), \qquad t>0,\ x\in\mathbb{R}^N,$$

for some $l_2\in\mathbb{N}$, where ξ_i $(i=1,\ldots,r)$ are again integer multiples of $\lambda_1,\ldots,\lambda_r$, and r_i, as functions of t, are polynomials with degree at most $m(i(B)-1)$. Now, the estimate (9.3.65) easily follows. This completes the proof. ■

Now, we consider the case when $p=1$. In such a situation the picture drastically changes. Indeed, as the following theorem shows, the spectrum of L_1 is no more a discrete set but it is a halfplane. The proof of this property is based on the results of the forthcoming Section 9.4, where the spectrum of the Ornstein-Uhlenbeck operator in L^p-spaces associated with the Lebesgue measure is characterized.

Theorem 9.3.24 *The spectrum of the Ornstein-Uhlenbeck operator in L^1_μ is the halfplane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \le 0\}$. Moreover, any $\lambda \in \mathbb{C}$ with negative real part is an eigenvalue of L_1.*

Proof. Let $\psi = 1/\rho$, where ρ is the density of the invariant measure μ with respect to the Lebesgue measure (see (9.3.2)). Denote by $I : L^1(\mathbb{R}^N) \to L^1_\mu$ the isometry defined by $If = f\psi$ for any $f \in L^1(\mathbb{R})$, and by $\tilde{L}_1$ the operator defined on $I^{-1}(D(L_1))$ by $\tilde{L}_1 = I^{-1}L_1 I$. As it is immediately seen $\tilde{L}_1$ is the generator of a strongly continuous semigroup of contractions in $L^1(\mathbb{R}^N)$ and $\sigma(L_1) = \sigma(\tilde{L}_1)$. Therefore, we can limit ourselves to characterize the spectrum of the operator $\tilde{L}_1$ in $L^1(\mathbb{R}^N)$. For this purpose, we observe that since $C^\infty_c(\mathbb{R}^N) \subset D(L_1)$ and I maps $C^\infty_c(\mathbb{R}^N)$ into itself, then $C^\infty_c(\mathbb{R}^N)$ is contained in $D(\tilde{L}_1)$. A straightforward computation shows that $\tilde{L}_1 = \tilde{A}$ in $C^\infty_c(\mathbb{R}^N)$, where

$$
\begin{aligned}
(\tilde{\mathcal{A}}u)(x) &= (\mathcal{A}u)(x) + \rho(x)\sum_{i,j=1}^N q_{ij} D_i\psi(x) D_j u(x) + \rho(x)u(x)(\mathcal{A}\psi)(x)\\
&= \frac{1}{2}\mathrm{Tr}\,(QD^2u(x)) + \langle (B+QQ_\infty^{-1})x, Du\rangle + (\rho\mathcal{A}\psi)(x)u(x),
\end{aligned}
$$

for any $x \in \mathbb{R}^N$ and any $u \in C^\infty_c(\mathbb{R}^N)$. Now, we observe that differentiating the formula (9.3.8) at $t = 0$ gives

$$
Q + BQ_\infty + Q_\infty B^* = 0. \tag{9.3.67}
$$

Using (9.3.67) we immediately deduce that

$$
2\langle Q_\infty^{-1}Bx, x\rangle = 2\langle BQ_\infty Q_\infty^{-1}x, Q_\infty^{-1}x\rangle = -\langle QQ_\infty^{-1}x, Q_\infty^{-1}x\rangle, \qquad x \in \mathbb{R}^N
$$

and, therefore,

$$
\begin{aligned}
\rho(x)(\mathcal{A}\psi)(x) &= \frac{1}{2}\mathrm{Tr}\,(QQ_\infty^{-1}) + \frac{1}{2}\langle QQ_\infty^{-1}x, Q_\infty^{-1}x\rangle + \langle B^*Q_\infty^{-1}x, x\rangle\\
&= \frac{1}{2}\mathrm{Tr}\,(QQ_\infty^{-1}) := k.
\end{aligned}
$$

Using again (9.3.67) to obtain

$$
B + QQ_\infty^{-1} = -Q_\infty B^* Q_\infty^{-1}, \tag{9.3.68}
$$

we finally get

$$
(\tilde{\mathcal{A}}u)(x) = \frac{1}{2}\mathrm{Tr}\,(QD^2u(x)) - \langle Q_\infty B^*Q_\infty^{-1}x, Du(x)\rangle + ku(x), \qquad x \in \mathbb{R}^N.
$$

Since $\hat{\mathcal{A}} := \tilde{\mathcal{A}} - kI$ is an Ornstein-Uhlenbeck, according to the forthcoming Propositions 9.4.1 and 9.4.2, the realization $\hat{L}_1$ of $\hat{\mathcal{A}}$ in $L^1(\mathbb{R}^N)$ is the generator of a strongly continuous semigroup of contractions and $C_c^\infty(\mathbb{R}^N)$ is a core of $\hat{L}_1$. We claim that $D(\tilde{L}_1) = D(\hat{L}_1)$. Indeed, since $(\tilde{L}_1, D(\tilde{L}_1))$ is a closed operator and $C_c^\infty(\mathbb{R}^N)$ is a core of $\hat{L}_1$, $D(\hat{L}_1) \subset D(\tilde{L}_1)$ and $\hat{L} = \tilde{L} - k$ in $D(\hat{L}_1)$. To prove the other inclusion, we observe that $\lambda - \tilde{L}_1$ and $\lambda I - \hat{L}_1$ are both invertible if λ is sufficiently large. Therefore,

$$L^1(\mathbb{R}^N) = (I - \tilde{L}_1)(D(\tilde{L}_1)) \supset (I - \tilde{L}_1)(D(\hat{L}_1)) = (\lambda I - \hat{L}_1)(D(\hat{L}_1)) = L^1(\mathbb{R}^N),$$

which implies that $D(\tilde{L}_1) = D(\hat{L}_1)$, so that $(\tilde{L}_1, D(\tilde{L}_1))$ and $(\hat{L}_1 - kI, D(\hat{L}_1))$ do coincide. It follows that $\sigma(\tilde{L}_1) = \{\lambda \in \mathbb{C} : \lambda + k \in \sigma(\hat{L}_1)\}$.
Now, since the spectrum of the matrix $-Q_\infty B^* Q_\infty^{-1}$ is contained in the halfplane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}$, according to the forthcoming Theorem 9.4.3, the spectrum of $\hat{L}_1$ is the halfplane $\{\lambda \in \mathbb{C} : \ \mathrm{Re}\lambda \leq \mathrm{Tr}\,(Q_\infty B^* Q_\infty^{-1})\}$. Moreover, any λ with $\mathrm{Re}\lambda < \mathrm{Tr}\,(Q_\infty B^* Q_\infty^{-1})$ is an eigenvalue of $\hat{L}_1$.

To complete the proof it suffices to show that $k = -\mathrm{Tr}\,(Q_\infty B^* Q_\infty^{-1})$ and this can be easily done taking the traces of both the sides of (9.3.68). ■

9.3.3 The sector of analyticity of the Ornstein-Uhlenbeck operator in L^p_μ

In this subsection we present a recent result due to R. Chill, E. Fasangova, G. Metafune and D. Pallara which gives a precise characterization of the sector of analyticity of the Ornstein-Uhlenbeck operator in L^p_μ $(p \in (1, +\infty))$, that is of the largest sector $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}$ where the function $z \mapsto T(z)$ is analytic with values in $L(L^p_\mu)$.

Theorem 9.3.25 *For any $p \in (1, +\infty)$ let $\theta_p \in (0, \pi/2]$ be defined by*

$$\cot \theta_p = \frac{\sqrt{(p-2)^2 + p^2 \|I + 2Q^{-\frac{1}{2}} Q_\infty B^* Q^{-\frac{1}{2}}\|_\infty^2}}{2\sqrt{p-1}}. \tag{9.3.69}$$

Then, Σ_{θ_p} is the biggest sector, with centre at the origin, where the Ornstein-Uhlenbeck semigroup can be analytically extended. Moreover, the so extended semigroup is contractive in L^p_μ. Finally, for $p = 2$ the set of analyticity of the function $z \mapsto T(z)$ is the set

$$\{z \in \mathbb{C} : \|Q_\infty^{\frac{1}{2}} e^{zB} Q_\infty^{-\frac{1}{2}}\|_\infty \leq 1\}, \tag{9.3.70}$$

and $\{T(z)\}$ is contractive in it.

Proof. See [31, Theorems 1, 2 and Remark 6]. ■

Remark 9.3.26 (i) The formula (9.3.69) gives a relation between the sector of analyticity of the semigroup $\{T(t)\}$ and the sector (with center at 0) Σ_{θ_0} of analyticity of the resolvent operator of the matrix $R := Q^{-1/2}Q_\infty B^* Q^{-1/2}$. Indeed,

$$\pi - \theta_0 = \arctan(\|I + 2Q^{-\frac{1}{2}}Q_\infty B^* Q^{-\frac{1}{2}}\|_\infty). \tag{9.3.71}$$

To check (9.3.71) we begin by observing that, due to the formula (9.3.67), $\frac{1}{2}I + Q^{-1/2}Q_\infty B^* Q^{-1/2}$ is a skew-symmetric matrix and, consequently, R is normal. Therefore $\|I + 2R\|_\infty$ coincides with the spectral radius $r(I + 2R)$ of $I + 2R$, i.e., the radius of the smallest ball centered at the origin which contains all the eigenvalues of the matrix $I + 2R$. Still using (9.3.67) it is easy to check all the eigenvalues of the matrix R lie on the straight lines $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda = -1/2\}$, and, therefore, $\arctan(r(I + 2R)) = \pi - \theta_0$.

(ii) The formula (9.3.69) also shows that when the Ornstein-Uhlenbeck semigroup is symmetric (i.e., when $BQ = QB^*$ or, equivalently, when $Q_\infty B^* = BQ_\infty$, see the proof of Proposition 9.3.10), the sector of analyticity of $\{T(t)\}$ depends only on p and it equals $\pi/2$ when $p = 2$. Indeed, in such a case, the formula (9.3.67) gives $Q^{-1/2}Q_\infty B^* Q^{-1/2} = -\frac{1}{2}I$.

Note that the symmetry of the Ornstein-Uhlenbeck semigroup in L^2_μ is also a necessary condition for its sector of analyticity to be $\Sigma_{\pi/2}$. Indeed, if $\{T(t)\}$ is analytic in $\Sigma_{\pi/2}$, then $\cot\theta_p = 0$. But this implies that $I + 2Q^{-1/2}Q_\infty B^* Q^{-1/2} = 0$ and, by virtue of (9.3.67), it follows that $Q_\infty B^* = B^* Q_\infty$, so that $\{T(t)\}$ is symmetric in L^2_μ, by Proposition 9.3.10.

(iii) In general the domain of analyticity of the functions $z \mapsto T(z)$ is larger than the sector Σ_{θ_p}. For instance this is the case of the one-dimensional Ornstein-Uhlenbeck semigroup and $p \neq 2$ (see [52]). But in general also in the multidimensional case, the domain of analyticity of $\{T(z)\}$ is larger than Σ_{θ_p}. We can easily see it when $p = 2$. Indeed, consider the two dimensional Ornstein-Uhlenbeck operator

$$(\mathcal{A}u)(x,y) = \frac{1}{2}\Delta u(x,y) - (ax - by)D_x u(x,y) - ayD_y u(x,y), \qquad (x,y) \in \mathbb{R}^2,$$

where a and b are real constants, with $a > 0$ and $b \neq 0$. A straightforward computation shows that the associated Ornstein-Uhlenbeck semigroup is not symmetric in L^2_μ. Therefore, θ_2 is strictly less than $\pi/2$. Moreover, since

$$e^{zB} = \begin{pmatrix} e^{-az} & bze^{-az} \\ 0 & e^{-az} \end{pmatrix}, \qquad z \in \mathbb{C},$$

then

$$\lim_{\substack{|z| \to +\infty \\ \|Q_\infty^{1/2} e^{zB^*} Q_\infty^{-1/2}\|_\infty \le 1}} = 0,$$

for any $\theta < \pi/2$, so that there exist some points $z \in \mathbb{C}$ which satisfy (9.3.70) and do not belong to Σ_{θ_2}.

9.3.4 Hermite polynomials

In this subsection we show some relations connecting the Hermite polynomials and the Ornstein-Uhlenbeck operators. For this purpose, we partially follow the approach in [109].

For any multi-index α, the N-dimensional Hermite polynomial $H_\alpha^{(N)}$ is defined by

$$H_\alpha^{(N)}(x) = (-1)^{|\alpha|} e^{\frac{1}{2}|x|^2} D^\alpha e^{-\frac{1}{2}|x|^2}, \qquad x \in \mathbb{R}^N.$$

Equivalently,

$$H_\alpha^{(N)}(x) = \prod_{k=1}^{N} H_{\alpha_k}^{(1)}(x_k), \qquad x \in \mathbb{R}^N, \tag{9.3.72}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$.

Throughout this first part of the subsection we denote by $\mathcal{A}$ the Ornstein-Uhlenbeck operator $\mathcal{A}$ defined in $W_\mu^{2,2}$ by

$$\mathcal{A}u(x) = \Delta u(x) - \langle x, Du(x)\rangle, \qquad x \in \mathbb{R}^N, \quad u \in W_\mu^{2,2}. \tag{9.3.73}$$

Here,

$$\mu(dx) = \mathcal{N}(0, I)(dx) = \frac{1}{\sqrt{(2\pi)^N}} e^{-\frac{1}{2}|x|^2} dx.$$

Let us first consider the one-dimensional case and we introduce the operators in $W_\mu^{1,2}$ defined by

$$\partial f(x) = f'(x), \qquad \partial^* f(x) = -f'(x) + x f(x), \qquad x \in \mathbb{R}, \qquad f \in W_\mu^{1,2}.$$

The notation is justified by the identity

$$\int_{\mathbb{R}} \partial f \, g d\mu = \int_{\mathbb{R}} f \partial^* g d\mu, \qquad f, g \in W_\mu^{1,2},$$

which follows integrating by parts, if $f, g \in C_c^\infty(\mathbb{R})$, and by approximation with compactly supported function for a general pair of functions $f, g \in W_\mu^{1,2}$ (see Lemmas 9.3.3 and 9.3.4).

With the notation now introduced, we can rewrite the Ornstein-Uhlenbeck operator $\mathcal{A}$ in the more compact form

$$\mathcal{A}u = -\partial^* \partial u, \qquad x \in \mathbb{R}^N, \qquad u \in D(L). \tag{9.3.74}$$

We summarize some basic properties of the Hermite polynomials. In particular we show that they are an orthonormal basis of L_μ^2 consisting of eigenfunctions of L.

Proposition 9.3.27 *For any $n \in \mathbb{N} \cup \{0\}$, the following properties are met:*

(*i*) *the Hermite polynomials $\{H_n^{(1)}\}$ can be equivalently defined by recurrence by*

$$H_0^{(1)} = 1, \qquad H_n^{(1)} = \partial^* H_{n-1}^{(1)}, \qquad n \geq 1;$$

(*ii*) $H_n^{(1)}$ *is a polynomial with degree n with leading term x^n;*

(*iii*) $\partial H_{n+1}^{(1)} = (n+1)H_n^{(1)}$*;*

(*iv*) $\mathcal{A}H_n^{(1)} = -nH_n^{(1)}$*;*

(*v*) $\{H_n^{(1)}/\sqrt{n!}\}$ *is an orthonormal basis of L^2_μ. Here, we set $0! = 1$.*

Proof. The properties (i), (ii) and (iii) can be proved by induction. In particular, to prove the property (iii) one can take advantage of the fact that $[\partial, \partial^*] = I$ in $W^{2,2}_\mu$.

The property (iv) is now a straightforward consequence of (i), (iii) and (9.3.74). Indeed, we have

$$\mathcal{A}H_n^{(1)} = -\partial^*\partial H_n^{(1)} = -\partial^*(nH_{n-1}^{(1)}) = -nH_n^{(1)}. \tag{9.3.75}$$

Let us now prove the property (v). For this purpose we observe that each polynomial of order $m \in \mathbb{N}$ belongs to span$\,(H_0^{(1)}, \ldots, H_m^{(1)})$. Therefore, to prove that $\{H_n^{(1)}/\sqrt{n!}\}$ is a basis of L^2_μ, it suffices to show that the set of all the polynomials are a dense subspace of L^2_μ. So, let us fix a function $f \in L^2_\mu$ such that

$$0 = \int_{\mathbb{R}} f(x)x^n d\mu = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)x^n e^{-\frac{1}{2}x^2}dx, \qquad n \in \mathbb{N} \cup \{0\}.$$

Then, the Fourier transform of the function $x \mapsto f(x)e^{-x^2/2}$ is an entire function, whose derivatives at 0 vanish. Therefore, $f(x)e^{-x^2/2} = 0$ for any $x \in \mathbb{R}$ and, consequently, $f = 0$.

To conclude the proof, let us prove that $\{H_n^{(1)}/\sqrt{n!}\}_{n\in\mathbb{N}}$ is an orthonormal system in L^2_μ. Fix $m, n \in \mathbb{N}$ with $m \leq n$ and denote by $(\cdot,\cdot)_{2,\mu}$ the Euclidean inner product in L^2_μ. Then, we have

$$\begin{aligned}(H_m^{(1)}, H_n^{(1)})_{2,\mu} &= (H_m^{(1)}, \partial^* H_{n-1}^{(1)})_{2,\mu} \\ &= (\partial H_m^{(1)}, H_{n-1}^{(1)})_{2,\mu} = -n(H_{m-1}^{(1)}, H_{n-1}^{(1)})_{2,\mu}\end{aligned}$$

and, iterating the argument, we finally get

$$(H_m^{(1)}, H_n^{(1)})_{2,\mu} = \frac{n!}{(n-m)!}(H_0^{(1)}, H_{n-m}^{(1)})_{2,\mu}.$$

Hence, if $n = m$, we get $\|H_n^{(1)}\|_2 = \sqrt{n!}$, while, if $m < n$, we have

$$(H_0^{(1)}, H_{n-m}^{(1)})_{2,\mu} = (\partial H_0^{(1)}, H_{n-m-1}^{(1)})_{2,\mu} = 0$$

and, consequently, $(H_m^{(1)}, H_n^{(1)})_{2,\mu} = 0$. The property (v) now follows. ■

We now consider the multidimensional case $N > 1$. According to the one-dimensional case, for any $k = 1, \ldots, N$, we define the operators ∂_k and ∂_k^* in $W^{1,2}_\mu$ by setting

$$\partial_k f(x) = \frac{\partial f}{\partial x_k}(x), \qquad \partial_k^* f(x) = -\frac{\partial f}{\partial x_k}(x) + x_k f(x), \qquad x \in \mathbb{R}^N.$$

With this notation the Ornstein-Uhlenbeck operator in (9.3.73) can be rewritten in the compact form $\mathcal{A} = -\sum_{k=1}^N \partial_k^* \partial_k$.

We can now prove the N-dimensional version of Proposition 9.3.27.

Proposition 9.3.28 *The following properties are met:*

(i) $H^{(N)}_{\alpha+e_k} = \partial_k^* H^{(N)}_\alpha$, *for any multi-index* α *and any* $k \in \{1, \ldots, N\}$;

(ii) for any multi-index α, $H^{(N)}_\alpha$ *is a polynomial with degree* $|\alpha|$, *whose leading term is given by* $x^\alpha = \prod_{i=1}^N x^{\alpha_i}$, *and it depends only on the variables* x_j *such that* $\alpha_j \neq 0$;

(iii) $\partial_k H^{(N)}_\alpha = \alpha_k H^{(N)}_{\alpha-e_k}$;

(iv) $\mathcal{A} H^{(N)}_\alpha = -|\alpha| H^{(N)}_\alpha$;

(v) $\{H^{(N)}_\alpha / \sqrt{\alpha!}\}_{\alpha\in\Lambda}$ *is an orthonormal basis of* L^2_μ.

Proof. The properties (i)-(iii) are immediate consequences of the results in Proposition 9.3.27 and formula (9.3.72), whereas the property (iv) can be checked adapting the proof of the property (v) in the quoted proposition. ■

Propositions 9.3.27 and 9.3.28 can be easily extended to the more general Ornstein-Uhlenbeck operator

$$\mathcal{A}u(x) = \frac{1}{2}\mathrm{Tr}\,(QD^2u(x)) + \langle Bx, Du(x)\rangle, \qquad x \in \mathbb{R}^N, \tag{9.3.76}$$

with $\sigma(B) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}$.

Corollary 9.3.29 *Let* μ *be the invariant measure of the Ornstein-Uhlenbeck semigroup associated with the operator* $\mathcal{A}$ *in* (9.3.76) *and let* R *be a non-singular matrix such that* $RQ_\infty R^* = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_N)$. *Then, the family of polynomials* $\{K^{(N)}_\alpha\}$ *defined by* $K^N_\alpha(x) = \tilde{H}^{(N)}_\alpha(Rx)$, *where*

$$\tilde{H}^{(N)}_\alpha(x) = \prod_{k=1}^N H^{(1)}_{\alpha_k}\left(\frac{x_k}{\sqrt{\lambda_k}}\right), \qquad x \in \mathbb{R}^N,$$

is a complete orthonormal set in L^2_μ.

Proof. Let $D = \mathrm{diag}\,(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_N})$ and set $M = D^{-1}R$. From the remarks before Lemma 9.3.15, it follows that the operator $\Phi_{M^{-1}}$, defined by $(\Phi_{M^{-1}}f)(x) = f(M^{-1}x)$ for any $x \in \mathbb{R}^N$ and any function $f : \mathbb{R}^N \to \mathbb{R}$, is a linear isometry between L^p_μ and $L^p_{\tilde{\mu}}$, where

$$\tilde{\mu}(dx) = \frac{1}{\sqrt{(2\pi)^N}} \exp\left(-\sum_{i=1}^N \frac{x_i^2}{2}\right) dx.$$

The assertion now follows from the property (iv) in Proposition 9.3.28. ■

9.4 The Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^N)$

To conclude the analysis of the Ornstein-Uhlenbeck operator, in this section we recall some results concerning the realization of the Ornstein-Uhlenbeck operator $\mathcal{A}$ in $L^p(\mathbb{R}^N)$. Here, B is any nonzero matrix and Q is strictly positive definite. The results that we present are taken from [112, 118].

Proposition 9.4.1 *For any $p \in [1, +\infty)$ the Ornstein-Uhlenbeck semigroup $\{T(t)\}$ can be extended to a strongly continuous semigroup (which we still denote by $\{T(t)\}$) to $L^p(\mathbb{R}^N)$. For any $f \in L^p(\mathbb{R}^N)$ and any $t > 0$, $T(t)f$ is given by the right-hand side of* (9.1.5). *Finally,*

$$\|T(t)\|_{L(L^p(\mathbb{R}^N))} \le e^{-\frac{t}{p}\mathrm{Tr}\,B}, \qquad t > 0. \tag{9.4.1}$$

Proof. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for any $p \in [1, +\infty)$, it suffices to prove that $T(t)f$ tends to f in $L^p(\mathbb{R}^N)$ as t tends to 0^+, and that

$$\|T(t)f\|_{L^p(\mathbb{R}^N)} \le e^{-\frac{t}{p}\mathrm{Tr}\,B}\|f\|_{L^p(\mathbb{R}^N)}, \qquad t > 0, \tag{9.4.2}$$

for any $f \in C_c^\infty(\mathbb{R}^N)$. Denote by $\{S(t)\}$ the semigroup in $L^p(\mathbb{R}^N)$ defined by $(S(t)f)(x) = f(e^{tB}x)$ for any $t > 0$, any $x \in \mathbb{R}^N$ and any $f \in L^p(\mathbb{R}^N)$. As it is easily seen, for any $f \in C_c^\infty(\mathbb{R}^N)$ and any $t > 0$, we can write $T(t)f = S(t) \circ G_t$, where

$$(G_t f)(x) = \frac{1}{\sqrt{(2\pi)^N \det Q_t}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Q_t^{-1}y, y\rangle} f(x-y)dy, \qquad t > 0,\ x \in \mathbb{R}^N.$$

Let us show that $\{S(t)\}$ is a strongly continuous (semi)group in $L^p(\mathbb{R}^N)$ and that

$$\|S(t)\|_{L(L^p(\mathbb{R}^N))} \le e^{-\frac{t}{p}\mathrm{Tr}\,B}, \qquad t > 0. \tag{9.4.3}$$

The estimate (9.4.3) easily follows by a change of variable in the integral defining $\|S(t)f\|_p$, recalling that $\det(e^{-tB}) = e^{-t\mathrm{Tr}\,B}$ for any $t \in \mathbb{R}$.

To show that $\{S(t)\}$ is a strongly continuous semigroup, we observe that if $f \in C_c^\infty(\mathbb{R}^N)$, then $S(t)f$ converges to f uniformly in $\mathbb{R}^N$ as t tends to 0^+. Since the support of $S(t)f$ is compact in $\mathbb{R}^N$ and there exists $R > 0$ such that $\mathrm{supp}(S(t)f) \subset B(R)$ for any $t \in [0,1]$, then $S(t)f$ converges to f also in $L^p(\mathbb{R}^N)$ as t tends to 0^+. Now, using the density of $C_c^\infty(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ and taking (9.4.3) into account, from Proposition A.1.2 we deduce that $S(t)f$ converges to f as t tends to 0^+ for any $f \in L^p(\mathbb{R}^N)$.

As far as $\{G_t\}$ is concerned, we observe that the Young inequality implies that, for any $t > 0$, G_t is a contraction in $L^p(\mathbb{R}^N)$. Moreover, standard arguments show that $G_t f$ converges to f in $L^p(\mathbb{R}^N)$, as t tends to 0^+, for any $f \in L^p(\mathbb{R}^N)$. Hence, the semigroup $\{T(t)\}$ is strongly continuous in $L^p(\mathbb{R}^N)$. Finally, (9.4.2) easily follows from (9.4.3) and the contractiveness of $\{G_t\}$. ■

In the rest of the section we denote by $A_p : D(A_p) \subset L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ the infinitesimal generator of $\{T(t)\}$ in $L^p(\mathbb{R}^N)$. The following proposition gives a complete characterization of the operator A_p and of its domain.

Proposition 9.4.2 *For any $p \in (1, +\infty)$,*

$$D(A_p) = \{u \in W^{2,p}(\mathbb{R}^N) : \mathcal{A}u \in L^p(\mathbb{R}^N)\}, \tag{9.4.4}$$

and $A_p u = \mathcal{A}u$ for any $u \in D(A_p)$. Moreover, there exist two positive constants C_1 and C_2 such that

$$C_1(\|u\|_p + \|\mathcal{A}u\|_p) \le \|u\|_{W^{2,p}(\mathbb{R}^N)} + \|x \mapsto \langle Bx, Du(x)\rangle\|_p \le C_2(\|u\|_p + \|\mathcal{A}u\|_p), \tag{9.4.5}$$

for any $u \in D(A_p)$. Finally, for any $p \in [1, +\infty)$, $C_c^\infty(\mathbb{R}^N)$ is a core of A_p.

Proof. First of all we observe that, using the same arguments as in Lemma 9.3.13, it can be easily checked that $C_c^\infty(\mathbb{R}^N)$ is a core of A_p for any $p \in [1, +\infty)$. In particular, this implies that A_p is the closure of $\mathcal{A} : C_c^\infty(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ for any $p \in [1, +\infty)$.

Now, let us prove the formulas (9.4.4) and (9.4.5) for any $p \in (1, +\infty)$. For this purpose, we adapt to the present situation the proof of Theorem 9.3.17. Let us introduce the operators $A_p^0 : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, defined by $A_p^0 u = \frac{1}{2}\mathrm{Tr}(QD^2u)$ for any $u \in W^{2,p}(\mathbb{R}^N)$, and $B_p^0 : \{u \in L^p(\mathbb{R}^N) : x \mapsto \langle Bx, Du(x)\rangle \in L^p(\mathbb{R}^N)\} \to L^p(\mathbb{R}^N)$, defined by $B_p^0 u = \langle B\cdot, Du\rangle$ for any $u \in D(B_p^0)$, where Du is meant in the sense of distributions. Note that B_p^0 is the infinitesimal generator of the strongly continuous group $\{S(t)\}$ defined in the proof of Proposition 9.4.1.

Let us observe that since $C_c^\infty(\mathbb{R}^N)$ is a core of $D(A_p)$, and $A_p = A_p^0 + B_p^0$ in $C_c^\infty(\mathbb{R}^N)$, then A_p is the closure of the operator $A_p^0 + B_p^0 : D(A_p^0) \cap D(B_p^0) = \{u \in W^{2,p}(\mathbb{R}^N) : \mathcal{A}u \in L^p(\mathbb{R}^N)\} \to L^p(\mathbb{R}^N)$. Therefore, to complete the proof, it suffices to show that the operator $A_p^0 + B_p^0 : D(A_p^0) \cap D(B_p^0) \to L^p(\mathbb{R}^N)$ is closed.

According to Proposition B.1.13(iv), the operator $\omega I - A_p^0$ admits bounded imaginary powers with power angle $\theta_{\omega I - A_p^0} = 0$ for any $\omega > 0$. Similarly, taking the estimate (9.4.3) into account and applying Proposition B.1.13(iii), we deduce that the operator $\omega I - B_p^0$ admits bounded imaginary powers for any $\omega > -(\mathrm{Tr}\, B)/p$, with power angle $\theta_{\omega I - B_p^0} \le \pi/2$.

To apply Theorem B.1.16, we need to show that the condition (B.1.6) is satisfied. For this purpose, following the arguments in the proof of Theorem 9.3.17, we replace the space $C_b^4(\mathbb{R}^N)$ with the space $\mathcal{G}_k$ defined by

$$\mathcal{G}_k = \left\{ u \in W^{k,2}(\mathbb{R}^N) : x \mapsto (1+|x|^2)^{k/2} D^\alpha u \in L^2(\mathbb{R}^N) \text{ for any } |\alpha| \le k \right\},$$

where $k = k(p)$ is to be properly chosen in order that $\mathcal{G}_k$ is contained in the domain of the commutator $[A_p^0, B_p^0]$. It is then easy to check that

$$[A_p^0, B_p^0]u = \mathrm{Tr}\,(QB^* D^2 u), \qquad u \in \mathcal{G}_k.$$

Therefore, the operator $[A_p^0, B_p^0]R(1, A_p^0)$ extends to a bounded linear operator in $L^p(\mathbb{R}^N)$.

To make the arguments in the proof of Theorem 9.3.17 work, it suffices to show that, for any $k \in \mathbb{N}$, $R(\lambda, A_p^0)$ and $R(\lambda, B_p^0)$ leave $\mathcal{G}_k$ invariant, if $\mathrm{Re}\lambda$ is sufficiently large. Showing that $\mathcal{G}_k$ is invariant for $R(\lambda, A_p^0)$ is an easy task, taking advantage of the Fourier transform. As far as $R(\lambda, B_p^0)$ is concerned, we begin by showing that the map

$$x \mapsto (\mathcal{E}_\alpha(S(t)u))(x) := (1+|x|^2)^{k/2}(D^\alpha S(t)u)(x)$$

belongs to $L^2(\mathbb{R}^N)$ for any multi-index with length at most k. Since,

$$DS(t)f = e^{tB^*}S(t)Df, \qquad t > 0,\ f \in W^{1,p}(\mathbb{R}^N), \tag{9.4.6}$$

we can limit ourselves to proving that $\mathcal{E}_0 u \in L^2(\mathbb{R}^N)$. For this purpose, we observe that

$$|x|^k |(S(t)u)(x)| \le \|e^{-tB}\|_\infty^k (S(t)(\mathcal{E}_0 u))(x), \qquad t > 0,\ x \in \mathbb{R}^N. \tag{9.4.7}$$

From (9.4.7) we easily deduce that $\mathcal{E}_0(S(t)f) \in L^2(\mathbb{R}^N)$. Now, iterating (9.4.6) and taking (9.4.7) into account, we can easily show that, up to replacing C_k and ω_k with larger constants,

$$\|\mathcal{E}_\alpha(S(t)u)\|_{L^2(\mathbb{R}^N)} \le C_k e^{\omega_k t} \|\mathcal{E}_0(|D^{|\alpha|}u|)\|_{L^2(\mathbb{R}^N)}, \qquad t > 0, \tag{9.4.8}$$

for any multi-index α with length k. Since $R(\lambda, B_p^0)$ is the Laplace transform of the semigroup $\{S(t)\}$ (see the formula (B.1.3)), from (9.4.8) we deduce that if $\mathrm{Re}\,\lambda$ is sufficiently large, then $R(\lambda, B_p^0)$ maps $\mathcal{G}_k$ into itself. This completes the proof. ∎

To conclude this section we characterize the spectrum of $(A_p, D(A_p))$. Since the proof is rather technical we prefer not to go into details referring the reader to the original paper [112] for the proof.

Theorem 9.4.3 *Suppose that either the eigenvalues of the matrix B or the eigenvalues of the matrix $-B$ have all negative real parts. Then, for any $p \in [1, +\infty)$, it holds that*

$$\sigma(A_p) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -(\mathrm{Tr}\, B)/p\}. \tag{9.4.9}$$

Proof. See [112, Theorems 4.4, 4.7, 4.11 & 4.12]. ∎

The following proposition is now a straightforward consequence of the formula (9.4.9).

Proposition 9.4.4 *For any $p \in [1, +\infty)$ the Ornstein-Uhlenbeck semigroup is not analytic in $L^p(\mathbb{R}^N)$.*

Remark 9.4.5 The results in Theorem 9.4.3 should be compared with the corresponding results in Theorem 9.3.22. In particular, it should be noticed that the spectrum of the Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^N)$ strictly depends on p (unless in the case when $\mathrm{Tr}\, B = 0$), whereas the spectrum in L^p_μ is p-invariant.

Chapter 10

A class of nonanalytic Markov semigroups in $C_b(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N, \mu)$

10.0 Introduction

In Chapter 9 we have seen that the Ornstein-Uhlenbeck semigroup is neither analytic nor strongly continuous in $BUC(\mathbb{R}^N)$. In this chapter we deal with Markov semigroups associated with the operator $\mathcal{A}$ defined on smooth functions by

$$\mathcal{A}u(x) = \Delta u(x) + \sum_{i,j=1}^{N} b_j(x) D_j u(x), \qquad x \in \mathbb{R}^N, \tag{10.0.1}$$

where b_j ($j = 1, \ldots, N$) are unbounded Lipschitz continuous functions in $\mathbb{R}^N$. We provide conditions on $b = (b_1, \ldots, b_N)$ implying that $\{T(t)\}$ is not analytic in $C_b(\mathbb{R}^N)$. Next, we consider the semigroup $\{T(t)\}$ in $L^p(\mathbb{R}^N, \mu)$ (μ being the invariant measure associated with the operator $\mathcal{A}$) and we still provide suitable growth conditions on b implying that $\{T(t)\}$ is not analytic in $L^p(\mathbb{R}^N, \mu)$. The results of this chapter are taken from [117].

10.1 Nonanalytic semigroups in $C_b(\mathbb{R}^N)$

The main result of this section is Theorem 10.1.1. We recall that $(\widehat{A}, D(\widehat{A}))$ denotes the weak generator of the semigroup $\{T(t)\}$. See Section 2.3.

Theorem 10.1.1 *Assume that the coefficients b_j ($j = 1, \ldots, N$) in (10.0.1) are locally Lipschitz continuous. Further, assume that there exist three sequences $\{r_n\}$, $\{\lambda_n\} \subset (0, +\infty)$ and $\{\sigma_n\} \subset \mathbb{R}^N$ such that*

$$r_n \le M, \qquad n \in \mathbb{N}, \tag{10.1.1}$$

for some positive constant M;

$$\lim_{n\to+\infty} \frac{r_n}{\lambda_n^2} = 0; \tag{10.1.2}$$

$$\lim_{n\to+\infty}\frac{r_n}{\lambda_n}b(\lambda_n x+\sigma_n)=h\in\mathbb{R}^N,\qquad h\neq 0,\tag{10.1.3}$$

uniformly with respect to x on compact subsets of $\mathbb{R}^N$. Then, $(\widehat{A},D(\widehat{A}))$ is not sectorial in $C_b(\mathbb{R}^N)$. In particular, if $r_n=1$ for any $n\in\mathbb{N}$, then the spectrum of $(\widehat{A},D(\widehat{A}))$ contains the imaginary axis.

Proof. To prove the assertion we argue by contradiction. We show that, if $(\widehat{A},D(\widehat{A}))$ were sectorial, then the semigroup $\{S(t)\}\subset L(C_0(\mathbb{R}^N))$, defined by $(S(t)f)(x)=f(x+th)$ for any $t>0$, any $x\in\mathbb{R}^N$ and any $f\in C_0(\mathbb{R}^N)$, should be analytic. But this cannot be the case since the infinitesimal generator of $\{S(t)\}$, is the operator $B:\left\{u\in C_0(\mathbb{R}^N):\frac{\partial u}{\partial h}\in C_0(\mathbb{R}^N)\right\}\to C_0(\mathbb{R}^N)$ defined by $Bu=\langle h,Du\rangle$ for any $u\in D(B)$, whose spectrum contains the imaginary axis. Here, h is as in (10.1.3).

So, let us assume that $(\widehat{A},D(\widehat{A}))$ is sectorial and let us introduce, for any $n\in\mathbb{N}$ the isometries $I_n:C_b(\mathbb{R}^N)\to C_b(\mathbb{R}^N)$ defined by

$$(I_nu)(x)=u\left(\frac{x-\sigma_n}{\lambda_n}\right),\qquad x\in\mathbb{R}^N,\quad u\in C_b(\mathbb{R}^N),\quad n\geq 1.$$

For any $n\in\mathbb{N}$ we denote by $\mathcal{A}_n$ the second-order differential operator defined on smooth functions by

$$(\mathcal{A}_nu)(x)=\frac{r_n}{\lambda_n^2}\Delta u(x)+\frac{r_n}{\lambda_n}\sum_{j=1}^N b_j(\lambda_nx+\sigma_n)D_ju(x),\qquad x\in\mathbb{R}^N,\tag{10.1.4}$$

and we denote by $\{T_n(t)\}$ the positive semigroup of contractions in $C_b(\mathbb{R}^N)$ associated with the operator $\mathcal{A}_n$. Since $\mathcal{A}_n=r_nI_n^{-1}\mathcal{A}I_n$, it is easy to check that, for any $n\in\mathbb{N}$, the weak generator $\widehat{A}_n$ of the semigroup $\{T_n(t)\}$ is the operator $\widehat{A}_n:D(\widehat{A}_n):=I_n^{-1}(D(\widehat{A}))\to C_b(\mathbb{R}^N)$ defined by $\widehat{A}_n=r_n(I_n^{-1}\widehat{A}I_n)$. Therefore, $\rho(\widehat{A}_n)=r_n\rho(\widehat{A})$ and

$$R(\lambda,\widehat{A}_n)=\frac{1}{r_n}I_n^{-1}R(r_n^{-1}\lambda,\widehat{A})I_n,\tag{10.1.5}$$

for any $n\in\mathbb{N}$.

Since we are assuming that $(\widehat{A},D(\widehat{A}))$ is sectorial, then there exist two positive constants C and K such that

$$\|R(\lambda,\widehat{A})\|_{L(C_b(\mathbb{R}^N))}\leq\frac{C}{|\lambda|},\qquad \mathrm{Re}\,\lambda>K.\tag{10.1.6}$$

Using (10.1.5), (10.1.6) and the assumptions on the sequence $\{r_n\}$, it is now immediate to check that

$$\|R(\lambda,\widehat{A}_n)\|_{L(C_b(\mathbb{R}^N))}\leq\frac{C}{|\lambda|},\qquad \mathrm{Re}\,\lambda>MK.\tag{10.1.7}$$

Now, we fix $f \in C_c^\infty(\mathbb{R}^N)$ and check that, for any $\lambda \in \mathbb{C}$, with $\mathrm{Re}\,\lambda > MK$, and any $x \in \mathbb{R}^N$, it holds that

$$\lim_{n\to+\infty} (R(\lambda, \widehat{A}_n)f)(x) = (R(\lambda, B)f)(x). \tag{10.1.8}$$

Of course, once (10.1.8) is proved, letting n tend to $+\infty$ in (10.1.7) we will get

$$\|R(\lambda, B)f\|_\infty \le \frac{C}{|\lambda|}\|f\|_\infty, \qquad \mathrm{Re}\ \lambda > MK,$$

first for any $f \in C_c^\infty(\mathbb{R}^N)$ and then, by density, for any $f \in C_0(\mathbb{R}^N)$. Theorem B.2.7 then will imply that B is sectorial: a contradiction.

To show (10.1.8) we modify the proof of the classical Trotter-Kato theorem (see Theorem B.1.11).

As a first step we prove that, for any $f \in C_c^\infty(\mathbb{R}^N)$, the function $T_n(\cdot)f$ converges to $S(\cdot)f$ in $[0,T]\times\mathbb{R}^N$ as n tends to $+\infty$, for any $T>0$. For this purpose, we begin by observing that, since $S(t)$ maps $C_c^\infty(\mathbb{R}^N)$ into itself for any $t \ge 0$, and $C_c^\infty(\mathbb{R}^N)$ is contained in $D(\widehat{A}_n)$ (see Proposition 2.3.6), by Proposition 2.3.5,

$$\frac{d}{dt}(T_n(t)S(s)f)(x) = (T_n(t)\widehat{A}_nS(s)f)(x), \qquad s,t>0, \quad x \in \mathbb{R}^N,$$

for any $f \in C_c^\infty(\mathbb{R}^N)$. Moreover, since $\{S(t)\}$ is a strongly continuous semigroup of contractions and the domain of its infinitesimal generator contains $C_c^\infty(\mathbb{R}^N)$, then the map $t \mapsto S(t)f$ is differentiable in $[0,+\infty)$ with values in $C_0(\mathbb{R}^N)$ and its derivative at t is $BS(t)f$. Therefore, we have

$$\begin{aligned}(T_n(t)f)(x) - (S(t)f)(x) &= -\int_0^t \frac{d}{ds}\{(T_n(t-s)S(s)f)(x)\}\,ds\\ &= \int_0^t (T_n(t-s)(\widehat{A}_n - B)S(s)f)(x)ds,\end{aligned}$$

for any $t>0$ and any $x \in \mathbb{R}^N$, which implies that

$$\|T_n(t)f - S(t)f\|_\infty \le T \sup_{s\in[0,T]} \|(\widehat{A}_n - B)S(s)f\|_\infty, \tag{10.1.9}$$

for any $t \in [0,T]$ and any $T>0$. Let us observe that the right-hand side of (10.1.9) converges to 0 as n tends to $+\infty$. Indeed, recalling that $\widehat{A}_nS(s)f = \mathcal{A}_nS(s)f$ (see again Chapter 2), we have

$$\begin{aligned}&|(\widehat{A}_nS(s)f)(x) - (BS(s)f)(x)|\\ \le\ &\frac{r_n}{\lambda_n^2}\|\Delta S(s)f\|_\infty + \left|\frac{r_n}{\lambda_n}b(\lambda_n x+\sigma_n) - h\right|\|DS(s)f\|_\infty \chi_{\mathrm{supp}(S(s)f)}(x),\end{aligned}$$

for any $s \in [0,T]$ and any $x \in \mathbb{R}^N$. Since $\mathrm{supp}(S(s)f) \subset \overline{B}(|h|T) + \mathrm{supp}(f)$ for any $s \in [0,T]$, using the conditions (10.1.2) and (10.1.3), we easily deduce that $\|(\widehat{A}_n - B)S(s)f\|_\infty$ tends to 0 as n tends to $+\infty$. Therefore, from (10.1.9) it follows that $T_n(\cdot)$ tends to $S(\cdot)f$ uniformly in $[0,T]\times\mathbb{R}^N$.

Now, by (2.2.14) and (B.1.3) we can write

$$(R(\lambda,\widehat{A}_n)f)(x) - (R(\lambda,B)f)(x) = \int_0^{+\infty} e^{-\lambda t}(T_n(t) - S(t)f)(x)dt,$$

for any $x\in\mathbb{R}^N$. Consequently,

$$\begin{aligned}&|(R(\lambda,\widehat{A}_n)f)(x) - (R(\lambda_0,B)f)(x)|\\ &\le \int_0^{+\infty} e^{-\mathrm{Re}\,\lambda t}|(T_n(t)f)(x) - (S(t)f)(x)|dt,\end{aligned} \tag{10.1.10}$$

for any $x\in\mathbb{R}^N$, and the right-hand side of (10.1.10) tends to 0 as n tends to $+\infty$, by the dominated convergence theorem. Therefore, (10.1.8) follows.

To conclude the proof, let us prove that if $r_n = 1$ for any $n\in\mathbb{N}$, then the spectrum of $(\widehat{A}, D(\widehat{A}))$ contains the imaginary axis. For this purpose, we observe that from (10.1.5) we get

$$\|R(\lambda,\widehat{A})\|_{L(C_b(\mathbb{R}^N))} = \|R(\lambda,\widehat{A}_n)\|_{L(C_b(\mathbb{R}^N))},$$

for any $\lambda\in\rho(\widehat{A}_n) = \rho(\widehat{A})$. Therefore, since $\rho(\widehat{A})$ contains the halfplane $\{\lambda\in\mathbb{C} : \mathrm{Re}\,\lambda > 0\}$, by (10.1.8) we have

$$\|R(\lambda,B)\|_{L(C_b(\mathbb{R}^N))} \le \limsup_{n\to+\infty}\|R(\lambda,\widehat{A}_n)\|_{L(C_b(\mathbb{R}^N))} = \|R(\lambda,\widehat{A})\|_{L(C_b(\mathbb{R}^N))}.$$

But, since $i\mathbb{R}\subset\sigma(B)$, it follows that

$$\lim_{\substack{\lambda\to is\\ \mathrm{Re}\lambda>0}} \|R(\lambda,A)\|_{L(C_b(\mathbb{R}^N))} = +\infty,$$

for any $s>0$, so that $i\mathbb{R}\subset\sigma(\widehat{A})$ (see Proposition A.3.4). ∎

Now we look for sufficient conditions under which (10.1.1)-(10.1.3) are satisfied.

Corollary 10.1.2 *Assume that there exist two sequences* $\{\tau_n\}\subset\mathbb{R}^N$ *and* $\{\lambda_n\}\subset(0,+\infty)$ *such that* $b(\tau_n)\neq 0$ *for any* $n\in\mathbb{N}$, $\{\lambda_n\}$ *is bounded,* $\lambda_n|b(\tau_n)|$ *tends to* $+\infty$ *as* n *tends to* $+\infty$ *and*

$$\lim_{n\to+\infty}\frac{|b(\tau_n+\lambda_n x) - b(\tau_n)|}{|b(\tau_n)|} = 0, \tag{10.1.11}$$

uniformly on compact subsets of $\mathbb{R}^N$. *Then,* $(\widehat{A}, D(\widehat{A}))$ *is not sectorial.*

Proof. Let $\{\sigma_n\}$ be a subsequence of $\{\tau_n\}$ such that $b(\sigma_n)/|b(\sigma_n)|$ converges to some $h \in \mathbb{R}^N$ as n tends to $+\infty$ Then, if we set $r_n = \lambda_n |b(\sigma_n)|^{-1}$ for any $n \in \mathbb{N}$, it is immediate to check that the three sequences $\{r_n\}$, $\{\lambda_n\}$ and $\{\sigma_n\}$ satisfy the conditions (10.1.1)-(10.1.3). ∎

Corollary 10.1.3 *Let $b_j \in C^1(\mathbb{R}^N)$ for any $j = 1, \ldots, N$. Further, assume that there exists a sequence $\{\tau_n\} \subset \mathbb{R}^N$, diverging to $+\infty$, such that $|b(\tau_n)| \geq K|\tau_n|^\alpha$ for some $K, \alpha > 0$ and any $n \in \mathbb{N}$. If*

$$\sum_{i,j=1}^{N} |D_i b_j(x)|^2 \leq K(1 + |x|^\beta)^2, \qquad x \in \mathbb{R}^N, \tag{10.1.12}$$

for some $\beta < \alpha$, then the assumptions of Corollary 10.1.2 *are satisfied.*

Proof. Without loss of generality, we can assume that $\lambda_n = 1$ for any $n \in \mathbb{N}$. The mean value theorem yields

$$\begin{aligned} |b(\tau_n + \lambda_n x) - b(\tau_n)| &\leq R \sup_{0 \leq \theta \leq 1} \left(\sum_{i,j=1}^{N} |D_i b_j(\tau_n + \theta \lambda_n x)|^2 \right)^{\frac{1}{2}} \\ &\leq KR(1 + (|\tau_n| + MR)^\beta), \end{aligned} \tag{10.1.13}$$

for any $|x| \leq R$. Dividing both the sides of (10.1.13) by $|b(\tau_n)|$, we easily see that the condition (10.1.11) is satisfied since, by assumption, $|b(\tau_n)|$ tend to $+\infty$ faster than $|\tau_n|^\beta$. ∎

Remark 10.1.4 The conditions in Corollary 10.1.3 are always satisfied in the case when b_j are polynomials for any $j = 1, \ldots, N$. Indeed, let b_i be the polynomial of maximal degree and define $\tau_n = nx_0$ for any $n \in \mathbb{N}$, where $x_0 \in \mathbb{R}^N$ is such that the homogeneous part of b_i of maximum degree does not vanish at x_0. It is immediate to check that

$$|b(\tau_n)| \geq |b_i(\tau_n)| \geq K n^{\deg(b_i)},$$

for some positive constant $K > 0$ and any n sufficiently large. Moreover, since $\sum_{i,j=1}^N |D_i b_j|^2$ is a polynomial of degree $2(\deg(b_i) - 1)$, the condition (10.1.12) is satisfied with $\beta = \deg(b_i) - 1 < \alpha := \deg(b_i)$.

Here we see, with techniques different from those in Chapter 9, that the nondegenerate Ornstein-Uhlenbeck semigroup is not analytic in $C_b(\mathbb{R}^N)$.

Corollary 10.1.5 *Let $b_j \in C^1(\mathbb{R}^N)$ $(j = 1, \ldots, N)$. Assume that there exist: (i) a sequence $\{\tau_n\} \subset \mathbb{R}^N$, such that $|b(\tau_n)|$ diverges to $+\infty$ as n tends to $+\infty$, (ii) positive numbers γ, δ, C such that*

$$\left(\sum_{i,j=1}^{N} |D_i b_j(x)|^2 \right)^{\frac{1}{2}} \leq \gamma |b(x)|^{\frac{3}{2}} + C,$$

for any $x \in \mathbb{R}^N$ *with* $|x-\tau_n| \le \delta |b(\tau_n)|^{-1/2}$. *Then* $(\widehat{A}, D(\widehat{A}))$ *is not sectorial in* $C_b(\mathbb{R}^N)$.

Proof. We begin the proof by observing that, without loss of generality, we can assume that $\delta < (4\gamma)^{-1}$ and that $b(\tau_n) \neq 0$ for any $n \in \mathbb{N}$. Next we fix an arbitrary $n \in \mathbb{N}$ and observe that, for any $s \in (0,\delta)$ and any $|x| \le s|b(\tau_n)|^{-1/2}$, we have

$$\begin{aligned} \frac{|b(\tau_n + x) - b(\tau_n)}{|b(\tau_n)|}| &\le \frac{|x|}{|b(\tau_n)|} \sup_{0\le\theta\le 1} \left(\sum_{i,j=1}^N |D_i b_j(\tau_n+\theta x)|^2 \right)^{\frac{1}{2}} \\ &\le \gamma s \sup_{0\le\theta\le 1} \frac{|b(\tau_n+\theta x)|^{\frac{3}{2}}}{|b(\tau_n)|^{\frac{3}{2}}} + Cs|b(\tau_n)|^{-\frac{3}{2}}. \end{aligned} \tag{10.1.14}$$

We set

$$F_n(s) = \sup_{|x|\le s|b(\tau_n)|^{-\frac{1}{2}}} \frac{|b(\tau_n+x)|}{|b(\tau_n)|}, \qquad s \in [0,\delta).$$

From (10.1.14) we get

$$F_n(s) \le 1 + \gamma s (F_n(s))^{\frac{3}{2}} + Cs|b(\tau_n)|^{-\frac{3}{2}}, \qquad s\in[0,\delta).$$

We claim that $F_n(s) < 4$ for any $s \in [0,\delta)$ and any n sufficiently large. To prove the claim, we begin by noting that, for n large enough, we have

$$1 + Cs|b(\tau_n)|^{-\frac{3}{2}} \le 1 + C\delta |b(\tau_n)|^{-\frac{3}{2}} \le 2.$$

It follows that

$$F_n(s) \le 2 + \gamma s (F_n(s))^{\frac{3}{2}}, \qquad s\in[0,\delta). \tag{10.1.15}$$

Recalling that $4\delta\gamma < 1$, we get

$$F_n(s) < 2 + \frac{1}{4}(F_n(S))^{\frac{3}{2}}, \qquad s\in[0,\delta).$$

Since F_n is continuous in $[0,+\infty)$ with $F_n(0)=1$ for any $n \in \mathbb{N}$, and $[0,4)$ is the biggest interval containing 1 in which the inequality $x < 2 + x^{3/2}/4$ holds, we easily deduce that $F_n(s) \in [0,4)$ for any $s \in [0,\delta)$, and the claim follows. Thus, for n large enough, the equation (10.1.14) yields, for $|x| \le s|b(\tau_n)|^{-1/2}$,

$$\frac{|b(\tau_n+x) - b(\tau_n)|}{|b(\tau_n)|} \le 8\gamma s + Cs|b(\tau_n)|^{-\frac{3}{2}}. \tag{10.1.16}$$

Now, fix $R > 0$ and let $\lambda_n = |b(\tau_n)|^{-\frac{3}{4}}$. For any $y \in B(R)$ and n sufficiently large such that $R|b(\tau_n)|^{-1/4} < \delta$, we have $|\lambda_n y| \le s|b(\tau_n)|^{-1/2}$ with $s = R|b(\tau_n)|^{-1/4}$. Applying (10.1.16), we get

$$\frac{|b(\tau_n+\lambda_n x) - b(\tau_n)|}{|b(\tau_n)|} \le 8R\gamma|b(\tau_n)|^{-\frac{1}{4}} + CR|b(\tau_n)|^{-\frac{7}{4}}.$$

Corollary 10.1.2 now allows us to conclude the proof. ∎

Next corollary gives suitable conditions on the drift term of the operator $\mathcal{A}$ implying that the spectrum of $\widehat{A}$ contains the imaginary axis.

Corollary 10.1.6 *Assume that $b_j \in C^1(\mathbb{R})$ for any $j = 1, \ldots, N$ and there exists a sequence $\{\tau_n\} \subset \mathbb{R}^N$, diverging to infinity, such that $|b(\tau_n)|$ tends to $+\infty$ as n tends to $+\infty$. Further, assume that $\sum_{i,j=1}^N |D_i b_j(x)|^2$ and $|b(x)|/|x|$ tend to 0 as $|x|$ tends to $+\infty$. Then, the spectrum of $(\widehat{A}, D(\widehat{A}))$ contains the imaginary axis. As a consequence, $(\widehat{A}, D(\widehat{A}))$ is not sectorial.*

Proof. Fix $R > 0$. By the mean value theorem we have

$$|b(\tau_n + \lambda_n x) - b(\tau_n)| \le R\lambda_n \sup_{0\le\theta\le 1}\left(\sum_{i,j=1}^N |D_i b_j(\tau_n + \theta\lambda_n x)|^2\right)^{\frac{1}{2}},$$

for any $|x| \le R$ and any sequence $\{\lambda_n\} \subset (0, +\infty)$.

We now take $\lambda_n = |b(\tau_n)|$ and $r_n = 1$ for any $n \in \mathbb{N}$, and we show that

$$\lim_{n\to+\infty} \sup_{0\le\theta\le 1}\left(\sum_{i,j=1}^N |D_i b_j(\tau_n + \theta\lambda_n x)|^2\right)^{\frac{1}{2}} = 0,$$

uniformly with respect to $x \in \overline{B}(R)$. For this purpose it suffices to show that $|\tau_n + \theta\lambda_n x|$ tends to $+\infty$ as n tends to $+\infty$, uniformly with respect to $\theta \in [0, 1]$ and $x \in \overline{B}(R)$. Since b is sublinear, we have

$$|\tau_n + \theta\lambda_n x| \ge |\tau_n|\left(1 - \frac{R|b(\tau_n)|}{|\tau_n|}\right) \ge \frac{1}{2}|\tau_n|,$$

for any $\theta \in [0, 1]$, any $x \in \overline{B}(R)$ and any n sufficiently large. Therefore,

$$\lim_{n\to+\infty} \inf_{\substack{\theta\in[0,1]\\ x\in\overline{B}(R)}} |\tau_n + \theta\lambda_n x| = +\infty.$$

Theorem 10.1.1 and Corollary 10.1.2 allow us to conclude. ∎

Up to now we have shown sufficient conditions implying that $(\widehat{A}, D(\widehat{A}))$ is not sectorial. In the next theorem we provide suitable conditions on the drift coefficient of the operator $\mathcal{A}$ which guarantee that $(\widehat{A}, D(\widehat{A}))$ is sectorial in $C_b(\mathbb{R}^N)$.

Theorem 10.1.7 *Let $b_j \in C^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, for some $p \in (N, +\infty)$ and any $j = 1, \ldots, N$, be such that* $\operatorname{div} b$ *is bounded from below. Then, $(\widehat{A}, D(\widehat{A}))$ is sectorial in $C_b(\mathbb{R}^N)$.*

Proof. As a first step we introduce the formal adjoint of $\mathcal{A}$, i.e., the operator

$$\mathcal{A}^* = \Delta - \sum_{i=1}^N b_i D_i - \operatorname{div} b,$$

and we prove that its realization A^* in $L^1(\mathbb{R}^N)$ with domain $D(A^*) = \{u \in L^1(\mathbb{R}^N) : \Delta u, (\operatorname{div} b)u \in L^1(\mathbb{R}^N)\}$ (Δu being meant in the sense of distributions) generates an analytic semigroup in $L^1(\mathbb{R}^N)$. Then, we use this result to show that $(\widehat{A}, D(\widehat{A}))$ is sectorial.

Let us observe that, since $\operatorname{div} b$ is bounded from below and it is locally bounded, according to [82, Theorem I, Lemmas 6 & 8] the operator

$$B_0 = \Delta - \operatorname{div} b + \inf_{\mathbb{R}^N}(\operatorname{div} b) := B + \inf_{\mathbb{R}^N}(\operatorname{div} b),$$

with domain $D(B_0) = D(A^*)$, generates an analytic semigroup in $L^1(\mathbb{R}^N)$. Moreover,

$$\|\Delta u\|_{L^1(\mathbb{R}^N)} + \|(\operatorname{div} b)u\|_{L^1(\mathbb{R})} \le C(\|u\|_{L^1(\mathbb{R}^N)} + \|Bu\|_{L^1(\mathbb{R}^N)}), \qquad (10.1.17)$$

for any $u \in D(A^*)$.

We now recall that if $u \in L^1(\mathbb{R}^N)$ is such that $\Delta u \in L^1(\mathbb{R}^N)$, then $u \in W^{1,q}(\mathbb{R}^N)$ for any $q \in [1, N/(N-1))$ and there exists a positive constant $C = C(N, q)$, independent of u, such that

$$\|Du\|_{L^q(\mathbb{R}^N)} \le C(N,q)\left(\|u\|_{L^1(\mathbb{R}^N)} + \|\Delta u\|_{L^1(\mathbb{R}^N)}\right), \qquad (10.1.18)$$

see [140, Theorem 5.8]. Applying the estimate (10.1.18), with $q = p/(p-1)$, to the function $v_\lambda : \mathbb{R}^N \to \mathbb{R}$ ($\lambda > 0$), defined by $v(x) = u(\lambda x)$ for any $x \in \mathbb{R}^N$, and minimizing with respect to $\lambda > 0$, we get the following inequality:

$$\|Du\|_{L^{\frac{p}{p-1}}(\mathbb{R}^N)} \le 2C(N,p)\|u\|_{L^1(\mathbb{R}^N)}^{\frac{p-N}{2p}} \|\Delta u\|_{L^1(\mathbb{R}^N)}^{\frac{N+p}{2p}}.$$

Hölder's inequality then gives

$$\begin{aligned}\left\|\sum_{i=1}^N b_i D_i u\right\|_{L^1(\mathbb{R}^N)} &\le \|b\|_{L^p(\mathbb{R}^N)}\|Du\|_{L^{\frac{p}{p-1}}(\mathbb{R}^N)} \\ &\le 2C(N,p)\|b\|_{L^p(\mathbb{R}^N)}\|u\|_{L^1(\mathbb{R}^N)}^{\frac{p-N}{2p}}\|\Delta u\|_{L^1(\mathbb{R}^N)}^{\frac{p+N}{2p}}, \qquad (10.1.19)\end{aligned}$$

for any $u \in L^1(\mathbb{R}^N)$ such that $\Delta u \in L^1(\mathbb{R}^N)$.

Now, from (10.1.17) and (10.1.19) it follows that the term $u \mapsto \langle b, Du\rangle$ is a small perturbation of $(B, D(B))$. Therefore, according to Theorem B.2.10, $(A^*, D(A^*))$ generates an analytic semigroup in $L^1(\mathbb{R}^N)$.

We are now in a position to prove that $(\widehat{A}, D(\widehat{A}))$ is sectorial in $C_b(\mathbb{R}^N)$. Possibly replacing A^* with $A^* - \omega$, with ω sufficiently large, we can assume that

$$\|R(\lambda, A^*)\|_{L(L^1(\mathbb{R}^N))} \le \frac{M}{|\lambda|},$$

for any $\lambda\in\mathbb{C}$ with positive real part. Let us set $D_\lambda=(\lambda-A^*)(C_c^\infty(\mathbb{R}^N))$ for such λ's. Since $C_c^\infty(\mathbb{R}^N)$ is a core of A^* (see [82, Theorem IV]), then D_λ is dense in $L^1(\mathbb{R}^N)$ for any $\operatorname{Re}\lambda>0$. Therefore, if $u\in D(\widehat{A})\subset C_b(\mathbb{R}^N)$ and λ is as above, then

$$\begin{aligned}\|u\|_\infty &= \sup\left\{\int_{\mathbb{R}^N} u\varphi\,dx : \varphi\in D_\lambda,\ \|\varphi\|_{L^1(\mathbb{R}^N)}\le 1\right\}\\ &\le \sup\left\{\int_{\mathbb{R}^N} u(\lambda-A^*)v\,dx : v\in C_c^\infty(\mathbb{R}^N),\ \|v\|_{L^1(\mathbb{R}^N)}\le \frac{M}{|\lambda|}\right\}\\ &= \sup\left\{\int_{\mathbb{R}^N} v(\lambda-\widehat{A})u\,dx : v\in C_c^\infty(\mathbb{R}^N),\ \|v\|_{L^1(\mathbb{R}^N)}\le \frac{M}{|\lambda|}\right\}\\ &\le \frac{M}{|\lambda|}\|(\lambda-\widehat{A})u\|_\infty.\end{aligned}$$

Therefore, $\|R(\lambda,\widehat{A})\|_{L(C_b(\mathbb{R}^N))}\le M/|\lambda|$ and the assertion follows from Theorem B.2.7. ∎

Example 10.1.8 Let $\mathcal{A}$ be the operator defined on smooth functions of two variables by

$$\mathcal{A}u(x,y)=D_x^2u(x,y)+D_y^2u(x,y)+b_0(\sqrt{x^2+y^2})(xu_y-yu_x),\qquad (x,y)\in\mathbb{R}^2,$$

where $b_0\in C^1(\mathbb{R})$. Note that the drift $b(x,y)=b_0(\sqrt{x^2+y^2})(-y,x)$ has divergence 0. The assumptions of Theorem 10.1.7 hold if

$$\int_0^{+\infty} r|b(r)|^q dr<+\infty,$$

for some $q>2$. In such a case, the semigroup associated with the operator $\mathcal{A}$ is analytic in $C_b(\mathbb{R}^N)$.

10.2 Nonanalytic semigroups in $L^p(\mathbb{R}^N,\mu)$

In this section we introduce a class of Markov semigroups $\{T(t)\}$, which admit an invariant measure μ, but which are not analytic in $L^p_\mu:=L^p(\mathbb{R}^N,\mu)$. Let $\mathcal{A}$ be the second-order elliptic operator defined by

$$\mathcal{A}\varphi=\Delta\varphi-\sum_{i=1}^N(D_iU+G_i)D_i\varphi,\qquad (10.2.1)$$

on smooth functions, where U and G satisfy the following hypotheses:

Hypotheses 10.2.1 (i) $U \in C^2(\mathbb{R}^N)$ and $e^{-U} \in L^1(\mathbb{R}^N)$;

(ii) $G \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and satisfies

$$\int_{\mathbb{R}^N} |G| e^{-U} dx < +\infty$$

and

$$\operatorname{div} G - \sum_{i=1}^{N} G_i D_i U = 0. \tag{10.2.2}$$

These conditions will be kept throughout this section and ensure that the semigroup $\{T(t)\}$ associated with $\mathcal{A}$ admits the measure

$$d\mu = M e^{-U(x)} dx, \qquad M = \left(\int_{\mathbb{R}^N} e^{-U(y)} dy \right)^{-1},$$

as its (unique) invariant measure (see Subsection 8.1.4). As it has been shown in Proposition 8.1.8, the semigroup $\{T(t)\}$ extends to a strongly continuous semigroup of contractions in L^p_μ for any $p \in [1, +\infty)$. Using the same notations as in Chapter 8, we still denote by $\{T(t)\}$ the so extended semigroup, and by $(L_p, D(L_p))$ its infinitesimal generator. Moreover, we write L instead of L_2.

Let us state the main result of this section.

Theorem 10.2.2 *Suppose that there exist some constants $k > 0$, $0 < \beta < \alpha$ and a sequence $\{\sigma_n\} \subset \mathbb{R}^N$, diverging to infinity, such that*

$$\begin{aligned} &(i)\ \ |G(\sigma_n)| \ge k|\sigma_n|^\alpha, \qquad n \ge 1; \\ &(ii)\ |\Delta U(x)| + |DU(x)| + |DG(x)| \le k(1 + |x|^\beta), \qquad x \in \mathbb{R}^N. \end{aligned} \tag{10.2.3}$$

Then, the semigroup $\{T(t)\}$ is not analytic in L^p_μ for any $p \in [1, +\infty)$.

Proof. We first prove the assertion in the case when $p = 2$. For this purpose, let us consider the isometry $R : L^2(\mathbb{R}^N) \to L^2_\mu$ defined by $Rf = f\psi$ for any $f \in L^2(\mathbb{R}^N)$, where

$$\psi(x) = M^{-\frac{1}{2}} e^{\frac{1}{2}U(x)}, \qquad x \in \mathbb{R}^N.$$

Further, let us define the operator $\widetilde{L}_2 : D(\widetilde{L}_2) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, with domain $D(\widetilde{L}_2) = \{g \in L^2(\mathbb{R}^N) : \psi g \in D(L_2)\}$, by setting

$$\widetilde{L}_2 u = R^{-1} L_2 R u, \qquad u \in D(\widetilde{L}_2).$$

Since, as it has already been recalled, the semigroup $\{T(t)\}$ is strongly continuous in L^2_μ, then it is immediate to check that $\widetilde{L}_2$ is the generator of a strongly continuous semigroup of contractions in $L^2(\mathbb{R}^n)$ (say $\{\widetilde{T}(t)\}$) defined by

$$\widetilde{T}(t)f = R^{-1}T(t)(Rf), \qquad t \ge 0, \quad f \in L^2(\mathbb{R}^N).$$

According to Propositions 2.3.6 and 8.1.9, $C_c^\infty(\mathbb{R}^N) \subset D(L_2)$. Therefore, $C_c^\infty(\mathbb{R}^N)$ is contained in $D(\widetilde{L}_2)$ and, since $L_2u = \mathcal{A}u$ for any $u \in C_c^\infty(\mathbb{R}^N)$, then

$$(\widetilde{L}_2 u)(x) = \Delta u(x) - \sum_{j=1}^N G_j(x)D_ju(x) - c(x)u(x), \qquad x \in \mathbb{R}^N, \quad (10.2.4)$$

for any $u \in C_c^\infty(\mathbb{R}^N)$, where the potential c is given by

$$c = \frac{1}{2}\left(\frac{1}{2}|DU|^2 - \Delta U + \sum_{i=1}^N G_iD_iU\right).$$

We claim that the operator $\widetilde{L}_2$ is not sectorial in $L^2(\mathbb{R}^N)$. Of course, this will imply that L_2 is not sectorial in L^2_μ as well.

To prove the claim, we adapt the technique in the proof of Theorem 10.1.1 to this situation. For this purpose, let us set $\gamma = \frac{1}{2}(\alpha+\beta)$ and introduce the sequences $\{\lambda_n\}$ and $\{r_n\}$ defined by

$$\lambda_n = \frac{1}{|\sigma_n|^\gamma + 1}, \qquad r_n = \frac{1}{|G(\sigma_n)|(|\sigma_n|^\gamma + 1)}, \qquad n \ge 1.$$

(Up to replacing $\{\sigma_n\}$ with a suitable subsequence, we can assume that $G(\sigma_n) \neq 0$ for any $n \in \mathbb{N}$, so that the sequence $\{r_n\}$ is well defined.) Moreover, for any $n \in \mathbb{N}$, we introduce the operator I_n defined by

$$(I_nu)(x) = u\left(\frac{x-\sigma_n}{\lambda_n}\right), \qquad x \in \mathbb{R}^N, \quad u \in L^2(\mathbb{R}^N), \quad n \ge 1$$

and $\widetilde{L}_{2,n} = r_nI_n^{-1}\widetilde{L}_2I_n$ with domain $D(\widetilde{L}_{2,n}) = I_n^{-1}(D(\widetilde{L}_2))$. As it is immediately seen, $\widetilde{L}_{2,n}$ is the infinitesimal generator of the strongly continuous semigroup of contractions $\{\widetilde{T}_n(t)\}$ in $L^2(\mathbb{R}^N)$, defined by $\widetilde{T}_n(t) = I_n^{-1}\widetilde{T}(r_nt)I_n$, for any $t \ge 0$.

Let us show that there exists $h \in \mathbb{R}^N$ such that, for any $u \in C_c^\infty(\mathbb{R}^N)$, the function $\widetilde{L}_{2,n}u$ converges to $\mathcal{B}_{-h}u := \sum_{i=1}^N h_iD_iu$ as n tends to $+\infty$, locally uniformly in $\mathbb{R}^N$. For this purpose, we observe that, by (10.2.4), we have

$$(\widetilde{L}_{2,n}u)(x) = \frac{r_n}{\lambda_n^2}\Delta u(x) - \frac{r_n}{\lambda_n}\sum_{i=1}^N G_i(\lambda_nx+\sigma_n)D_iu(x) - r_nc(\lambda_nx+\sigma_n)u(x),$$

for any $x \in \mathbb{R}^N$, any $n \in \mathbb{N}$ and any $u \in C_c^\infty(\mathbb{R}^N)$.

Due to the choice of the sequences $\{\sigma_n\}$ and $\{r_n\}$, it is immediate to check that r_n/λ_n^2 tends to 0 as n tends to $+\infty$. Moreover, by the mean value theorem and (10.2.3)(ii) we have

$$|G(\lambda_n x+\sigma_n)|\le |G(\sigma_n)|+kR(1+(R+|\sigma_n|)^\beta), \tag{10.2.5}$$

for any $x\in \overline{B}(R)$ and any $R>0$. Hence,

$$\lim_{n\to+\infty}\frac{|G(\sigma_n+\lambda_n x)-G(\sigma_n)|}{|G(\sigma_n)|}=0,$$

locally uniformly in $\mathbb{R}^N$. Up to replacing $\{\sigma_n\}$ with a suitable subsequence, we can assume that $G(\sigma_n)/|G(\sigma_n)|$ converges to some vector $h\in\mathbb{R}^N$. Therefore,

$$\frac{r_n}{\lambda_n^2}\Delta u-\frac{r_n}{\lambda_n}\sum_{i=1}^N G_i(\lambda_n\cdot+\sigma_n)D_iu$$

converges to $\mathcal{B}_h u$ uniformly in $\mathbb{R}^N$ as n tends to $+\infty$ and, hence, in $L^2(\mathbb{R}^N)$, since the support of u is compact.

Let us now show that $c(\lambda_n\cdot+\sigma_n)u$ converges to 0 in $L^2(\mathbb{R}^N)$. For this purpose, we note that, thanks to (10.2.3)(ii),

$$|c(x)|\le C\left((1+|x|^\beta)^2+(1+|x|^\beta)|G(x)|\right),\qquad x\in\mathbb{R}^N, \tag{10.2.6}$$

for some positive constant C. Estimates (10.2.6) and (10.2.5) imply that the sequence $\{r_n|c(\lambda_n\cdot+\sigma_n)|\}$ converges to 0, locally uniformly in $\mathbb{R}^N$, and we are done.

Now the proof follows immediately, arguing by contradiction. Indeed, if $\widetilde{L}_2$ were sectorial in $L^2(\mathbb{R}^N)$, then the operators $\widetilde{L}_{2,n}$ should be sectorial as well, and there should exist $K,C>0$, independent of n (observe that $R(\lambda,\widetilde{L}_{2,n})=r_n^{-1}I_n^{-1}R(r_n^{-1}\lambda,\widetilde{L}_2)I_n$), such that

$$\|R(\lambda,\widetilde{L}_{2,n})u\|_{L^2(\mathbb{R}^N)}\le\frac{C}{|\lambda|}\|u\|_{L^2(\mathbb{R}^N)},$$

for any $\lambda\in\mathbb{C}$, with $\mathrm{Re}\,\lambda>K$ and any $u\in C_c^\infty(\mathbb{R}^N)$. Since the realization B_h of the operator $\mathcal{B}_h$ generates a strongly continuous semigroup in $L^2(\mathbb{R}^N)$ and $C_c^\infty(\mathbb{R}^N)$ is a core of B_h, using Trotter-Kato theorem (see Theorem B.1.11), it should follow that

$$\|R(\lambda,B_h)u\|_{L^2(\mathbb{R}^N)}\le\frac{C}{|\lambda|}\|u\|_{L^2(\mathbb{R}^N)},$$

for any $u\in L^2(\mathbb{R}^N)$, which, of course, is a contradiction.

To prove the assertion with a general $p\in[1,+\infty)$ it now suffices to observe that, if $T(t)$ were analytic in L^p_μ for some p as above ($p\neq 2$), then, by Stein interpolation theorem, it should be analytic in L^2_μ as well. For more details, we refer the reader to [45, Theorem 1.4.2]. ∎

Example 10.2.3 We now provide two concrete examples of differential operators in $\mathbb{R}^2$, such that the associated semigroups $\{T(t)\}$ are not analytic in L^2_μ. For this purpose, let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the operators defined on smooth functions by

$$\begin{aligned}\mathcal{A}_1u(x,y) = \Delta u(x,y) &- \left(x + y(x^2+y^2)^{r/2}\right) D_xu(x,y)\\ &- \left(y - x(x^2+y^2)^{r/2}\right) D_yu(x,y)\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_2u(x,y) = \Delta u(x,y) &- (x^2+y^2)\left(x - y(x^2+y^2)^{r/2}\right) D_xu(x,y)\\ &-(x^2+y^2)\left(y - x(x^2+y^2)^{r/2}\right) D_yu(x,y),\end{aligned}$$

for some $r>1$ and any $(x,y)\in\mathbb{R}^2$. We can rewrite the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ in the compact form (10.2.1) taking $U_j(x,y) = (2j)^{-1}(x^2+y^2)^j$, $G_j(x,y) = (x^2+y^2)^{r/2+(j-1)}(y,-x)$ for any $(x,y)\in\mathbb{R}^2$ and $j=1,2$. It is immediate to check that the condition (10.2.2) is satisfied. Moreover, if we take $\sigma_n=(n,0)$ for any $n\in\mathbb{N}$, also the assumptions of Theorem 10.2.2 are satisfied with $\alpha = r-1+2j$ and $\beta = r-2+2j$, $j=1,2$. Therefore, the semigroup $\{T_j(t)\}$ associated with the operator $\mathcal{A}_j$ is not analytic in $L^p_{\mu_j}$ ($p\in[1,+\infty)$) either for $j=1$ or for $j=2$, where

$$d\mu_1 = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}dxdy,$$

$$d\mu_2 = \left(\int_{\mathbb{R}^2} e^{-\frac{1}{4}(x^2+y^2)^2}\,dxdy\right)^{-1} e^{-\frac{1}{4}(x^2+y^2)^2}\,dxdy.$$

Part II

Markov semigroups in unbounded open sets

Chapter 11

The Cauchy-Dirichlet problem

11.0 Introduction

In this chapter we consider the parabolic Cauchy-Dirichlet problem

$$\begin{cases} D_t u(t,x) - \mathcal{A}u(t,x) = 0, & t > 0, \quad x \in \Omega, \\ u(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \\ u(0,x) = f(x), & \qquad\quad\; x \in \Omega, \end{cases} \tag{11.0.1}$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded connected smooth open set; f is a bounded and continuous function on Ω; $\mathcal{A}$ is the usual second-order uniformly elliptic operator

$$\mathcal{A} = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i=1}^{N} b_i D_i + c, \tag{11.0.2}$$

with unbounded coefficients

We show that, under rather weak assumptions of the smoothness of the coefficients q_{ij}, b_j $(i,j = 1,\dots,N)$ and c and assuming the existence of a Lyapunov function associated with the operator $\mathcal{A}$, the Cauchy-Dirichlet problem (11.0.1) admits a unique classical solution (i.e., a function $u \in C^{1,2}((0,+\infty) \times \Omega)$, bounded in $(0,+\infty) \times \Omega$ and continuous in $[0,+\infty) \times \overline{\Omega} \setminus (\{0\} \times \partial\Omega)$ which solves (11.0.1)). This, as usually, will allow us to define a semigroup $\{T(t)\}$ of bounded linear operators on $C_b(\Omega)$.

Next, assuming somewhat heavier assumptions on the growth rate of the coefficients at infinity, we show that such a semigroup satisfies an uniform gradient estimate similar to that proved in Chapter 6. The results of this chapter have been recently proved in [59].

The chapter is split into sections as follows. First, in Section 11.1, we show two maximum principles for the solution to the problem (11.0.1). In the first one, which is rather similar to that proved in Chapter 4, we deal with classical solutions to the problem (11.0.1) which are continuous in $[0,+\infty) \times \overline{\Omega}$. In the second one, we weaken a bit the assumptions on the continuity of u. Indeed, we just assume that u is continuous in $[0,+\infty) \times \overline{\Omega} \setminus (\{0\} \times \partial\Omega)$. This latter maximum principle is the keystone to prove the uniform gradient estimates.

In Section 11.2, by virtue of the previous two maximum principles and some classical Schauder estimates, we prove the existence and uniqueness of the classical bounded solution u to (11.0.1), and, then, we define the semigroup $\{T(t)\}$.

Next, in Section 11.3, under somewhat heavier assumptions on the growth rate of the coefficients of the operator $\mathcal{A}$, we prove that the gradient estimate

$$\|DT(t)f\|_\infty \le C\frac{e^{\omega t}}{\sqrt{t}}\|f\|_\infty, \qquad t>0, \tag{11.0.3}$$

holds for any $\omega>0$, some positive constant $C=C(\omega)$ and any $f\in C_b(\Omega)$. First, in Subsection 11.3.1 we prove (11.0.3) for classical solutions to the problem (11.0.1) enjoying some more regularity properties. This is done applying the Bernstein method as we did in Chapter 6. Unfortunately, it is not immediate to adapt the procedure in the proof of Theorem 6.1.7 to our situation, due to the fact that no *a priori* information of the sign of Du on $(0,+\infty)\times\partial\Omega$ is available. To overcome such a difficulty, we first prove the gradient estimate (11.0.3) at the boundary of Ω by a comparison argument with a suitable one-dimensional parabolic problem. This and the maximum principle in Proposition 11.1.3 then allow us to prove the uniform estimate (11.0.3).

Finally, in Subsections 11.3.2 and 11.3.3, adapting to our situation the approach introduced by P. Cannarsa and V. Vespri in [26, 27], we prove (11.0.3) for any classical solution to the problem (11.0.1). This is done in the following way. The operator $\mathcal{A}$ is approximated by a family of uniformly elliptic operators $\mathcal{A}_\varepsilon$ such that, for any $p\ge 2$, their realizations in $L^p(\Omega)$ generate analytic semigroups $\{S_\varepsilon(t)\}$. Then, using the Sobolev embedding theorems, it is shown that, for any $f\in C_c^\infty(\Omega)$, the function $u_\varepsilon=S_\varepsilon(\cdot)f$ is the classical solution to the problem (11.0.1), with $\mathcal{A}$ being replaced with the operator $\mathcal{A}_\varepsilon$. Then, letting ε go to 0^+ and using a compactness argument based on classical Schauder estimates, it follows that u_ε converges to $T(t)f$, which satisfies all the additional assumptions needed, in Proposition 11.3.7, to prove the gradient estimate (11.0.3). Hence, (11.0.3) follows in this particular case. Finally, in the general case when $f\in C_b(\Omega)$, the estimate (11.0.3) is obtained by approximating f with a suitable sequence of $C_c^\infty(\Omega)$-smooth functions converging, locally uniformly and in a dominated way, to f.

Unfortunately, to the best of our knowledge, estimates for the higher-order derivatives of the function $T(t)f$ when $f\in C_b(\Omega)$ seem not to be available for a general domain Ω. As it is readily seen, differently from what happens when $\Omega=\mathbb{R}^N$ (see Chapters 6 and 7), the Bernstein method cannot be easily extended to estimate second- and third-order derivatives of the function $T(t)f$ and this is essentially due to the boundary conditions.

This fact prevents us to make the machinery of Chapter 6 work to study the nonhomogeneous elliptic problem and the nonhomogeneous Cauchy-Dirichlet problem in Ω and to prove optimal Schauder estimates. To the best of our knowledge there are only a few results in this direction (see Remark 11.3.16)

We conclude this introduction with the following remark.

Remark 11.0.1 Differently from what happens in the whole of $\mathbb{R}^N$, there never exists an invariant measure μ associated with the semigroup $\{T(t)\}$ considered in this chapter. Indeed, if this were the case, then

$$\int_\Omega T(t)d\mu = \int_\Omega f d\mu, \tag{11.0.4}$$

for any $f \in C_b(\Omega)$. But, according to Corollary 11.2.2, it holds that $0 \le T(t)\mathbb{1} \le \mathbb{1}$ for any $t > 0$. Hence, from (11.0.4), it should follow that $T(t)\mathbb{1} \equiv \mathbb{1}$ which is a contradiction.

11.1 Two maximum principles

Here, we prove two maximum principles for the classical solutions to the problem (11.0.1). Throughout this section we assume that the following conditions are satisfied:

Hypotheses 11.1.1 (i) Ω is an unbounded connected open set with uniformly $C^{2+\alpha}$-boundary for some $\alpha \in (0,1)$;

(ii) $q_{ij} \equiv q_{ji} \in C^1_b(\Omega)$ for any $i,j = 1,\ldots,N$ and there exists $\kappa_0 > 0$ such that

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \ge \kappa_0|\xi|^2, \qquad x \in \Omega,\ \xi \in \mathbb{R}^N; \tag{11.1.1}$$

(iii) $c(x) \le 0$ for any $x \in \Omega$;

(iv) $q_{ij}, b_i, c \in C^{1+\alpha}(\Omega \cap B(R))$ for any $i,j = 1,\ldots,N$ and any $R > 0$;

(v) there exist a positive function $\varphi \in C^2(\overline{\Omega})$ and a positive constant λ_0 such that

$$\lim_{\substack{|x|\to+\infty \\ x\in\overline{\Omega}}} \varphi(x) = +\infty$$

and

$$\sup_{x\in\Omega}(\mathcal{A}\varphi(x) - \lambda_0\varphi(x)) < +\infty.$$

The following proposition provides us a maximum principle for classical solutions to the Cauchy-Dirichlet problem (11.0.1), which are continuous in $[0,T] \times \overline{\Omega}$.

Proposition 11.1.2 *Assume that Hypotheses* 11.1.1 *are satisfied. Fix* $T>0$ *and let* $u\in C([0,T]\times\overline{\Omega})\cap C^{1,2}((0,T]\times\Omega)$ *be a bounded function satisfying*

$$\begin{cases} D_tu(t,x)-\mathcal{A}u(t,x)\le 0, & t\in(0,T],\ x\in\Omega,\\ u(t,x)\le 0, & t\in(0,T],\ x\in\partial\Omega,\\ u(0,x)\le 0, & x\in\Omega. \end{cases} \tag{11.1.2}$$

Then, $u\le 0$. *In particular, if* $u\in C([0,T]\times\overline{\Omega})\cap C^{1,2}((0,T]\times\Omega)$ *is a bounded solution to the Cauchy problem*

$$\begin{cases} D_tu(t,x)-\mathcal{A}u(t,x)=0, & 0<t\le T,\ x\in\Omega,\\ u(t,x)=0, & 0<t\le T,\ x\in\partial\Omega,\\ u(0,x)=0, & x\in\Omega, \end{cases}$$

then, $u=0$.

Proof. The proof is close to that of Theorem 4.1.3. Of course we can limit ourselves to proving the first part since the second part can then be deduced applying the first one to the functions u and $-u$.

We set $v=e^{-\lambda_0 t}u$ and we prove that $v\le 0$ in $[0,T]\times\overline{\Omega}$. For this purpose, for any $n\in\mathbb{N}$, we introduce the function $v_n:[0,T]\times\overline{\Omega}\to\mathbb{R}$ defined by

$$v_n(t,x)=v(t,x)-\frac{\varphi(x)}{n},\qquad t\in[0,T],\quad x\in\overline{\Omega}.$$

Replacing φ with $\varphi+C$ for some suitable constant $C>0$, we can assume that $\mathcal{A}\varphi-\lambda_0\varphi\le 0$. Now, a straightforward computation shows that v_n satisfies (11.1.2) with the operator $\mathcal{A}$ being replaced with $\mathcal{A}-\lambda_0$. Due to our assumption on φ, the function v_n attains its maximum at some point (t_n,x_n). Repeating the same arguments as in the proof of Theorem 4.1.3, we can see that $v_n\le 0$ in $(0,T)\times\Omega$. Thus, letting n go to $+\infty$ gives $v\le 0$ in $[0,T]\times\overline{\Omega}$. ∎

The next proposition provides us with a maximum principle for classical solutions to the problem (11.0.1) which are not continuous in $\{0\}\times\partial\Omega$.

We introduce a few notation which will be used in the proof of the next proposition and, more generally, throughout this chapter. By $d:\overline{\Omega}\to\mathbb{R}$ and Ω_δ ($\delta>0$) we denote, respectively, the distance function

$$d(x)=\mathrm{dist}(x,\partial\Omega),\qquad x\in\overline{\Omega}$$

and the set

$$\Omega_\delta=\{x\in\overline{\Omega}:d(x)<\delta\}. \tag{11.1.3}$$

In view of the assumptions on Ω, the function d is C^2-smooth, with bounded second-order derivatives in Ω_δ, if $\delta>0$ is sufficiently small. See [66, Lemma 14.16] and also Appendix D.

Proposition 11.1.3 *Suppose that Hypotheses* 11.1.1 *are satisfied and let* $u \in C^{1,2}((0,T)\times\Omega)\cap C([0,T)\times\overline{\Omega}\setminus(\{0\}\times\partial\Omega))$ *be bounded in* $(0,T)\times\Omega$. *If* $D_t u \le \mathcal{A}u$ *in* $(0,T)\times\Omega$ *and* $u\le 0$ *in* $(0,T)\times\partial\Omega\cup(\{0\}\times\Omega)$, *then* $u\le 0$ *in* $(0,T)\times\Omega$.

Finally, if $D_t u \le \mathcal{A}u$ *and* u *is bounded in* $(0,T)\times\partial\Omega\cup(\{0\}\times\Omega)$ $(T>0)$, *then*

$$\sup_{(t,x)\in(0,T)\times\Omega} u(t,x) \le \|u(0,\cdot)\|_\infty + \sup_{(t,\xi)\in(0,T)\times\partial\Omega}|u(t,\xi)|.$$

Proof. We limit ourselves to proving the first part of the proposition, since the last assertion then easily follows applying this part to the functions $u-K$, where $K=\sup_{(t,\xi)\in(0,T)\times\partial\Omega}|u(t,\xi)|+\|u(0,\cdot)\|_\infty$.

To begin with, we observe that there exists a function $g\in C^2(\overline{\Omega})$ such that

$$g\ge 0 \;\text{ in }\; \Omega, \qquad |Dg|=1 \;\text{ on }\; \partial\Omega, \qquad \partial\Omega=\{x\in\overline{\Omega}: g(x)=0\}.$$

Indeed, let η be a smooth function such that $0\le\eta\le\mathbb{1}$ in Ω, $\eta\equiv\mathbb{1}$ in $\Omega_{\delta/2}$ and $\eta\equiv 0$ outside $\Omega_{3\delta/4}$. It is easy to check that the function $g=\eta d+\mathbb{1}-\eta$ satisfies the claim.

Now, we introduce the function $v=e^{-\lambda_0 t}u$, where λ_0 is given by Hypothesis 11.1.1(v), and we prove that $v\le 0$ in $(0,T)\times\Omega$. For this purpose, we fix $R>1$ and consider the set

$$\Omega_R=\Omega\cap B(R)=\{g>0\}\cap\{R^2-|x|^2>0\}.$$

Moreover, we define the function $v_R:(0,T)\times\overline{\Omega}_R\to\mathbb{R}$ by setting

$$v_R(t,x)=v(t,x)-\frac{\|v\|_\infty}{C_R}\varphi(x), \qquad t\in(0,T),\;\; x\in\overline{\Omega}_R,$$

where $C_R=\inf_{[0,T]\times(\partial B(R)\cap\Omega)}\varphi$. As in the proof of Proposition 11.1.2,we can assume that $\mathcal{A}\pi-\lambda_0\varphi\le 0$. By Hypothesis 11.1.1(v), C_R tends to $+\infty$ as R tends to $+\infty$. Further, it is easy to see that v_R satisfies

$$\begin{cases} D_t v_R-\mathcal{A}v_R+\lambda_0 v_R\le 0, & \text{in } (0,T)\times\Omega_R,\\ v_R\le 0, & \text{in } (0,T)\times\partial\Omega_R,\\ v_R\le 0, & \text{in } \{0\}\times\Omega_R.\end{cases}$$

We claim that $v_R\le 0$ in $(0,T)\times\Omega_R$. Once the claim is proved, letting R tend to $+\infty$, we will obtain that $v\le 0$ in $(0,T)\times\Omega$ and we will be done.

To prove that $v_R\le 0$ in $(0,T)\times\overline{\Omega}_R$, for any $\varepsilon,\lambda>0$ we introduce the function $\psi:(0,T)\times\Omega_R\to\mathbb{R}$ defined by

$$\psi(t,x)=\frac{1}{t^{\varepsilon\kappa_0}}\left[\exp\left(\lambda t-\frac{\varepsilon(g(x))^2}{t}\right)+\exp\left(\lambda t-\frac{\varepsilon(h(x))^2}{t}\right)\right],$$

for any $t > 0$ and any $x \in \Omega_R$, where κ_0 is given by (11.1.1) and $h(x) = R^2 - |x|^2$ for any $x \in \mathbb{R}^N$. A straightforward computation shows that

$$\begin{aligned}
D_t\psi(t,\cdot) - \mathcal{A}\psi(t,\cdot) = &\ \frac{1}{t^2}\Big\{(\lambda - c)t^2 - \varepsilon\left[\kappa_0 - 2\langle QDg, Dg\rangle - 2g(\mathcal{A}-c)g\right]t \\
&\qquad + \varepsilon g^2\left[1 - 4\varepsilon\langle QDg, Dg\rangle\right]\Big\}\psi(t,\cdot) \\
&+\frac{1}{t^2}\Big\{(\lambda - c)t^2 - \varepsilon\left[\kappa_0 - 2\langle QDh, Dh\rangle - 2h(\mathcal{A}-c)h\right]t \\
&\qquad + \varepsilon h^2\left[1 - 4\varepsilon\langle QDh, Dh\rangle\right]\Big\}\psi(t,\cdot) \\
:= &\ \frac{1}{t^2}\left(\Lambda_1(t,\cdot) + \Lambda_2(t,\cdot)\right),
\end{aligned} \tag{11.1.4}$$

for any $t \in (0,+\infty)$. We now claim that we can fix $\varepsilon > 0$ sufficiently small and λ sufficiently large such that $(D_t - \mathcal{A})\psi \geq 0$ in $(0,+\infty)\times\Omega_R$. To prove the claim, we begin by considering the first term in the last side of (11.1.4). Since $|Dg| \geq 1$ on $\partial\Omega$, we can fix $\delta > 0$ such that $|Dg(x)| \geq 3/4$ if $x \in \Omega_R$ and $g(x) \leq \delta$. Then, we split $\Omega_R = \{x \in \Omega_R : g(x) > \delta\} \cup \{x \in \Omega_R : g(x) \leq \delta\} := \Omega_{R,1} \cup \Omega_{R,2}$. If $x \in \Omega_{R,1}$, we have

$$\Lambda_1(t,x) \geq \lambda t^2 - \varepsilon\left(\kappa_0 + 2\sup_{\Omega_R}|g(\mathcal{A}-c)g|\right)t + \varepsilon\delta^2\left(1 - 4\varepsilon\|Q\|_\infty\sup_{\Omega_R}|Dg|^2\right). \tag{11.1.5}$$

It is now clear that, if we fix $\varepsilon = (8\|Q\|_\infty\sup_{\Omega_R}|Dg|^2)^{-1}$ and, then, $\lambda = \lambda(\varepsilon,\delta)$ large enough, we can make the right-hand side of (11.1.5) nonnegative.

Now, let $x \in \Omega_{R,2}$. Then,

$$\begin{aligned}
&\kappa_0 - 2\langle Q(x)Dg(x), Dg(x)\rangle - 2g(x)\left((\mathcal{A}-c)g\right)(x) \\
\leq &\ \kappa_0 - 2\kappa_0|Dg(x)|^2 + 2g(x)\sup_{\Omega_R}|(\mathcal{A}-c)g| \\
\leq &\ -\frac{1}{2}\kappa_0 + \frac{1}{2t}(g(x))^2 + \frac{t}{2}\sup_{\Omega_R}|(\mathcal{A}-c)g|^2.
\end{aligned}$$

Therefore,

$$\Lambda_1(t,x) \geq \left(\lambda - \frac{1}{2}\sup_{\Omega_R}|(\mathcal{A}-c)g|^2\right)t^2 + \frac{1}{2}\varepsilon\kappa_0 t + \varepsilon g^2\left(\frac{1}{2} - 4\varepsilon\|Q\|_\infty\sup_{\Omega_R}|Dg|^2\right). \tag{11.1.6}$$

Again, we can fix $\varepsilon = (16\|Q\|_\infty\sup_{\Omega_R}|Dg|^2)^{-1}$ and, then, $\lambda = \lambda(\varepsilon)$ large enough such that the right-hand side of (11.1.6) is nonnegative. Similarly, we can argue with the function Λ_2. Indeed, the keystone to prove that Λ_1 is nonnegative in $(0,+\infty)\times\Omega_R$ was the fact that $|Dg| \geq 1$ when $g = 0$. Now, if we consider the function h it is easy to check that, since $R > 1$, then $|Dh| \geq 1$

when $h = 0$. Summing up, we have shown that we can fix the parameters λ and ε such that $D_t\psi - \mathcal{A}\psi \geq 0$ in $(0, +\infty) \times \Omega_R$.

Let us now suppose by contradiction that $0 < M = \sup_{(0,T)\times\Omega_R} v_R$ for some $T > 0$. For any $a > 0$ we introduce the function $v_{a,R}$ defined by

$$v_{a,R}(t, x) = v_R(t, x) - Ma^{\varepsilon\kappa_0}\psi(t + a, x),$$

for any $(t, x) \in [0, T] \times \overline{\Omega}_R \setminus (\{0\} \times \Omega \cap \partial B(R))$. Clearly $(D_t - \mathcal{A} + \lambda_0)v_{a,R} \leq 0$ in $(0, T) \times \Omega_R$. Take $\eta > 0$ such that $\lambda a - \varepsilon\eta^2 a^{-1} > 0$ and introduce the set

$$I_{\eta,R} = \{x \in \overline{\Omega}_R : \min\{g(x), h(x)\} \leq \eta\}.$$

For any $x \in I_{\eta,R}$ we have

$$a^{\varepsilon\kappa_0}\psi(a, x) \geq 2\exp\left(\lambda a - \frac{\varepsilon\eta^2}{a}\right) > 1.$$

By continuity, there exists $\delta > 0$ such that, for any $t \in [0, \delta]$ and any $x \in I_{\eta,R}$,

$$a^{\varepsilon\kappa_0}\psi(t + a, x) > 1.$$

It follows that $v_{a,R} < M - M = 0$ in $[0, \delta] \times I_{\eta,R}$.

Since $v_R(0, \cdot) \leq 0$ in $\Omega_R \setminus I_{\eta,R}$, we have $v_{a,R}(0, \cdot) < 0$ in $\Omega_R \setminus I_{\eta,R}$. The boundedness of Ω_R and the continuity of u in $[0, T) \times \Omega_R \setminus I_{\eta,R}$ imply that $v_{a,R} \leq 0$ in $[0, \delta] \times (\Omega_R \setminus I_{\eta,R})$ for some $\delta > 0$. Hence, $v_{a,R}$ is nonpositive in $[0, \delta] \times \overline{\Omega}_R \setminus (\{0\} \times \partial\Omega \cap B(R))$.

Applying the classical maximum principle in $[\delta, T) \times \overline{\Omega}_R$, we get that $v_{a,R} \leq 0$ in $[\delta, T) \times \overline{\Omega}_R$ and, consequently, we obtain that $v_{a,R}$ is nonpositive in $[0, T) \times \overline{\Omega}_R \setminus (\{0\} \times \partial\Omega_R)$.

Finally, letting a go to 0^+, we find that $v_R \leq 0$ in $[0, T) \times \overline{\Omega}_R \setminus (\{0\} \times \partial\Omega_R)$, and we are done, letting R tend to $+\infty$. ∎

Remark 11.1.4 Observe that the above theorem covers also the case of certain nonsmooth domains, whose boundaries can be described by a finite number of functions g_i as in the statement. For further details, we refer the reader to [59, Theorem A.2].

11.2 Existence and uniqueness of the classical solution

In this section we prove that, under Hypotheses 11.1.1, the Cauchy-Dirichlet problem (11.0.1) has a unique bounded classical solution u (i.e., a function $u \in C^{1,2}((0, +\infty) \times \Omega)$ which is continuous in $[0, +\infty) \times \overline{\Omega} \setminus (\{0\} \times \partial\Omega)$, bounded in $(0, +\infty) \times \Omega$ and solves the Cauchy problem (11.0.1)).

Theorem 11.2.1 *Suppose that Hypotheses* 11.1.1 *are satisfied. Then, for any* $f \in C_b(\Omega)$*, the Cauchy-Dirichlet problem* (11.0.1) *admits a unique bounded classical solution* u*. Moreover,*

$$\|u\|_\infty \le \|f\|_\infty$$

and $u \ge 0$ *if* $f \ge 0$*.*

Further, if $f \in C_c^{2+\alpha}(\Omega)$*, then* u *belongs to* $C^{1+\alpha/2,2+\alpha}((0,T)\times(\Omega\cap B(R)))$ *for any* $R, T > 0$*. Moreover,* Du *belongs to* $C^{1+\alpha/2,2+\alpha}((\varepsilon,T)\times\Omega')$ *for any* $0 < \varepsilon < T$ *and any bounded open set* $\Omega' \subset\subset \Omega$*. In particular,* $Du \in C^{1,2}((0,+\infty)\times\Omega)$*.*

Proof. Uniqueness follows from Proposition 11.1.3. So, let us prove that the Cauchy-Dirichlet problem (11.0.1) actually admits a solution with the claimed regularity properties. We split the proof into two steps. First in Step 1, we consider the case when $f \in C_c^{2+\alpha}(\Omega)$ and then, in Step 2, we deal with the general case.

Step 1. Let $f \in C_c^{2+\alpha}(\Omega)$. To prove that the problem (11.0.1) admits a classical solution, we use an approximation argument different from that used in the proof of Theorem 2.2.1. Instead of approximating the domain, we rather approximate the coefficients of the operator $\mathcal{A}$. For this purpose, we consider the sequence of uniformly elliptic operators $\mathcal{A}^{(n)}$ with bounded coefficients in $C^\alpha(\Omega)$ defined by

$$\mathcal{A}^{(n)} = \sum_{i,j=1}^N q_{ij}D_{ij} + \sum_{i=1}^N b_i^{(n)}D_i + c^{(n)}u, \qquad n \in \mathbb{N},$$

where the coefficients $b_i^{(n)}$ $(i = 1, \ldots, N)$ and $c^{(n)}$ coincide, respectively, with b_i and c in $\Omega \cap B(n)$, and $c^{(n)} \le 0$ in Ω. Let $u_n \in C^{1+\alpha/2,2+\alpha}((0,T)\times\Omega) \cap C([0,T]\times\overline{\Omega})$, for any $T > 0$, be the classical solution of (11.0.1), with $\mathcal{A}^{(n)}$ instead of $\mathcal{A}$ (see Proposition C.3.2). The maximum principle in Proposition 11.1.2 yields $\|u_n\|_\infty \le \|f\|_\infty$ for any $n \in \mathbb{N}$. Let us fix $R > 0$ and observe that, since Ω is unbounded and connected, $\mathrm{dist}(\Omega \setminus B(R+1), \Omega \cap B(R)) > 0$. Since $\mathcal{A}^{(n)} = \mathcal{A}^{(m)} = \mathcal{A}$ in $\Omega \cap B(R+1)$ for $n, m \ge R+1$, from the local Schauder estimates in Theorem C.1.5 (see (C.1.19)), for any $T > 0$, there exists a positive constant C, independent of n and m, such that

$$\begin{aligned}\|u_n - u_m\|_{C^{1+\alpha/2,2+\alpha}((0,T)\times(\Omega\cap B(R)))} &\le C\|u_n - u_m\|_{C((0,T)\times(\Omega\cap B(R+1)))}\\ &\le 2C\|f\|_\infty.\end{aligned}$$

Now, using a compactness argument as in the proof of Theorem 2.2.1, we can determine a subsequence $\{u_{n_k}\}$ converging in $C^{1,2}(F)$, for any compact set $F \subset [0,T]\times\overline{\Omega}$, to a function $u \in C^{1+\alpha/2,2+\alpha}_{\rm loc}([0,T]\times\overline{\Omega})$, which solves the Cauchy-Dirichlet problem (11.0.1) and satisfies $\|u\|_\infty \le \|f\|_\infty$. Moreover, by the maximum principle in Proposition 11.1.2, we deduce that u is nonnegative, whenever f is. Note that, applying the same arguments as above to any subsequence of $\{u_n\}$, one can actually show that the whole sequence $\{u_n\}$ converges to u in $C^{1,2}(F)$, for any compact set $F \subset (0,T) \times \overline{\Omega}$.

The further regularity properties of the function u follow from Theorem C.1.4(ii).

Step 2. Now, we consider the general case when $f \in C_b(\Omega)$. Let $\{f_n\} \in C_c^{2+\alpha}(\Omega)$ be a sequence of smooth functions converging to f uniformly on compact subsets of Ω and such that $\|f_n\|_\infty \le \|f\|_\infty$ for any $n \in \mathbb{N}$. Moreover, let $u_n \in C^{1+\alpha/2,2+\alpha}((0,T) \times (\Omega \cap B(R)))$, for any $R, T > 0$, be the solution of (11.0.1) given by Step 1, with f_n instead of f. Let us fix $\varepsilon > 0$. According to Theorem C.1.5, for any $\varepsilon \in (0,T)$ we can determine a positive constant C, independent of n and m, such that

$$\|u_n - u_m\|_{C^{1+\alpha/2,2+\alpha}((\varepsilon,T)\times(\Omega\cap B(R)))} \le 2C\|f\|_\infty, \qquad n, m \in \mathbb{N}.$$

Now, as in Step 1, we can find out a subsequence $\{u_{n_k}\}$ converging to a function $u \in C^{1+\alpha/2,2+\alpha}((\varepsilon,T) \times (\Omega \cap B(R)))$ for any $\varepsilon, R > 0$, $\varepsilon < T$, which solves the equation $D_t u - \mathcal{A}u = 0$ in $(0,T) \times \Omega$ and vanishes in $(0,T) \times \partial\Omega$.

To conclude the proof, it remains to show that $u(t,\cdot)$ converges to f as t tends to 0^+, uniformly on compact sets of Ω.

For notational convenience, in the rest of the proof we denote by u_g the classical solution to (11.0.1) corresponding to $g \in C_c^{2+\alpha}(\Omega)$.

We fix a compact set $K \subset \Omega$ and a function $\vartheta \in C_c^{2+\alpha}(\Omega)$ such that $0 \le \vartheta \le \mathbb{1}$ in Ω and $\vartheta \equiv \mathbb{1}$ in K. Moreover, we split

$$u_{n_k} = u_{\vartheta f_{n_k}} + u_{(1-\vartheta)f_{n_k}}, \qquad k \in \mathbb{N}, \tag{11.2.1}$$

and we claim that there exists a positive constant C, independent of k such that

$$|u_{(1-\vartheta)f_{n_k}}(t,\cdot)| \le C\big(\mathbb{1} - u_\vartheta(t,\cdot)\big), \qquad t > 0. \tag{11.2.2}$$

To prove (11.2.2), it suffices to observe that, for any $m \in \mathbb{N}$ and any $t > 0$,

$$|T_m(t)((\mathbb{1} - \vartheta)f_{n_k})| \le \|f_{n_k}\|_\infty \left(\mathbb{1} - T_m(t)\vartheta\right) \le C\left(\mathbb{1} - T_m(t)\vartheta\right),$$

where $\{T_m\}$ is the semigroup associated with the operator $\mathcal{A}^{(m)}$, and let m go to $+\infty$, using the fact that $(1-\vartheta)f_{n_k}$ and ϑ are compactly supported in $\overline{\Omega}$. Now, subtracting ϑf from both the sides of (11.2.1) and letting k go to $+\infty$ give

$$|u(t,\cdot) - \vartheta f| \le |u_{\vartheta f} - \vartheta f| + C(1 - u_\vartheta),$$

for any $t > 0$. Since $\vartheta f, \vartheta \in C_c(\overline{\Omega})$, then $u_{\vartheta f}$ and u_ϑ tends, respectively, to ϑf and ϑ as t tends to 0, uniformly in $\overline{\Omega}$. Since $\vartheta \equiv 1$ on K, it follows that $u(t,\cdot)$ tends to f uniformly on K as t tends to 0^+. This shows that actually u is the classical solution to (11.0.1). The proof is now complete. ∎

Corollary 11.2.2 *The family $\{T(t)\}$ defined by $T(t)f = u(t, \cdot)$ for any $t > 0$ and any $f \in C_b(\Omega)$, where u is the classical solution to the problem* (11.0.1)*, is a semigroup of positive contractions in $C_b(\Omega)$.*

Proof. The semigroup law (i.e., the equality $T(t+s) = T(t)T(s)$ for any $s, t > 0$) and the positivity of the semigroup (i.e., $T(t)f \geq 0$ whenever $f \geq 0$) follow immediately from the maximum principle in Proposition 11.1.3 and Theorem 11.2.1. ∎

11.3 Gradient estimates

This section is devoted to prove that, under Hypotheses 11.1.1 and suitable additional growth assumptions on the coefficients of the operator $\mathcal{A}$, for any $\omega > 0$ there exists a positive constant $C = C(\omega)$ such that

$$\|DT(t)f\|_\infty \leq C\frac{e^{\omega t}}{\sqrt{t}}, \qquad t > 0, \tag{11.3.1}$$

for any $f \in C_b(\Omega)$. Of course, it suffices to prove (11.3.1) in some time-domain $(0, T)$, and then to use the semigroup rule to extend it to all the positive times (see the proof of Theorem 6.1.7). As mentioned at the very beginning of this chapter the arguments that we need to prove (11.3.1) differ from those used in Chapter 6 in the case when $\Omega = \mathbb{R}^N$.

Throughout this section we assume the following additional assumptions on the coefficients of the operator $\mathcal{A}$. We recall that Ω_δ is defined in (11.1.3).

Hypothesis 11.3.1 There exist some constants $k, M, \beta, d_1, d_2 \in \mathbb{R}$, $s < 1/2$ and $\delta > 0$ such that

$$\sum_{i,j=1}^N D_i b_j(x)\xi_i\xi_j \leq (-s\,c(x) + k)|\xi|^2, \qquad x \in \Omega,\ \xi \in \mathbb{R}^N, \tag{11.3.2}$$

$$\sum_{j=1}^N b_j(x) D_j d(x) \leq M, \qquad x \in \Omega_\delta, \tag{11.3.3}$$

$$|Dc(x)| \leq \beta(1 - c(x)), \qquad x \in \Omega, \tag{11.3.4}$$

$$|b(x)| \leq d_1 e^{d_2|x|}, \qquad x \in \Omega. \tag{11.3.5}$$

Remark 11.3.2 Observe that if $s = 0$ in (11.3.2) and Ω is star-shaped with respect to 0, then Hypothesis 11.1.1(v) is a straightforward consequence of the positivity of $-c$ and the boundedness of q_{ij} $(i, j = 1, \ldots, N)$. Indeed, since

$$\begin{aligned}\langle b(x), x\rangle &= \langle b(0), x\rangle + \int_0^1 \frac{d}{dt}\langle b(tx), x\rangle dt\\ &= \langle b(0), x\rangle + \int_0^1 \sum_{i,j=1}^N D_i b_j(tx) x_i x_j dt,\end{aligned}$$

for any $x \in \Omega$, from (11.3.2) we get

$$\langle b(x), x\rangle \leq \langle b(0), x\rangle + k|x|^2, \qquad x \in \Omega.$$

Hence, if we take $\varphi(x) = 1 + |x|^2$ for any $x \in \Omega$, we get

$$\begin{aligned}\mathcal{A}\varphi(x) &= 2\mathrm{Tr}(Q(x)) + 2\langle b(x), x\rangle + c(x)(1 + |x|^2)\\ &\leq 2\mathrm{Tr}(Q(x)) + 2|b(0)||x| + k|x|^2 + c(x)(1 + |x|^2).\end{aligned}$$

Since $c \leq 0$ and the q_{ij}'s are bounded for any $i, j = 1. \ldots, N$, we can determine λ_0 sufficiently large such that $\mathcal{A}\varphi - \lambda_0\varphi \leq 0$ and Hypothesis 11.1.1(v) is satisfied taking as a Lyapunov function the function $\varphi : \mathbb{R}^N \to \mathbb{R}$ defined by $\varphi(x) = 1 + |x|^2$ for any $x \in \mathbb{R}^N$.

Remark 11.3.3 As we will see in Subsection 11.3.4, the gradient estimate (11.3.1) may fail if we do not assume Hypothesis 11.3.1.

Let us state the main result of this section. Its proof will be given in Subsection 11.3.3.

Theorem 11.3.4 *Suppose that the coefficients of the operator $\mathcal{A}$ satisfy Hypotheses* 11.1.1 *and* 11.3.1. *Then, the classical solution u to the Cauchy-Dirichlet problem* (11.0.1) *satisfies the gradient estimates* (11.3.1).

11.3.1 *A priori* gradient estimates

In this subsection we prove the gradient estimate (11.3.1) assuming some more regularity on the classical solution u to the problem (11.0.1). To begin with, let us prove the following lemma which plays a crucial role in obtaining the *a priori* gradient estimates.

Lemma 11.3.5 *Let a, b, δ be positive constants and let $g : [0, +\infty) \times [0, \delta] \to \mathbb{R}$ be the classical solution to the Cauchy-Dirichlet problem*

$$\begin{cases} g_t(t, r) = a g_{rr}(t, r) + b g_r(t, r), & t > 0, \quad r \in (0, \delta),\\ g(t, 0) = 0, \quad g(t, \delta) = 1, & t > 0,\\ g(0, r) = 1, & r \in (0, \delta), \end{cases} \tag{11.3.6}$$

i.e., $g \in C([0,T]\times[0,\delta]\setminus(\{0\}\times\{0,\delta\}))\cap C^{1,2}((0,T)\times[0,\delta])$*. Then* $g_r \ge 0$, $g_{rr}\le 0$ *in* $(0,+\infty)\times(0,\delta)$ *and there exists a positive constant* C *such that*

$$0 \le g(t,r) \le \frac{C}{\sqrt{t}} r, \qquad t>0,\ r\in(0,\delta). \tag{11.3.7}$$

Proof. The existence and the uniqueness of the classical solution g to the problem (11.3.6) follows from the results in Section 11.2, where we replace Ω with $(0,\delta)$.

To prove that g satisfies the claimed properties, we find out an explicit representation formula for g. For this purpose, let us consider the differential operator $(B,D(B))$ defined by

$$Bu = au'' + bu' \qquad D(B)=\{u\in C^2([0,\delta]):\ u(0)=u(\delta)=0\}.$$

According to Theorem C.3.6(iv), $(B,D(B))$ generates an analytic semigroup $\{R(t)\}$ of positive contractions in $C([0,\delta])$. Let us introduce the function $\psi : [0,\delta]\to\mathbb{R}$ defined by

$$\psi(r) = q\int_0^r e^{-b\,s/a}ds, \qquad r\in[0,\delta],$$

where $q = a(1-e^{-b\delta/a})/b$. As it is immediately seen, $B\psi=0$, $\psi(0)=0$ and $\psi(\delta)=1$. A straightforward computation shows that $g(t,\cdot)=R(t)(\mathbb{1}-\psi)+\psi$ for any $t>0$. Since the function $t\mapsto R(t)(\mathbb{1}-\psi)$ is analytic in $(0,+\infty)$ with values in $D(B^n)$ for any $n\in\mathbb{N}$ (see Theorem B.2.2(iv)), we immediately deduce that g is a C^∞-smooth function in $(0,+\infty)\times[0,\delta]$. To show that g_r is nonpositive in $(0,+\infty)\times(0,\delta)$ we begin by observing that, according to the maximum principle, we have

$$0\le g(t,\cdot)\le \mathbb{1}, \qquad t>0. \tag{11.3.8}$$

Hence $R(t)(\mathbb{1}-\psi)\le \mathbb{1}-\psi$ for any $t>0$. Moreover, from (11.3.8) we also deduce that

$$\begin{aligned} g(t+s,\cdot) &= R(t+s)(\mathbb{1}-\psi)+\psi \\ &= R(t)R(s)(\mathbb{1}-\psi)+\psi \\ &\le R(t)(\mathbb{1}-\psi)+\psi = g(t,\cdot), \end{aligned}$$

for any $s,t>0$. Therefore, the function $g(\cdot,r)$ is decreasing for any $r\in(0,\delta)$. This implies that $g_t(t,r)\le 0$ for any $t\in(0,+\infty)$ and any $r\in(0,\delta)$. From the differential equation solved by g, we now deduce that

$$ag_{rr}(t,r)+bg_r(t,r)\le 0, \qquad t>0,\ r\in(0,\delta),$$

or, equivalently, that

$$\frac{d}{dr}\left(e^{br/a}g_r\right)\le 0.$$

Hence, the function $r \mapsto e^{br/a} g_r(t,r)$ is decreasing in $(0,\delta)$ for any $t>0$. Since $g(t,\delta)=1$ and $0 \le g \le \mathbb{1}$, we deduce that $g_r(t,\delta) \ge 0$ and, consequently, $g_r(t,\cdot) \ge 0$ for any $t>0$.

Now, since

$$g_{rr} = \frac{1}{a}(g_t - bg_r),$$

by difference, we immediately deduce that $g_{rr} \le 0$ in $(0,+\infty)\times(0,\delta)$.

To conclude the proof, let us check (11.3.7). For this purpose, we observe that since $\{R(t)\}$ is an analytic semigroup, for any $T>0$ there exists a positive constant $\widetilde{C} = \widetilde{C}(T)$ such that

$$\|ag_{rr}(t,\cdot) + bg_r(t,\cdot)\|_\infty \le \frac{\widetilde{C}}{t}, \qquad t \in (0,T). \tag{11.3.9}$$

Now, since $C^1([0,\delta])$ belongs to the class $J_{1/2}$ between $C([0,\delta])$ and $C^2([0,\delta])$ (see Proposition A.4.4), then there exists a positive constant C such that

$$\begin{aligned}\|v\|_{C^1([0,\delta])} &\le C\|v\|_\infty^{\frac{1}{2}}\left(\|v\|_{C^1([0,\delta])}^{\frac{1}{2}} + \|v''\|_\infty^{\frac{1}{2}}\right)\\ &\le \frac{C}{2\varepsilon}\|v\|_\infty + C\varepsilon\|v\|_{C^1([0,\delta])} + C\varepsilon\|v''\|_\infty, \end{aligned} \tag{11.3.10}$$

for any $v \in C^2([0,\delta])$ and any $\varepsilon > 0$. Now, choosing $C\varepsilon < 1$ we deduce that

$$\|v\|_{C^1([0,\delta])} \le C(\varepsilon)\|v\|_\infty + \frac{C\varepsilon}{1-C\varepsilon}\|v''\|_\infty, \tag{11.3.11}$$

for some positive constant $C(\varepsilon)$, blowing up as ε tends to 0^+. Hence, from (11.3.9) and (11.3.11), we deduce that

$$\|g_{rr}(t,\cdot)\|_\infty \le C_1\left(\frac{1}{t} + \|g_r(t,\cdot)\|_\infty\right) \le \frac{C_2(\varepsilon)}{t} + \frac{C\varepsilon}{1-C\varepsilon}\|g_{rr}(t,\cdot)\|_\infty,$$

for any $t \in (0,T)$, any $T>0$ and some positive constants C_1 and C_2, independent of $t \in (0,T)$. Hence, up to replacing ε with a smaller constant, we finally obtain that, for any $T>0$, there exists a positive constant $\widetilde{C} = \widetilde{C}(T)$ such that

$$\|g_{rr}(t,\cdot)\|_\infty \le \frac{C_2}{t}, \qquad t \in (0,T).$$

Now, from the first part of (11.3.10) and (11.3.11), we get

$$\|g_r(t,\cdot)\|_\infty \le \overline{C}\|g(t,\cdot)\|_\infty^{\frac{1}{2}}\|g_{rr}(t,\cdot)\|_\infty^{\frac{1}{2}} \le \overline{C}\|g_{rr}(t,\cdot)\|_\infty^{\frac{1}{2}}, \qquad t>0,$$

where $\overline{C}$ is independent of $t>0$, and we immediately get (11.3.7). This finishes the proof. ∎

Taking advantage of Lemma 11.3.5, we can prove the following *a priori* gradient estimates on the boundary of Ω.

Proposition 11.3.6 *Assume that Hypotheses* 11.1.1 *and the estimate* (11.3.3) *are satisfied. Then, there exists a positive constant* C_0 *depending on* κ_0, M, δ, T *and the sup-norm of the diffusion coefficients* q_{ij} $(i, j = 1, \dots, N)$ *such that any bounded classical solution* u *of* (11.0.1)*, differentiable with respect to the space variables in* $(0, T) \times \overline{\Omega}$*, satisfies the estimate*

$$|Du(t, \xi)| \le \frac{C_0}{\sqrt{t}} \|f\|_\infty, \qquad t \in (0, T),\ \xi \in \partial\Omega. \tag{11.3.12}$$

Proof. For any $x \in \Omega_\delta$ (see (11.1.3)) let $\xi(x)$ be the unique point on $\partial\Omega$ satisfying $|x - \xi(x)| = d(x)$. Then, we can write

$$x = \xi(x) - d(x)\nu(\xi(x)),$$

where $\nu(\xi)$ is the outer unit normal to $\partial\Omega$ at ξ. Moreover, $Dd(x) = -\nu(\xi(x))$ for any $x \in \Omega_\delta$. To proceed, we observe that, since $u = 0$ on $\partial\Omega$,

$$|Du(t, \xi)| = \left| \frac{\partial u}{\partial \nu}(t, \xi) \right|, \qquad t > 0,\ \xi \in \partial\Omega.$$

Hence, in order to prove the claim it is enough to show that

$$(T(t)\mathbb{1})(x) \le \frac{C_0}{\sqrt{t}} d(x), \qquad t \in (0, T),\ \ x \in \Omega_\delta, \tag{11.3.13}$$

for some C_0 as in the statement of the proposition, where $\{T(t)\}$ is the semigroup defined in Corollary 11.2.2. Indeed, once (11.3.13) is proved, using the fact that the semigroup is order preserving, for any $f \in C_b(\Omega)$, any $\xi \in \partial\Omega$ and $x = \xi + d(x)\nu(\xi)$, we can write

$$|(T(t)f)(x) - (T(t)f)(\xi)| = |(T(t)f)(x)| \le \|f\|_\infty (T(t)\mathbb{1})(x) \le \frac{C_0}{\sqrt{t}} d(x) \|f\|_\infty, \tag{11.3.14}$$

and (11.3.12) easily follows dividing both the first and the last side of (11.3.14) by d and letting x go to ξ.

To prove (11.3.13) we use a comparison argument. For this purpose, let $z : (0, T) \times \Omega_\delta \to \mathbb{R}$ be the function defined by

$$z(t, x) = g(t, d(x)), \qquad t > 0,\ x \in \Omega_\delta,$$

$g : [0, +\infty) \times [0, \delta] \to \mathbb{R}$ being the solution to the problem (11.3.6), where we take $a = \kappa_0$ and $b = M_0 := \sup_{x\in\Omega}(\mathcal{A} - c)d$ (which is finite, since the diffusion coefficients are bounded and the drift b satisfies (11.3.3)). By (11.3.7) we know that

$$|z(t, x)| = g(t, d(x)) \le \frac{C}{\sqrt{t}} d(x), \qquad t > 0,\ x \in \Omega_\delta,$$

for some constant C depending on the same quantities as in the statement of the proposition. Thus, to prove (11.3.13) we have only to show that

$$(T(t)\mathbb{1})(x) \le z(t,x), \qquad t>0,\ x\in\Omega_\delta. \tag{11.3.15}$$

Let us consider the function $v = z - T(\cdot)\mathbb{1}$ in $(0,+\infty)\times\Omega_\delta$. In view of the assumptions on Ω, the function d is in $C^2(\Omega_\delta)$ and, consequently, $v \in C^{1,2}((0,+\infty)\times\Omega_\delta)$, it is continuous in $[0,+\infty)\times\overline{\Omega}_\delta\setminus(\{0\}\times\partial\Omega_\delta)$, bounded in $(0,+\infty)\times\Omega_\delta$, and nonnegative in $(0,+\infty)\times\partial\Omega_\delta\cup(\{0\}\times\Omega_\delta)$. Moreover,

$$\begin{aligned} D_t v - \mathcal{A}v = {} & \left(\kappa_0 - \sum_{i,j=1}^N q_{ij}D_i d D_j d\right) g_{rr}(\cdot, d(\cdot)) \\ & + (M_0 - \mathcal{A}d + c\,d)g_r(\cdot,d(\cdot)) - c\,g(\cdot,d(\cdot)) \ge 0, \end{aligned}$$

since $g, g_r, -g_{rr}$ are nonnegative. The maximum principle in Proposition 11.1.3 now yields (11.3.15) and it concludes the proof. ∎

The following proposition provides us an *a priori* estimate of Du.

Proposition 11.3.7 *Let Hypotheses* 11.1.1 *and* 11.3.1*, but estimate* (11.3.5)*, be satisfied. Then, for any $T>0$, there exists a positive constant C, depending on κ_0, k, s, β, T and $\|DQ\|_\infty$, such that*

$$\|Du(t,\cdot)\|_\infty \le \frac{C}{\sqrt{t}}\|f\|_\infty, \qquad t\in(0,T), \tag{11.3.16}$$

for any bounded classical solution u of (11.0.1) *such that*

(*i*) *Du belongs to $C^{1,2}((0,+\infty)\times\Omega)$;*

(*ii*) *$\sqrt{t}\,|Du|$ is continuous in $[0,+\infty)\times\overline{\Omega}\setminus(\{0\}\times\partial\Omega)$, bounded in $(0,T)\times\Omega$ for any $T>0$ and it satisfies*

$$\lim_{t\to 0^+}\sqrt{t}|Du(t,x)| = 0, \qquad x\in\Omega,$$

as well as the gradient estimate (11.3.12).

Proof. Changing c to $c-1$ (hence u to $e^{-t}u$) we may assume that $|Dc| \le -\beta c$ (see (11.3.4)). To get (11.3.16) we apply the Bernstein method. For this purpose, we define the function

$$v(t,x) = |u(t,x)|^2 + a\,t\,|Du(t,x)|^2, \qquad t>0,\ x\in\Omega,$$

where the constant $a>0$ will be chosen later. Due to assumptions (i) and (ii), v belongs to $C^{1,2}((0,+\infty)\times\Omega)$, it is continuous in $[0,+\infty)\times\overline{\Omega}\setminus(\{0\}\times\partial\Omega)$, bounded in $(0,+\infty)\times\Omega$ and $v(0,\cdot) = f^2$. We claim that we can fix $a>0$, depending only on $\kappa_0, h, k, s, \beta, T$ and $\|DQ\|_\infty$ (which is finite, since $q_{ij} \in C^1_b(\Omega)$ for any $i,j = 1,\dots,N$), such that

$$D_t v(t,x) - \mathcal{A}v(t,x) \le 0, \qquad t \in (0,T),\ x \in \Omega. \tag{11.3.17}$$

Proposition 11.1.3 and estimate (11.3.12) will imply that

$$v(t,x) \le \|f\|_\infty + \sup_{(t,\xi)\in(0,T)\times\partial\Omega} at\,|Du(t,\xi)|^2 \le (1 + aC_0^2)\|f\|_\infty^2,$$

for any $t \in (0,T)$ and any $x \in \Omega$. Therefore, the estimate (11.3.16) will follow with $C = (C_0^2 + a^{-1})^{1/2}$.

To prove (11.3.17) we observe that v satisfies the equation

$$D_t v - \mathcal{A}v = a|Du|^2 - 2\sum_{i,j=1}^N q_{ij}\, D_i u\, D_j u + g_1 + g_2,$$

where

$$\begin{aligned} g_1 &= a\,t\left(2\sum_{i,j=1}^N D_i b_j\, D_i u\, D_j u + 2u\sum_{j=1}^N D_j c D_j u + c|Du|^2\right) + cu^2,\\ g_2 &= 2\,a\,t\left(\sum_{i,j,h=1}^N D_h q_{ij} D_h u D_{ij} u - \sum_{i,j,h=1}^N q_{ij} D_{ih} u D_{jh} u\right). \end{aligned}$$

Using Hypothesis 11.3.1 and the inequality $2ab \le \varepsilon a^2 + \varepsilon^{-1} b^2$, which holds for any $a, b, \varepsilon > 0$, and recalling that $c \le 0$, we can easily show that, for any $\varepsilon > 0$,

$$\begin{aligned} D_t v - \mathcal{A}v \le{}& c\left(1 - \frac{a\beta}{\varepsilon}t\right)u^2\\ &+\left\{a - 2\kappa_0 + at\left[2k - (2s-1)c + \frac{\|DQ\|_\infty}{\varepsilon} - \beta\varepsilon c\right]\right\}|Du|^2\\ &+(\varepsilon\|DQ\|_\infty - 2\kappa_0)at\,|D^2u|^2. \end{aligned} \tag{11.3.18}$$

Recalling that $2s < 1$ and choosing $\varepsilon < \min\{2\kappa_0\|DQ\|_\infty, (1-2s)\beta^{-1}\}$, from (11.3.18) we get

$$D_t v - \mathcal{A}v \le c\left(1 - \frac{a\beta}{\varepsilon}T\right)u^2 + \left\{a - 2\kappa_0 + aT\left[2k + \frac{\|DQ\|_\infty}{\varepsilon}\right]\right\}|Du|^2.$$

Now, taking a small enough, we get immediately (11.3.17). ■

11.3.2 An auxiliary problem

As a second step, in order to prove Theorem 11.3.4, in this subsection we introduce an auxiliary problem set in L^p-spaces. We still consider an elliptic operator $\mathcal{A}$ of the type (11.0.2), but we assume that its coefficients satisfy the following set of hypotheses.

Hypotheses 11.3.8 (i) the coefficients q_{ij}, b_j $(i,j=1,\ldots,N)$ and c satisfy Hypotheses 11.1.1(ii) and 11.1.1(iv);

(ii) there exist $0<\sigma<\sqrt{2\kappa_0}$ and $\beta>0$ such that

$$c(x)\le -1, \qquad |Dc(x)|\le \beta|c(x)|, \qquad |b(x)|\le \sigma|c(x)|^{\frac{1}{2}}, \qquad x\in\Omega.$$

We are going to show that, under the previous hypotheses, the operator $\mathcal{A}$ generates an analytic semigroup in $L^p(\Omega)$, for any $p\ge 2$. For this purpose we adapt the ideas of [26, 27, 119], where the case $\Omega=\mathbb{R}^N$ is considered.

We introduce the space D_p defined by

$$D_p=\{u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega): c\,u\in L^p(\Omega)\},$$

and we endow it with the norm

$$\|u\|_{D_p}=\|u\|_{W^{2,p}(\Omega)}+\|c\,u\|_{L^p(\Omega)}, \qquad u\in D_p. \tag{11.3.19}$$

As it is immediately seen, D_p is a Banach space when it is endowed with the norm in (11.3.19). Moreover, we consider the dense subspace D of D_p defined as follows:

$$D=\{u\in C^\infty(\overline{\Omega}): u_{|\partial\Omega}=0,\ \mathrm{supp}\,u\subset\subset\overline{\Omega}\}.$$

Finally, we denote by $\mathcal{A}_0$ the operator defined on smooth functions by

$$\mathcal{A}_0\varphi=\sum_{i,j=1}^N q_{ij}D_{ij}\varphi,$$

and by A_p the realization of the operator $\mathcal{A}$ in D, defined by $A_pu=\mathcal{A}u$ for any $u\in D$.

In what follows to shorten the notation we denote by $\|\cdot\|_p$ and $\|\cdot\|_{j,p}$ $(j=1,2)$ the usual norms of $L^p(\Omega)$ and $W^{j,p}(\Omega)$, respectively.

Lemma 11.3.9 *For any $p\ge 2$ there exists a positive constant C, depending on N,p,β,R and the coefficients q_{ij} $(i,j=1,\ldots,N)$, such that*

$$\||c|^{1/2}Du\|_p\le \varepsilon\|\mathcal{A}_0u\|_p+C\varepsilon^{-1}(\|u\|_p+\|cu\|_p), \tag{11.3.20}$$

for any $0<\varepsilon<R$ and any $u\in D$.

Proof. Fix $u \in D$. Integrating by parts and using the fact that $u = 0$ on $\partial\Omega$ and $p \geq 2$ we obtain

$$\begin{aligned}\int_\Omega |c|^{\frac{p}{2}} |Du|^p dx &= \int_\Omega |c|^{\frac{p}{2}} |Du|^{p-2}|Du|^2 dx \\ &= \frac{p}{2}\int_\Omega |c|^{\frac{p}{2}-1}\langle Dc, Du\rangle |Du|^{p-2} u dx \\ &\quad -(p-2)\int_\Omega |c|^{\frac{p}{2}} u|Du|^{p-4}\sum_{i,j=1}^N D_{ij}^2 u D_i u D_j u dx \\ &\quad -\int_\Omega |c|^{\frac{p}{2}} u|Du|^{p-2}\Delta u\, dx.\end{aligned}$$

Using the Hölder inequality, and observing that, by Hypothesis (11.3.8)(ii), $|Dc| \leq \beta|c| \leq \beta|c|^{\frac{3}{2}}$, we get

$$\begin{aligned}&\int_\Omega |c|^{\frac{p}{2}}|Du|^p dx \\ &\leq \frac{\beta p}{2}\int_\Omega |u||Du|^{p-1}|c|^{\frac{p-1}{2}}|c| dx \\ &\quad +(p-2+\sqrt{N})\int_\Omega |c|^{\frac{p}{2}-1}|Du|^{p-2}|c||u||D^2u| dx \\ &\leq \frac{\beta p}{2}\bigg(\int_\Omega |c|^{\frac{p}{2}}|Du|^p dx\bigg)^{1-\frac{1}{p}}\bigg(\int_\Omega |c|^p|u|^p dx\bigg)^{\frac{1}{p}} \\ &\quad +(p-2+\sqrt{N})\bigg(\int_\Omega |c|^{\frac{p}{2}}|Du|^p dx\bigg)^{1-\frac{2}{p}}\bigg(\int_\Omega |c|^p|u|^p dx\bigg)^{\frac{1}{p}}\bigg(\int_\Omega |D^2u|^p dx\bigg)^{\frac{1}{p}},\end{aligned}$$

or, equivalently,

$$\||c|^{\frac{1}{2}}Du\|_p^2 \leq \frac{\beta p}{2}\|c\,u\|_p\||c|^{\frac{1}{2}}Du\|_p + (p-2+\sqrt{N})\|c\,u\|_p\|D^2u\|_p.$$

It follows that

$$\begin{aligned}\||c|^{\frac{1}{2}}Du\|_p &\leq \frac{\beta p}{2}\|c\,u\|_p + \sqrt{p-2+\sqrt{N}}\,\|c\,u\|_p^{\frac{1}{2}}\|D^2u\|_p^{\frac{1}{2}} \\ &\leq C\varepsilon^{-1}\|c\,u\|_p + \varepsilon\|D^2u\|_p,\end{aligned}$$

for any $\varepsilon \in (0,1)$, with C depending on β, p, and the statement follows with $\|D^2u\|_p$ instead of $\|\mathcal{A}_0u\|_p$. To complete the proof it suffices to observe that

$$\|D^2u\|_p \leq C(\|u\|_p + \|\mathcal{A}_0u\|_p), \tag{11.3.21}$$

for some positive constant C, independent of u (see [66, Lemma 9.17]). ∎

In what follows to simplify the notation we set

$$q(\xi,\eta)=\sum_{i,j=1}^N q_{ij}\xi_i\eta_j,\qquad \xi,\eta\in\mathbb{R}^N. \tag{11.3.22}$$

Moreover, we introduce the function $b':\mathbb{R}^N\to\mathbb{R}$ defined by

$$b_i'(x)=b_i(x)-\sum_{j=1}^N D_jq_{ij}(x),\qquad x\in\overline{\Omega}.$$

From Hypotheses 11.3.8 we obtain that

$$|b'(x)|\le\sigma|c(x)|^{\frac{1}{2}}+K, \tag{11.3.23}$$

for a positive constant K depending only on the sup-norm of the gradient of the diffusion coefficients.

Proposition 11.3.10 *For any $p\ge 2$ there exist two constants λ_0 and λ_1 such that*

$$(\lambda-\lambda_0)\|u\|_p\le\|\lambda u-A_pu\|_p,\qquad \lambda>\lambda_0, \tag{11.3.24}$$

$$C_1\|u\|_{D_p}\le\|\lambda u-A_pu\|_p\le C_2\|u\|_{D_p},\qquad \lambda>\lambda_1, \tag{11.3.25}$$

for any $u\in D$ and some positive constants C_1,C_2 depending only on λ, N, p, β, σ, κ_0 and the sup-norm of the coefficients q_{ij} $(i,j=1,\ldots,N)$.

Proof. To shorten the notation, we denote by C any positive constant depending at most on λ,N,p,β,σ and the coefficients q_{ij} $(i,j=1,\ldots,N)$, which may vary from line to line.

To prove (11.3.24) we multiply the identity $f=\lambda u-A_pu$ by $u|u|^{p-2}$ and integrate by parts over Ω. We get

$$\begin{aligned}&\int_\Omega(\lambda-c)|u|^pdx+(p-1)\int_\Omega|u|^{p-2}q(Du,Du)dx\\ &=\int_\Omega u|u|^{p-2}fdx+\int_\Omega u|u|^{p-2}\langle b',Du\rangle dx.\end{aligned} \tag{11.3.26}$$

Using (11.3.23), the Cauchy-Schwarz and Young inequalities we deduce that

$$\begin{aligned}\int_\Omega u|u|^{p-2}\langle b',Du\rangle dx&\le\sigma\int_\Omega|c|^{\frac{1}{2}}|u|^{p-1}|Du|dx+K\int_\Omega|u|^{p-1}|Du|dx\\ &\le\sigma\Big(\int_\Omega|c||u|^pdx\Big)^{\frac{1}{2}}\Big(\int_\Omega|u|^{p-2}|Du|^2dx\Big)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&+K\left(\int_\Omega |u|^{p-2}|Du|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega |u|^p dx\right)^{\frac{1}{2}}\\
\leq{}& \frac{1}{2}\int_\Omega |c||u|^p dx+\frac{\sigma^2}{2}\int_\Omega |u|^{p-2}|Du|^2dx\\
&+\varepsilon\int_\Omega |u|^{p-2}|Du|^2dx+\frac{K^2}{4\varepsilon}\int_\Omega |u|^p dx. \qquad (11.3.27)
\end{aligned}
$$

Combining (11.3.26) and (11.3.27) gives

$$
\begin{aligned}
&\left(\lambda-\frac{K^2}{4\varepsilon}\right)\int_\Omega |u|^p dx+\frac{1}{2}\int_\Omega |c||u|^p dx\\
&+\left((p-1)\kappa_0-\frac{\sigma^2}{2}-\varepsilon\right)\int_\Omega |u|^{p-2}|Du|^2dx\\
\leq{}& \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}\left(\int_\Omega |u|^p dx\right)^{1-\frac{1}{p}}. \qquad (11.3.28)
\end{aligned}
$$

Since $0<\sigma<\sqrt{2\kappa_0}$, $p\geq 2$ and $c\leq -1$, we can fix ε sufficiently small and, then, take $\lambda>\lambda_0:=K^2/(4\varepsilon)$ to make all the terms in the left-hand side of (11.3.28) nonnegative. The estimate (11.3.24) now easily follows.

Let us now show the first inequality in (11.3.25). We begin by estimating $c\,u$. For this purpose, we multiply the identity $\lambda u-A_pu=f$ by $|c|^{p-1}u|u|^{p-2}$ and integrate by parts over Ω. Taking (11.1.1) into account, we obtain

$$
\begin{aligned}
&\int_\Omega (\lambda|c|^{p-1}+|c|^p)|u|^p dx+\kappa_0(p-1)\int_\Omega |c|^{p-1}|u|^{p-2}|Du|^2dx\\
\leq{}& \int_\Omega (\lambda|c|^{p-1}+|c|^p)|u|^p dx+(p-1)\int_\Omega |c|^{p-1}|u|^{p-2}q(Du,Du)dx\\
={}&(p-1)\int_\Omega |c|^{p-2}u|u|^{p-2}q(Du,Dc)dx+\int_\Omega |c|^{p-1}u|u|^{p-2}\langle b',Du\rangle dx\\
&+\int_\Omega |c|^{p-1}u|u|^{p-2}f dx. \qquad (11.3.29)
\end{aligned}
$$

Let us estimate the last side of (11.3.29). Arguing as in the proof of (11.3.27), we get

$$
\begin{aligned}
&\left|\int_\Omega |c|^{p-1}u|u|^{p-2}\langle b',Du\rangle dx\right|\\
\leq{}& \left(\frac{\sigma^2}{2}+\varepsilon\right)\int_\Omega |c|^{p-1}|u|^{p-2}|Du|^2dx+\frac{1}{2}\int_\Omega |c|^p|u|^p dx+\frac{K^2}{4\varepsilon}\int_\Omega |c|^{p-1}|u|^p dx,
\end{aligned}
$$
(11.3.30)

for any $\varepsilon > 0$. Similarly, using Hypothesis 11.3.8(ii), we get

$$\begin{aligned}
&\int_\Omega |c|^{p-2}|u|^{p-1}|q(Du, Dc)|dx \\
&\le \|Q\|_\infty \int_\Omega |c|^{p-2}|Dc||u|^{p-1}|Du|dx \\
&\le \|Q\|_\infty \beta \bigg(\int_\Omega |c|^{p-1}|u|^{p-2}|Du|^2 dx\bigg)^{\frac{1}{2}} \bigg(\int_\Omega |c|^{p-1}|u|^p dx\bigg)^{\frac{1}{2}} \\
&\le \varepsilon \int_\Omega |c|^{p-1}|u|^{p-2}|Du|^2 dx + \frac{\|Q\|_\infty^2 \beta^2}{4\varepsilon} \int_\Omega |c|^{p-1}|u|^p dx. \qquad (11.3.31)
\end{aligned}$$

Summing up, from (11.3.29)-(11.3.31) we get

$$\begin{aligned}
&\bigg(\lambda - \frac{K^2}{4\varepsilon} - (p-1)\frac{\|Q\|_\infty^2\beta^2}{4\varepsilon}\bigg) \int_\Omega |c|^{p-1}|u|^p dx + \frac{1}{2}\int_\Omega |c|^p|u|^p dx \\
&+ \bigg((p-1)(\kappa_0 - \varepsilon) - \frac{\sigma^2}{2} - \varepsilon\bigg) \int_\Omega |c|^{p-1}|u|^{p-2}|Du|^2 dx \\
&\le \bigg(\int_\Omega |f|^p dx\bigg)^{\frac{1}{p}} \bigg(\int_\Omega |c|^p|u|^p dx\bigg)^{1-\frac{1}{p}}.
\end{aligned}$$

Hence, choosing ε sufficiently small and, then,

$$\lambda > \lambda_1 := \frac{K^2 + (p-1)\|Q\|_\infty^2\beta^2}{4\varepsilon},$$

we get

$$\|c\, u\|_p \le 2\|f\|_p. \qquad (11.3.32)$$

We now use Lemma 11.3.9 with $\varepsilon = (2\sigma)^{-1}$ to estimate the second-order derivatives of u. We have

$$\begin{aligned}
\|\langle b, Du\rangle\|_p &\le \sigma \||c|^{\frac{1}{2}} Du\|_p \\
&\le \frac{1}{2}\|\mathcal{A}_0 u\|_p + 2\sigma^2 C\|u\|_p + 2\sigma^2 C\|cu\|_p \\
&\le \frac{1}{2}\|f\|_p + \frac{1}{2}\|\langle b, Du\rangle\|_p + \bigg(\frac{1}{2} + 2\sigma^2 C\bigg)\|cu\|_p + \bigg(\frac{\lambda}{2} + 2\sigma^2 C\bigg)\|u\|_p.
\end{aligned}$$

Hence, taking (11.3.24) and (11.3.32) into account, we get $\|\langle b, Du\rangle\|_p \le C\|f\|_p$ and, then, by difference, $\|\mathcal{A}_0 u\|_p \le C\|f\|_p$. From (11.3.21) we get $\|D^2 u\|_p \le C\|f\|_p$. Moreover, since $c \le -1$, from (11.3.20) we deduce that

$$\|Du\|_p \le C\|f\|_p.$$

Now, the first inequality in (11.3.25) follows.

To prove the other estimate in (11.3.25) it suffices to show that

$$\|\langle b, Du\rangle\|_p \leq C\|u\|_{D_p}, \tag{11.3.33}$$

for any $u \in D$, and this can easily be done, taking (11.3.20), with $\varepsilon = 1$, and Hypothesis (11.3.8)(ii) into account. ∎

Remark 11.3.11 The estimate (11.3.33) and the density of D in D_p show that the operator A_p is well defined in D_p. We still denote by A_p the operator so extended to D_p. Of course, such an operator satisfies all the results in Proposition 11.3.10. In particular, by (11.3.25), the graph norm of A_p is equivalent to the norm of D_p.

Proposition 11.3.12 *For any $p \in [2, +\infty)$, (A_p, D_p) generates a strongly continuous analytic semigroup $\{S(t)\}$ in $L^p(\Omega)$.*

Proof. Without loss of generality, we replace $\mathcal{A}$ with the operator $B_p = A_p - kI$, defined in $D(B_p) = D_p$, where $k \geq 1$ is a suitable positive constant to be fixed.

We split the proof into two steps. First we prove that B_p generates a strongly continuous semigroup in $L^p(\Omega)$ and then, in Step 2, we show that such a semigroup is analytic.

Step 1. To begin with, we fix $\sigma' \in (\sigma, \sqrt{2\kappa_0})$ and $k > 0$ sufficiently large such that

$$|b'(x)| \leq \sigma'|c'(x)|^{\frac{1}{2}}, \qquad x \in \Omega, \tag{11.3.34}$$

where $c' = c - k$. Since now we can take $K = 0$ in (11.3.23), the estimate (11.3.28) implies that B_p is dissipative in $L^p(\Omega)$.

Let us show $\lambda - B_p$ is surjective for some $\lambda > 0$. The Lumer-Phillips theorem (see Theorem B.1.7), then, will imply that B_p is the generator of a strongly continuous semigroup of contractions. For this purpose, for any $\varepsilon > 0$, we set

$$b_\varepsilon = \frac{b}{\sqrt{1-\varepsilon c}}, \qquad c'_\varepsilon = \frac{c}{1-\varepsilon c} - k. \tag{11.3.35}$$

As it is immediately seen, b_ε and c_ε satisfy Hypothesis 11.3.8, uniformly with respect to $\varepsilon > 0$ with the same constants σ and β, provided that $k > 1$. Since b_ε and c_ε are bounded, the realization $B_{\varepsilon,p}$ of the operator $\mathcal{B}_\varepsilon = \mathcal{A}_0 + \langle b_\varepsilon, D\rangle + c'_\varepsilon$ in $L^p(\Omega)$, with domain $D(B_{\varepsilon,p}) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, generates an analytic semigroup (see Theorem C.3.6(ii)). Moreover, (11.3.24) implies that there exists λ_0 such that

$$(\lambda - \lambda_0)\|u\|_p \leq \|\lambda u - B_{\varepsilon,p}u\|_p, \qquad u \in D(B_{\varepsilon,p}), \quad \lambda > \lambda_0,$$

for any $\varepsilon > 0$. Since $D(B_{\varepsilon,p})$ is dense in $L^p(\Omega)$, applying the Lumer-Phillips theorem to the operator $B_{\varepsilon,p} - \lambda_0 I$, we easily deduce that the resolvent of $B_{\varepsilon,p}$ contains the half-line $(\lambda_0, +\infty)$ for any $\varepsilon \in (0, 1)$.

Now, given $f \in L^p(\Omega)$, let $u_\varepsilon \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ be such that $(2\lambda_0 - B_{\varepsilon,p})u_\varepsilon = f$. According to Proposition 11.3.10 we know that

$$\|u_\varepsilon\|_{2,p} + \|c_\varepsilon u_\varepsilon\|_p \leq C\|f\|_p,$$

for some positive constant C, independent of ε. By a weak compactness argument, we can determine a sequence $\{\varepsilon_n\}$ converging to 0 such that u_{ε_n} tends weakly to a function u in $W^{2,p}(\Omega)\cap W^{1,p}_0(\Omega)$. Moreover, we can assume that u_{ε_n} tends to u both strongly in $W^{1,p}_{\rm loc}(\Omega)$ and a.e. in Ω. Applying Fatou's lemma, we easily see that $\|cu\|_p \leq C\|f\|_p$. Hence, $u \in D(B_p)$. Similarly, it is easy to check that u solves the differential equation $\lambda u - B_p u = f$. Therefore, the operator $2\lambda_0 - B_p$ is surjective, and the Lumer-Phillips theorem applies.

Step 2. We now prove that the semigroup $\{R(t)\}$ generated by B_p is analytic. For this purpose, we fix $u \in D$ and set $u^* := \overline{u}|u|^{p-2}$. An integration by parts shows that

$$-\mathrm{Re}\int_\Omega u^*\, B_p u\, dx = \mathrm{Re}\int_\Omega \left(\frac{p}{2}|u|^{p-2}q(Du, D\overline{u}) + \frac{p-2}{2}\overline{u}^2|u|^{p-4}q(Du,Du)\right) dx$$
$$-\mathrm{Re}\int_\Omega \overline{u}|u|^{p-2}\,\langle b', Du\rangle\, dx - \int_\Omega c'|u|^p dx, \qquad (11.3.36)$$

where q is given by (11.3.22). Since

$$\mathrm{Re}\,(|u|^2 q(Du, D\overline{u})) = q(\mathrm{Re}(\overline{u}Du), \mathrm{Re}(\overline{u}Du)) + q(\mathrm{Im}(\overline{u}Du), \mathrm{Im}(\overline{u}Du)) \qquad (11.3.37)$$

and

$$\mathrm{Re}\,(\overline{u}^2 q(Du, Du)) = q(\mathrm{Re}(\overline{u}Du), \mathrm{Re}(\overline{u}Du)) - q(\mathrm{Im}(\overline{u}Du), \mathrm{Im}(\overline{u}Du)), \qquad (11.3.38)$$

from (11.3.36)-(11.3.38) we easily deduce that

$$-\mathrm{Re}\int_\Omega u^*\, B_p u dx = (p-1)\int_\Omega |u|^{p-4}q(\mathrm{Re}(\overline{u}Du), \mathrm{Re}(\overline{u}Du))dx$$
$$+\int_\Omega |u|^{p-4}q(\mathrm{Im}(\overline{u}Du), \mathrm{Im}(\overline{u}Du))dx$$
$$-\int_\Omega |u|^{p-2}\,\langle b', \mathrm{Re}(\overline{u}Du)\rangle dx - \int_\Omega c'|u|^p dx. \qquad (11.3.39)$$

The condition (11.3.34) and the Hölder inequality imply that

$$\left|\int_\Omega |u|^{p-2}\,\langle b', \mathrm{Re}(\overline{u}Du)\rangle dx\right|$$
$$\leq \sigma' \int_\Omega |c'|^{\frac{1}{2}}|\mathrm{Re}(\overline{u}Du)||u|^{p-2}dx$$

$$\leq \sigma' \left(\int_\Omega |c'||u|^p dx\right)^{\frac{1}{2}} \left(\int_\Omega |u|^{p-4}|\mathrm{Re}(\overline{u}Du)|^2 dx\right)^{\frac{1}{2}}$$

$$\leq \frac{\sigma'}{\sqrt{\kappa_0}} \left(\int_\Omega |c'||u|^p dx\right)^{\frac{1}{2}} \left(\int_\Omega |u|^{p-4} q(\mathrm{Re}(\overline{u}Du), \mathrm{Re}(\overline{u}Du)) dx\right)^{\frac{1}{2}}. \quad (11.3.40)$$

If we set

$$F^2 := \int_\Omega |u|^{p-4} q(\mathrm{Re}(\overline{u}Du), \mathrm{Re}(\overline{u}Du)) dx,$$

$$G^2 := \int_\Omega |u|^{p-4} q(\mathrm{Im}(\overline{u}Du), \mathrm{Im}(\overline{u}Du)) dx,$$

$$H^2 := \int_\Omega |c'||u|^p dx,$$

from (11.3.39) and (11.3.40) we get

$$-\mathrm{Re} \int_\Omega u^* B_p u \, dx \geq \left(p - 1 - \frac{(\sigma')^2}{2\kappa_0}\right) F^2 + G^2 + \frac{1}{2} H^2. \quad (11.3.41)$$

Now, we consider the imaginary part of $\int_\Omega u^* B_p u dx$. Integrating by parts, we can easily show that

$$\mathrm{Im} \int_\Omega u^* B_p u \, dx = -\mathrm{Im} \int_\Omega \left(\frac{p}{2}|u|^{p-2} q(Du, D\overline{u}) + \frac{p-2}{2}\overline{u}^2 |u|^{p-4} q(Du, Du)\right) dx$$
$$+\mathrm{Im} \int_\Omega \overline{u}|u|^{p-2} \langle b', Du\rangle \, dx. \quad (11.3.42)$$

Since $\mathrm{Im}(q(Du, D\overline{u})) = 0$ and $\mathrm{Im}(q(\overline{u}Du, \overline{u}Du)) = 2q(\mathrm{Re}(\overline{u}Du), \mathrm{Im}(\overline{u}Du))$, we get

$$\mathrm{Im} \int_\Omega u^* B_p u \, dx = (p-2) \int_\Omega |u|^{p-4} q(\mathrm{Re}(\overline{u}Du), \mathrm{Im}(\overline{u}Du)) dx$$
$$+\mathrm{Im} \int_\Omega \overline{u}|u|^{p-2} \langle b', Du\rangle \, dx. \quad (11.3.43)$$

Using, first, the Cauchy-Schwarz inequality for the inner product induced by the matrix Q and, then, the Hölder inequality, we can show that

$$\left|\int_\Omega |u|^{p-4} q(\mathrm{Re}(\overline{u}Du), \mathrm{Im}(\overline{u}Du)) dx\right| \leq F\, G,$$

whereas the other term in the right-hand side of (11.3.43) can be estimated as in (11.3.40), replacing everywhere the real part of $\overline{u}Du$ with its imaginary part. Summing up, we get

$$\left|\mathrm{Im} \int_\Omega u^* B_p u \, dx\right| \leq (p-2) FG + \frac{\sigma'}{\sqrt{\kappa_0}} GH. \quad (11.3.44)$$

Combining (11.3.41) and (11.3.44), it follows that we can determine a positive constant C such that

$$\left|\mathrm{Im}\int_\Omega u^*\, B_p u\, dx\right| \le C\left(-\mathrm{Re}\int_\Omega u^*\, B_p u\, dx\right), \tag{11.3.45}$$

for any $u \in D$. By density, (11.3.45) can be extended to any $u \in D(B_p)$.

Formula (11.3.45) immediately implies that the numerical range $r(B_p)$ of B_p is contained in the sector $\{\lambda \in \mathbb{C} : |\mathrm{Im}\,\lambda| \le -C\,\mathrm{Re}\,\lambda\}$ so that, applying Theorem A.3.6, we can complete the proof. ∎

To conclude this subsection we show some regularity properties of the function $S(\cdot)f$.

Proposition 11.3.13 *Let $p > N+1$. Then, for any $f \in C_c^\infty(\Omega)$ the function $u = S(\cdot)f$ is the bounded classical solution of the Cauchy-Dirichlet problem (11.0.1). Moreover, $u \in C^{1+\alpha/2,2+\alpha}((0,T)\times(\Omega\cap B(R)))$ for any $R > 0$. Further, $Du \in C_b([0,T]\times\overline{\Omega}) \cap C^{1+\alpha/2,2+\alpha}((\varepsilon,T)\times\Omega')$ for any $0 < \varepsilon < T$, any $\Omega' \subset\subset \Omega$ and any $R > 0$ and it belongs to $C^{1+\alpha/2,2+\alpha}((0,T)\times(\Omega\cap B(R)))$.*

Proof. Fix $f \in C_c^\infty(\Omega) \subset D(A_p)$. Since $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, according to Proposition 11.3.10 and Remark B.2.6, the function $t \mapsto S(t)f$ is continuous in $[0,+\infty)$ with values in $W^{2,p}(\Omega)$. Since $p > N+1$, the Sobolev embedding theorems (see [2, Theorem 5.4]) and Remark B.2.6 imply that both u and Du are bounded and continuous in $[0,T]\times\overline{\Omega}$ for any $T > 0$. To conclude that u is a classical solution to (11.0.1), we have to show that $u \in C^{1,2}((0,+\infty)\times\Omega)$. Since $\{S(t)\}$ is analytic, the function u is continuously differentiable in $[0,+\infty)$ with values in $W^{2,p}(\Omega)$ (see again Remark B.2.6). The Sobolev embedding theorems yield $D_t u \in C([0,+\infty); C_b(\overline{\Omega}))$.

Now, fix $\varepsilon \in (0,T)$ and set

$$\tau = \sup_{\varepsilon\le t\le T}\Big(\|u(t,\cdot)\|_{W^{2,p}(\Omega)} + \|D_t u(t,\cdot)\|_{W^{2,p}(\Omega)}\Big). \tag{11.3.46}$$

Since $u(t,\cdot) \in D(A_p^n)$ for any $t \in [\varepsilon,T]$ and any $n \in \mathbb{N}$, the function $A_p u(t,\cdot)$ belongs to $W^{2,p}(\Omega)$. Observing that

$$\begin{aligned}\mathrm{Tr}(QD^2u(t,\cdot)) &= A_p u(t,\cdot) - \langle b, Du(t,\cdot)\rangle - cu(t,\cdot)\\ &= D_t u(t,\cdot) - \langle b, Du(t,\cdot)\rangle - cu(t,\cdot), \qquad t \in (\varepsilon,T],\end{aligned}$$

and $u(t,\cdot) \in W^{2,p}(\Omega)$, by difference, we deduce that $\mathrm{Tr}(QD^2u(t,\cdot)) \in W^{1,p}_{\mathrm{loc}}(\Omega)$. By local regularity results for elliptic equations in L^p-spaces, we deduce that $u(t,\cdot) \in W^{3,p}_{\mathrm{loc}}(\Omega)$ for any $t \in [\varepsilon,T]$ (see Theorem C.1.1). Moreover, since the coefficients of the operator $\mathcal{A}$ are locally bounded in Ω, from (11.3.46) we deduce that, for any pair of bounded open sets $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, it holds that

$$\begin{aligned}\|u(t,\cdot)\|_{W^{3,p}(\Omega_1)} &\le C_1\left(\|A_p u(t,\cdot) - \langle b, Du(t,\cdot)\rangle - cu(t,\cdot)\|_{W^{1,p}(\Omega_2)} + \|u\|_{L^p(\Omega_2)}\right)\\ &\le C_2\tau,\end{aligned} \tag{11.3.47}$$

for some positive constants $C_1 = C_1(\Omega_1, \Omega_2)$ and $C_2 = C_2(\Omega_1, \Omega_2)$ and any $t \in [\varepsilon, T]$. We have thus proved that $D^2u(t, \cdot) \in W^{1,p}(\Omega_1)$ for any $t \in [\varepsilon, T]$. Since $p > N + 1$, the Sobolev embedding theorems imply that $D^2u(t, \cdot) \in C^\theta(\Omega_1)$ for some $\theta \in (0, 1)$ and, by virtue of (11.3.47),

$$\sup_{\varepsilon \le t \le T} \|D^2u(t, \cdot)\|_{C^\theta(\Omega_1)} < +\infty. \tag{11.3.48}$$

Now, without loss of generality, we assume that Ω_1 has a smooth boundary. Therefore, applying Proposition A.4.6 with $X = C(\overline{\Omega}_1)$, $Y = C^{2+\theta}(\overline{\Omega}_1)$, $Z = C^2(\overline{\Omega}_1)$, $I = [\varepsilon, T]$ and $\theta = 0$ (and taking Proposition A.4.4 into account), we immediately deduce that $D^2u \in C([\varepsilon, T] \times \overline{\Omega}_1)$ and the arbitrariness of ε, T and Ω_1 implies that $D^2u \in C((0, +\infty) \times \Omega)$. We have so proved that u is a classical solution to the problem (11.0.1).

The last part of the assertion follows from Proposition 11.1.3 and Theorem 11.2.1. This finishes the proof. ∎

11.3.3 Proof of Theorem 11.3.4

To prove that the bounded classical solution to the Cauchy-Dirichlet problem (11.0.1) satisfies the gradient estimate (11.0.3) we use an approximation argument. First we prove the assertion in the case when $f \in C_c^\infty(\Omega)$ and, then, in the general case.

Step 1. Let $f \in C_c^\infty(\Omega)$. For any $\varepsilon \in (0, 1)$, let $c_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$c_\varepsilon(x) = -\varepsilon \exp\left(4d_2\sqrt{1 + |x|^2}\right), \qquad x \in \mathbb{R}^N.$$

Taking (11.3.5) into account, it can be easily shown that for any $\sigma > 0$ there exists $C_{\sigma,\varepsilon} > 0$ such that

$$|b| \le \sigma|c_\varepsilon + c|^{1/2} + C_{\sigma,\varepsilon}. \tag{11.3.49}$$

Moreover, using (11.3.4), we get

$$|D(c_\varepsilon + c)| \le \beta_0(1 - c_\varepsilon - c), \tag{11.3.50}$$

where $\beta_0 := \max(\beta, 4d_2)$, β and d_2 being given by (11.3.4) and (11.3.5).

Let $\mathcal{A}_\varepsilon = \mathcal{A} + c_\varepsilon - k$. Taking (11.3.49) and (11.3.50) into account, one can check that, if k is sufficiently large, the operator $\mathcal{A}_\varepsilon$ satisfies Hypotheses 11.3.8 with β, σ being replaced by β_0 and some $\sigma' \in (\sigma_0, \sqrt{2\kappa_0})$. Hence, it generates an analytic semigroup $\{S_\varepsilon(t)\}$ in $L^p(\Omega)$ for any $p \ge 2$. By Proposition 11.3.13 we know that if $p > N + 1$, the function $u_\varepsilon = S_\varepsilon(\cdot)f$ is the bounded classical solution to the problem (11.0.1), where we replace $\mathcal{A}$ with the operator $\mathcal{A}_\varepsilon$. Moreover, Du_ε is bounded and continuous in $[0, T] \times \overline{\Omega}$, for any $T > 0$, and it belongs to $C^{1,2}((0, +\infty) \times \Omega)$. Hence, u_ε satisfies the assumptions of Proposition 11.3.7. Since the coefficients of the operator $\mathcal{A}_\varepsilon$ satisfy Hypotheses 11.3.1

with constants being independent of $\varepsilon \in (0,1)$ (see (11.3.49) and (11.3.50)), we can find out a positive constant $C = C(T)$, independent of ε, such that

$$\|Du_\varepsilon(t,\cdot)\|_\infty \le \frac{C}{\sqrt{t}}\|f\|_\infty, \qquad t \in (0,T). \tag{11.3.51}$$

Since, for any $\alpha \in (0,1)$, the C^α-norms of the coefficients of the operator $\mathcal{A}_\varepsilon$ are bounded, in bounded subsets of Ω, uniformly with respect to ε, arguing as in Step 1 of the proof of Theorem 11.2.1, we can show that, up to a subsequence, u_ε converges to $T(\cdot)f$ in $C^{1,2}([0,T]\times(\Omega\cap B(R)))$, for any $R>0$, as ε tends to 0^+. Moreover, letting ε go to 0^+ in (11.3.51), we get

$$\|DT(t)f\|_\infty \le \frac{C}{\sqrt{t}}\|f\|_\infty, \qquad t \in (0,T), \tag{11.3.52}$$

and the assertion follows in this case.

Step 2. Given a function $f \in C_b(\Omega)$, we approximate it by a sequence $\{f_n\} \in C_c^\infty(\Omega)$ of smooth functions converging to f uniformly on compact subsets of Ω and such that $\|f_n\|_\infty \le \|f\|_\infty$ for any $n \in \mathbb{N}$. Arguing as in Step 2 of the proof of Theorem 11.2.1, we easily see that, up to a subsequence, $T(\cdot)f_n$ converges to $T(\cdot)f$ in $C^{1,2}(F)$ for any compact set $F \subset (0,T)\times\Omega$ and any $T>0$. By (11.3.52), we know that

$$\|DT(t)f_n\|_\infty \le \frac{C}{\sqrt{t}}\|f_n\|_\infty \le \frac{C}{\sqrt{t}}\|f\|_\infty, \qquad t \in (0,T).$$

Letting n go to $+\infty$, we get the assertion. ∎

11.3.4 A counterexample to the gradient estimates

In this subsection we show that the gradient estimate (11.0.3) fails, in general, if Hypothesis 11.3.1 is not satisfied. The following result generalizes an example in [144].

Example 11.3.14 Let us consider the following Cauchy-Dirichlet problem in $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2,\ x>0\}$

$$\begin{cases} D_t u(t,x,y) = \Delta u(t,x,y) + g(y)D_x u(t,x,y), & t>0,\ (x,y)\in\mathbb{R}^2_+,\\ u(t,0,y) = 0, & t>0,\ y\in\mathbb{R},\\ u(0,x,y) = 1, & (x,y)\in\mathbb{R}^2_+, \end{cases} \tag{11.3.53}$$

where $g(y) = \sqrt{1+y^2}$ for any $y \in \mathbb{R}$.

Observe that (11.3.3) fails. Indeed, in this situation $d(x,y) = x$ and, consequently, the condition (11.3.3) reads as

$$\sqrt{1+y^2} \le M, \qquad (x,y) \in \Omega_\delta = \{(x,y) \in \mathbb{R}^2 : x \in (0,\delta)\},$$

which, of course, leads us to a contradiction, letting y go to ∞. However, Theorem 11.2.1 guarantees that the problem (11.3.53) admits a unique bounded classical solution u. Since Ω is convex, to prove that the gradient estimate does not hold, it suffices to show that, for any $t > 0$, $u(t, \cdot)$ is not uniformly continuous in Ω. This can be done showing that, for any $t, x > 0$,

$$\sup_{y>0} u(t, x, y) = 1. \tag{11.3.54}$$

Indeed, if $u(t, \cdot)$ were uniformly continuous in $\mathbb{R}^2_+$, then, for any $t > 0$, the function $x \mapsto w_t(x) = \sup_{y>0} u(t, x, y)$ would be continuous in $[0, +\infty)$. Since $w_t(0) = 0$, we would get $w_t(x) < 1$ for any x in a sufficiently small neighborhood of $x = 0$, contradicting (11.3.54).

To prove (11.3.54), we fix $n > 0$ and take $r_n = \sqrt{n^2 - 1}$. Moreover, we denote by $v = v_n$ the classical solution to the problem

$$\begin{cases} D_t v(t, x, y) = \Delta v(t, x, y) + n D_x v(t, x, y), & t > 0, \quad (x, y) \in R_n, \\ v(t, x, y) = 0, & t > 0, \quad (x, y) \in \partial R_n, \\ v(0, x, y) = 1, & \qquad\quad (x, y) \in R_n, \end{cases}$$

where $R_n = (0, +\infty) \times (r_n, +\infty)$. We are going to show that for any $t, x > 0$

$$(i) \lim_{n\to+\infty} \sup_{y>r_n} v_n(t, x, y) = 1; \qquad (ii)\ u(t, x, y) \ge v_n(t, x, y), \qquad n \in \mathbb{N}. \tag{11.3.55}$$

Clearly (i) and (ii) give (11.3.54). Indeed, they give $\sup_{y>0} u(t, x, y) \ge \mathbb{1}$ and the maximum principle in Proposition 11.1.3 implies that $u \le \mathbb{1}$.

Let us prove (11.3.55)(i). We look for the solution v_n in the form $v_n(t, x, y) = a_n(t, x) b_n(t, y)$, with $a = a_n$ and $b = b_n$ solving, respectively, the Cauchy problems

$$\begin{cases} D_t a(t, x) = D_{xx} a(t, x) + n D_x a(t, x), & t > 0, \quad x > 0, \\ a(t, 0) = 0, & t > 0, \quad x > 0, \\ a(0, x) = 1, & \qquad\quad x > 0 \end{cases}$$

and

$$\begin{cases} D_t b(t, y) = D_{yy} b(t, y), & t > 0, \quad y > r_n, \\ b(t, r_n) = 0, & t > 0, \\ b(0, y) = 1, & \qquad\quad y > r_n. \end{cases} \tag{11.3.56}$$

To obtain an explicit formula for a_n, we begin by observing that $a_n(t, x) = a_1(n^2 t, nx)$. Next, setting $z(t, x) = e^{x/2} e^{t/4} a_1(t, x)$, we see that z solves the

Cauchy problem

$$\begin{cases} D_t z(t,x) = D_{xx} z(t,x), & t>0, \quad x>0, \\ z(t,0)=0, & t>0, \\ z(0,x) = e^{\frac{x}{2}}, & \qquad x>0. \end{cases} \tag{11.3.57}$$

Due to the well known representation formulas for the solutions to (11.3.56) and (11.3.57), we can write

$$\begin{aligned} a_n(t,x) &= \frac{e^{-\frac{n^2 t}{4}}}{n\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{|nx-z|^2}{4n^2 t}} - e^{-\frac{|nx+z|^2}{4n^2 t}} \right) e^{\frac{z-nx}{2}} dz \\ &= \frac{1}{\sqrt{\pi}} \int_{-\frac{x+nt}{2\sqrt{t}}}^{+\infty} e^{-s^2} ds - \frac{e^{-nx}}{\sqrt{\pi}} \int_{\frac{x-nt}{2\sqrt{t}}}^{+\infty} e^{-s^2} ds, \end{aligned} \tag{11.3.58}$$

$$b_n(t,y) = \frac{1}{\sqrt{\pi t}} \int_0^{y-r_n} e^{-\frac{z^2}{4t}} dz,$$

for any $t, x > 0$, any $y > r_n$ and any $n \in \mathbb{N}$. Hence,

$$\sup_{y>r_n} a_n(t,x,y) = a_n(t,x) \sup_{y>r_n} b_n(t,y) = a_n(t,x).$$

Letting n go to $+\infty$, from (11.3.58) we immediately get (11.3.55).

To prove (11.3.55)(ii) we use the maximum principle in Proposition 11.1.3. Set $w = u - v_n$ in $(0,+\infty)\times R_n$. We have $w = 0$ in $\{0\}\times R_n$. Moreover, $w \ge 0$ in $(0,+\infty)\times \partial R_n$. Indeed, the quoted maximum principle implies that $u \ge 0$ in $(0,+\infty)\times\Omega$. To conclude that $u \ge v_n$ in $(0,+\infty)\times\mathbb{R}^2_+$, it suffices to show that

$$D_t w(t,x,y) \ge \Delta w(t,x,y) + g(y) D_x w(t,x,y), \qquad t>0, \quad (x,y)\in R_n. \tag{11.3.59}$$

For this purpose we observe that

$$D_t w(t,x,y) = \Delta w(t,x,y) + g(y) D_x w(t,x,y) + [g(y)-n] D_x a_n(t,x) b_n(t,y),$$

for any $t>0$ and any $(x,y)\in R_n$. From (11.3.58) we easily deduce that $a_n(t,\cdot)$ is increasing in $(0,+\infty)$. Therefore, since $\sqrt{1+y^2} \ge n$ for any $y \in (r_n,+\infty)$, it follows that $[g(\cdot)-n] D_x a_n b_n \ge 0$ in $(0,+\infty)\times R_n$ and (11.3.59) follows.

Remark 11.3.15 We remark that, if we take $g(y) = -\sqrt{1+y^2}$ in (11.3.53), then Hypotheses 11.1.1 and 11.3.1 are satisfied. Therefore, the gradient estimate (11.0.3) holds. This shows that (11.3.2)-(11.3.5) are not merely conditions on the growth rate at infinity of the coefficients.

Remark 11.3.16 As it has been claimed in the introduction of this chapter, as far as we know, uniform estimates for higher order derivatives of the function $T(t)f$ seem to be not available in a general domain Ω. This prevents us to perform the same techniques as in Chapter 6 to prove optimal Schauder estimates for both the nonhomogeneous elliptic problem with Dirichlet boundary conditions and the nonhomogeneous Cauchy-Dirichlet problem associated with the operator $\mathcal{A}$.

The situation is different when Ω is an exterior domain. Indeed, very recently, estimates for the higher-order derivatives of the function $T(t)f$ have been proved in [71].

We remark that in the particular case when Ω is the halfspace $\Omega = \{x \in \mathbb{R}^N : \langle x, v\rangle > 0\}$ (v being an unitary vector in $\mathbb{R}^N$) and $\mathcal{A}$ is the operator defined by

$$\mathcal{A}u(x) = \Delta u(x) + \sum_{i,j=1}^{N} b_{ij}x_jD_iu(x), \qquad x \in \Omega,$$

where $B = (b_{ij})$ is a matrix such that v is an eigenvector of both B and B^*, optimal Schauder estimates for the elliptic problem

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) = f(x), & x \in \Omega,\\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{11.3.60}$$

($\lambda \geq 0$), are available and they have been proved by E. Priola in [127]. To state them, we need to introduce some notation. We denote by Λ any orthogonal matrix such that $\Lambda(\Omega) := \mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 > 0\}$. Moreover we set $\widetilde{B} = \Lambda B\Lambda^*$ and we denote by $\widetilde{B}_1 \in L(\mathbb{R}^{N-1})$ the matrix obtained from $\widetilde{B}$ by erasing the first line and the first row.

Assuming that $f \in C^\alpha_b(\mathbb{R}^N_+)$ for some $\alpha \in (0,1)$, E. Priola shows that, for any $\lambda \geq 0$, the problem (11.3.60) admits a unique solution $u \in C^{2+\alpha}_b(\mathbb{R}^N_+)$ if and only if

$$[[f]] := \sup_{\substack{t>0\\ y\in\mathbb{R}^{N-1}}} t^{-\frac{\alpha}{2}}|f(\Lambda^{-1}(0, e^{t\widetilde{B}_1y})) - f(\Lambda^{-1}(0,y))| < +\infty.$$

In such a case, there exists a positive constant C, independent of f, such that

$$\|u\|_{C^{2+\alpha}_b(\Omega)} \leq C([[f]] + \|f\|_{C^\alpha_b(\mathbb{R}^N)}).$$

This result shows that, differently from what happens when $\Omega = \mathbb{R}^N$, when we have Dirichlet boundary conditions on $\partial\Omega$ (and Ω is not an exterior domain), we cannot expect, in general, to prove optimal Schauder estimates for the solution to the elliptic equation in Ω, only assuming that the data are Hölder continuous in Ω: we need to assume some additional conditions on them.

Chapter 12

The Cauchy-Neumann problem: the convex case

12.0 Introduction

Let Ω be a smooth convex unbounded domain in $\mathbb{R}^N$. In this chapter we consider both the parabolic problem with homogeneous Neumann boundary conditions

$$\begin{cases} D_t u(t,x) - \mathcal{A}u(t,x) = 0, & t>0, \quad x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, & t>0, \quad x \in \partial\Omega, \\ u(0,x) = f(x), & \qquad\quad x \in \Omega \end{cases} \tag{12.0.1}$$

and the elliptic problem

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) = f(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(x) = 0, & x \in \partial\Omega, \end{cases} \tag{12.0.2}$$

when $f \in C_b(\overline{\Omega})$. Here, $\mathcal{A}$ is given, as usual, by

$$\mathcal{A}\varphi(x) = \sum_{i,j=1}^N q_{ij}(x) D_{ij}\varphi(x) + \sum_{j=1}^N b_j(x) D_i\varphi(x) + c(x)\varphi(x), \qquad x \in \Omega, \tag{12.0.3}$$

on smooth functions and $\nu(x)$ denotes the outer unit normal to $\partial\Omega$ at $x \in \partial\Omega$.

Under suitable assumptions on the coefficients of the operator $\mathcal{A}$, we show that the Cauchy-Neumann problem (12.0.1) admits a unique classical solution u (i.e., a function $u \in C([0,+\infty)\times\overline{\Omega}) \cap C^{0,1}((0,+\infty)\times\overline{\Omega}) \cap C^{1,2}((0,+\infty)\times\Omega)$ solving the Cauchy-Neumann problem (12.0.1) pointwise) which is bounded in $(0,T)\times\Omega$ for any $T>0$). This will allow us to associate a semigroup $\{T(t)\}$ of bounded operators with the Cauchy-Neumann problem (12.0.1) by setting $T(t)f = u(t,\cdot)$ for any $t>0$.

As in Chapter 2, the solution to the problem (12.0.1) is obtained by approximating our problem with a sequence of Cauchy-Neumann problems in (convex) bounded domains Ω_n ($n \in \mathbb{N}$). The Neumann boundary condition gives some problems. Indeed, differently to what happens in the case considered in Chapter 2, it is not immediate to show that the solutions u_n to the Cauchy-Neumann problems in the bounded sets Ω_n converge to a solution to the problem (12.0.1). To overcome such a difficulty, we prove an *a priori* gradient estimate for the functions u_n, with constants being independent of n. This forces us to assume stronger hypotheses on the coefficients than those in Chapter 2. In particular, we have to assume some dissipativity and growth conditions on the coefficients of the operator $\mathcal{A}$.

Once this gradient estimate and a suitable maximum principle for bounded classical solutions to the Cauchy-Neumann problem (12.0.1) is proved, one can show that the sequence $\{u_n\}$ converges to a bounded classical solution u to the problem (12.0.1) and that $T(t)f$ satisfies the following estimate:

$$\|DT(t)f\|_\infty \le \frac{C_T}{\sqrt{t}}\|f\|_\infty, \qquad t\in(0,T), \tag{12.0.4}$$

for any $f\in C_b(\overline{\Omega})$, any $T>0$ and some positive constant C_T, independent of f. Moreover, one can also show that

$$\|DT(t)f\|_\infty \le C\|f\|_{C^1_b(\overline{\Omega})}, \qquad t>0, \tag{12.0.5}$$

for any $f\in C^1_\nu(\overline{\Omega})$ and C, as above, is a positive constant independent of f.

The gradient estimates

$$\|Du_n(t,\cdot)\|_\infty \le \frac{C_T}{\sqrt{t}}\|f\|_\infty, \qquad t\in(0,T),\;\; f\in C_b(\overline{\Omega}_n)$$

and

$$\|Du_n(t,\cdot)\|_\infty \le C\|f\|_{C^1_\nu(\overline{\Omega}_n)}, \qquad t\in(0,T),\;\; f\in C^1_\nu(\overline{\Omega}_n)$$

are proved by using the Bernstein method as we did in Chapter 6. Here $c_0=\sup_\Omega c$ and C_T and C are two positive constants independent, respectively, of $t\in(0,T)$ and of $t>0$. Moreover, they are independent of n as well.

Concerning the elliptic problem (12.0.2), we show that, for any $\lambda>c_0:=\sup_\Omega c(x)$ and any $f\in C_b(\overline{\Omega})$, it admits a unique solution $u\in D(\mathcal{A})$, where

$$\begin{aligned} D(\mathcal{A}) = \Bigg\{ u\in C_b(\overline{\Omega})\cap \bigcap_{1\le p<+\infty} W^{2,p}(\Omega\cap B(R)) \;\text{ for any } R>0: \\ \mathcal{A}u\in C_b(\overline{\Omega}),\; \frac{\partial u}{\partial\nu}(x)=0 \;\text{ for any } x\in\partial\Omega\Bigg\}. \end{aligned} \tag{12.0.6}$$

Moreover, using (12.0.4) and (12.0.5), we then show that $D(\mathcal{A})\subset C^1_b(\mathbb{R}^N)$ and that $C^1_b(\overline{\Omega})$ is of class $J_{1/2}$ between $C_b(\overline{\Omega})$ and $D(\mathcal{A})$, that is

$$\|Du\|_\infty \le M_\omega \|u\|_\infty^{\frac{1}{2}}\|(\mathcal{A}-\omega)u\|_\infty^{\frac{1}{2}}, \qquad f\in D(\mathcal{A}),$$

for any $\omega>0$.

In Section 12.3 we come back to the parabolic Cauchy-Neumann problem and, under somewhat heavier assumptions on the growth of the coefficients at infinity, we prove the pointwise estimate

$$|(DT(t)f)(x)|^p \le e^{\overline{\sigma}_p t}(T(t)|Df|^p)(x), \qquad t>0,\ x\in\overline{\Omega},\ f\in C^1_\nu(\overline{\Omega}), \tag{12.0.7}$$

where $\overline{\sigma}_p$ are suitable real constants. But, here, it must be $T(t)\mathbb{1}\equiv\mathbb{1}$, otherwise (12.0.7) fails for $f\equiv\mathbb{1}$. Thus, we suppose that $c\equiv 0$. Next, using (12.0.7) we prove the pointwise estimates

$$|(DT(t)f)(x)|^p \le \left(\frac{\overline{\sigma}_2}{2\kappa_0(1-e^{-\overline{\sigma}_2 t})}\right)^{\frac{p}{2}}(T(t)(|f|^p))(x), \qquad p\in[2,+\infty), \tag{12.0.8}$$

$$|(DT(t)f)(x)|^p \le \frac{c_p\overline{\sigma}_p t^{1-\frac{p}{2}}}{\kappa_0(1-e^{-\overline{\sigma}_p t})}(T(t)(|f|^p))(x), \qquad p\in(1,2), \tag{12.0.9}$$

for any $t>0$ and any $x\in\overline{\Omega}$, where $\overline{\sigma}_p/(1-e^{-\overline{\sigma}_p t})$ is replaced with $1/t$ when $\overline{\sigma}_p=0$.

As in Chapter 7, such estimates lead to some interesting results. First, they allow to prove that

$$\|DT(t)f\|_\infty \le \left(\frac{\overline{\sigma}_2}{2\kappa_0(1-e^{-\overline{\sigma}_2 t})}\right)^{\frac{1}{2}}\|f\|_\infty, \qquad t>0,\ f\in C_b(\overline{\Omega}),$$

which improves (12.0.4), since it shows that the constant can be taken independent of T and, when $k_0<0$, it shows that the sup-norm of $DT(t)f$ decreases exponentially to 0 as t tends to $+\infty$.

Next, when $\overline{\sigma}_2\le 0$, (12.0.8) and (12.0.9) provide us with a Liouville type theorem. Namely, they imply that the constants are the only distributional solutions to $\mathcal{A}u=0$ which belong to $D(\mathcal{A})$.

We also show that, under the same assumptions which allow us to prove the uniform gradient estimates,

$$|(DT(t)f)(x)|^p \le \left(\frac{2-p}{p}\right)^{1-\frac{p}{2}}\left(\frac{1}{p(p-1)\kappa_0 t}+1\right)^{\frac{p}{2}}(T(t)(|f|^p))(x),$$

for any $t>0$, any $x\in\Omega$, any $f\in C_b(\overline{\Omega})$ and any $p>1$. Again, we can use this estimate to improve the gradient estimates proved in Section 12.1, when t approaches $+\infty$.

Finally, in Section 12.4, we briefly generalize some results of Chapter 8 to the case of the invariant measure associated with the semigroup $\{T(t)\}$ considered in this chapter. As in the case when $\Omega=\mathbb{R}^N$, whenever an invariant measure exists, the semigroup can be extended to a strongly continuous semigroup defined in the L^p space related to the measure μ, for any $p\in[1,+\infty)$.

Using the pointwise estimates of Section 12.3 we prove gradient estimates for the semigroup in $L^p(\Omega,\mu)$. Such estimates allow us to give a partial characterization of the domain $D(L_p)$ of the infinitesimal generator of $\{T(t)\}$ in $L^p(\Omega,\mu)$.

In the last part of the section we deal with the case when

$$\mathcal{A}\varphi = \frac{1}{2}\Delta\varphi - \langle DU, D\varphi\rangle,$$

on smooth functions φ, and U is a convex function. In such a situation a complete description of $D(L_2)$ is available. Moreover, the Poincaré inequality can be proved, provided a suitable dissipative condition is satisfied.

12.1 Construction of the semigroup and uniform gradient estimates

In this section we prove that for any $f \in C_b(\overline{\Omega})$, the Cauchy-Neumann problem (12.0.1) admits a unique classical solution (i.e., a function $u \in C([0,+\infty) \times \overline{\Omega}) \cap C^{1,2}((0,+\infty) \times \overline{\Omega})$ that solves pointwise the problem (12.0.1)), which is bounded in $[0,T] \times \overline{\Omega}$ for any $T > 0$. This will allow us to associate a semigroup of bounded linear operators with the problem (12.0.1), as explained in the introduction. To carry out our programme we assume the following set of hypotheses.

Hypotheses 12.1.1 (i) Ω is a convex unbounded open set of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$;

(ii) the coefficients q_{ij} and b_j $(i,j = 1,\ldots,N)$ and c belong to $C^{1+\alpha}_{\rm loc}(\overline{\Omega})$ and

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \geq \kappa(x)|\xi|^2, \qquad x \in \Omega, \quad \xi \in \mathbb{R}^N, \tag{12.1.1}$$

for some function $\kappa : \overline{\Omega} \to \mathbb{R}$ such that $0 < \kappa_0 := \inf_{x\in\mathbb{R}^N} \kappa(x)$;

(iii) there exist some constants $q_0, \gamma > 0$, $c_0, k_0, \beta \in \mathbb{R}$, $\beta < 1/2$ such that

$$|Dq_{ij}(x)| \leq q_0\kappa(x), \qquad x \in \Omega, \tag{12.1.2}$$

$$\sum_{i,j=1}^N D_i b_j(x)\xi_i\xi_j \leq (-\beta c(x) + k_0)|\xi|^2, \qquad x \in \Omega,\ \xi \in \mathbb{R}^N, \tag{12.1.3}$$

$$\sup_{x\in\Omega} c(x) = c_0, \tag{12.1.4}$$

$$|Dc(x)| \leq \gamma(1 + |c(x)|), \qquad x \in \Omega; \tag{12.1.5}$$

(iv) there exist a function $\varphi \in C^2(\overline{\Omega})$ and a constant $\lambda_0 > c_0$ such that

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty, \qquad \sup_{\Omega}(\mathcal{A}\varphi - \lambda_0\varphi) < +\infty, \qquad \frac{\partial\varphi}{\partial\nu}(x) \geq 0, \quad x \in \partial\Omega. \tag{12.1.6}$$

Remark 12.1.2 Suppose that $0 \in \Omega$. Then $\langle x, \nu(x)\rangle \ge 0$ for any $x \in \partial\Omega$. Therefore, one might look for a function φ satisfying (12.1.6) in the form $\varphi(x) = g(|x|^2)$, for any $x \in \Omega$, where $g : [0,+\infty) \to \mathbb{R}$ is a smooth and increasing function such that $\lim_{t\to+\infty} g(t) = +\infty$.

We now introduce a nested sequence $\{\Omega_n\}$ of bounded convex domains, with $C^{2+\alpha}$-smooth boundaries, such that

$$\bigcup_{n>0} \Omega_n = \Omega, \qquad B(n) \cap \partial\Omega \subset \partial\Omega_n, \quad n > 0. \tag{12.1.7}$$

Moreover, we denote by $\{T_n(t)\}$ the strongly continuous analytic semigroup generated by the realization A_n of the operator $\mathcal{A}$ with homogeneous Neumann boundary conditions in Ω_n, i.e., by the operator $A_n : D(A_n) \subset C_b(\overline{\Omega}) \to C_b(\overline{\Omega})$ defined by

$$A_n u = \mathcal{A}u, \qquad u \in D(A_n), \tag{12.1.8}$$

where

$$D(A_n) = \left\{u \in \bigcap_{1\le p<+\infty} W^{2,p}(\Omega_n) : \ \mathcal{A}u \in C(\overline{\Omega}_n), \ \frac{\partial u}{\partial \nu}(x) = 0, \quad x \in \partial\Omega_n\right\}, \tag{12.1.9}$$

(see Theorem C.3.6(v)). According to Proposition C.3.3, for any $f \in C(\overline{\Omega}_n)$, the function $u_n = T_n(\cdot)f$ belongs to $C([0,+\infty)\times\overline{\Omega}_n)\cap C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Omega}_n)$ for any $0 < \varepsilon < T$ and it is the unique classical solution of the Cauchy problem

$$\begin{cases} D_t u_n(t,x) = \mathcal{A}u_n(t,x), & t>0,\ x \in \Omega_n,\\[1ex] \dfrac{\partial u_n}{\partial \nu}(t,x) = 0, & t>0,\ x\in\partial\Omega_n,\\[1ex] u_n(0,x) = f(x), & x \in \Omega_n. \end{cases} \tag{12.1.10}$$

Moreover,

$$\|T_n(t)\|_{L(C(\overline{\Omega}_n))} \le e^{c_0 t}, \qquad t > 0, \tag{12.1.11}$$

and $T_n(t)f \ge 0$ for any $t \ge 0$ and any nonnegative function $f \in C(\overline{\Omega}_n)$.

We can now state the main result of this section.

Theorem 12.1.3 *Under Hypotheses* 12.1.1 *there exists a unique classical solution* $u \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\overline{\Omega})$ *to the Cauchy-Neumann problem* (12.0.1), *which is bounded in* $[0,T]\times\overline{\Omega}$ *for any* $T > 0$. *Moreover,* $u_n = T_n(\cdot)f$ *converges to* u *uniformly on compact sets of* $(0,+\infty)\times\overline{\Omega}$. *Further, there exists a semigroup* $\{T(t)\}$ *in* $C_b(\overline{\Omega})$ *such that* $u(t,\cdot) = T(t)f$ *for any* $t \ge 0$ *and*

$$\|T(t)f\|_\infty \le e^{c_0 t}\|f\|_\infty, \qquad f \in C_b(\overline{\Omega}).$$

Finally, for any $\omega > 0$, *there exists a positive constant* C_ω *such that*

$$\|DT(t)f\|_\infty \le C_\omega \frac{e^{\omega t}}{\sqrt{t}}\|f\|_\infty, \qquad t > 0, \tag{12.1.12}$$

for any $f \in C_b(\overline{\Omega})$.

Proof. Fix two open sets Ω' and Ω'' such that $\Omega' \subset \Omega''$ and $\mathrm{dist}(\Omega', \Omega\backslash\Omega'') > 0$, and let $n_0 \in \mathbb{N}$ be such that $\Omega'' \subset \Omega_n$ for any $n \geq n_0$. According to Theorem C.1.5, there exists a positive constant C, independent of n, such that

$$\|u_n\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\Omega')} \leq C\|f\|_\infty,$$

for any $n \geq n_0$. This estimate implies that there exist a subsequence $T_{n_k}(\cdot)f$ and a continuous function $u : (0, +\infty) \times \overline{\Omega} \to \mathbb{R}$ such that

$$\lim_{k\to+\infty} (T_{n_k}(t)f)(x) = u(t, x), \qquad t > 0,\ x \in \overline{\Omega}.$$

Moreover, $T_{n_k}(\cdot)f$ converges to u in $C^{1+\beta/2,2+\beta}([\varepsilon, T] \times \Omega')$ for any $\beta < \alpha$, any bounded set $\Omega' \subset \Omega$ and any $0 < \varepsilon < T$. As it is immediately seen, u is a solution of the equation $D_t u - \mathcal{A}u = 0$ in $(0, +\infty) \times \Omega$ and its normal derivative vanishes on $(0, +\infty)\times\partial\Omega$. Moreover, $u \in C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((0, +\infty)\times\overline{\Omega})$. By (12.1.11) we get

$$|u(t, x)| \leq e^{c_0 t}\|f\|_\infty, \qquad t > 0,\ x \in \overline{\Omega}. \tag{12.1.13}$$

Hence, u is bounded in $(0, T] \times \overline{\Omega}$ for any $T > 0$.

In contrast to what we showed in Section 2.2, when we dealt with the Cauchy problem in the whole of $\mathbb{R}^N$, in general, the sequence $\{T_n(t)f\}$ is not increasing for any fixed $t > 0$ and any nonnegative function $f \in C_b(\overline{\Omega})$. This lack of monotonicity is strictly related to the choice of Neumann boundary conditions.

To prove the continuity of u on $\{0\} \times \overline{\Omega}$, we observe that the arguments used in Step 2 in the proof of Theorem 2.2.1, based on a localization argument, cannot be immediately adapted to our situation, since they only allow us to prove the continuity of $T(\cdot)f$ on $\{0\} \times \Omega$. To overcome this difficulty, we apply a different technique which requires us to prove some gradient estimates for the function $T_n(t)f$, with constants being independent of n, and a maximum principle for bounded solutions to the problem (12.0.1). To prove such a gradient estimate we strongly take advantage of the convexity of Ω.

Proposition 12.1.4 *Let Hypotheses* 12.1.1 *be satisfied and fix* $T > 0$. *Then, there exists a constant* $C_T > 0$, *independent of* $n \in \mathbb{N}$, *such that*

$$\|DT_n(t)f\|_\infty \leq \frac{C_T}{\sqrt{t}}\|f\|_\infty, \qquad t \in (0, T], \tag{12.1.14}$$

for any $f \in C_b(\overline{\Omega})$. *Moreover*

$$\|DT_n(t)f\|_\infty \leq C_1 e^{c_0 \frac{t}{2}}\|f\|_{C^1_b(\overline{\Omega})}, \qquad t \geq 0,\ x \in \Omega_n, \tag{12.1.15}$$

for any $f \in C^1_b(\Omega_n)$ *such that* $\partial f/\partial\nu = 0$ *on* $\partial\Omega_n$ *and some positive constant* C_1, *independent of* n. *Here,* c_0 *is given by* (12.1.4).

Proof. Let us begin the proof by checking the estimate (12.1.14). Without loss of generality, we can suppose that $c_0 \le -1$. Indeed, once the assertion is proved in this situation, the general case will follow by replacing $\mathcal{A}$ with the operator $\mathcal{A}' = \mathcal{A} - (c_0+1)I$. Moreover, we can suppose that $f \in D(A_n)$. Indeed, for a general $f \in C_b(\overline{\Omega})$ it suffices to split $T_n(t)f = T_n(t/2)T_n(t/2)f$ and to observe that $T_n(t/2)f \in D(A_n)$.

Let $n > 0$, set $u_n = T_n(\cdot)f$ and define

$$v_n(t,x) = |u_n(t,x)|^2 + at|Du_n(t,x)|^2, \qquad t \in (0,T], \;\; x \in \overline{\Omega}_n, \tag{12.1.16}$$

where $a > 0$ will be chosen later. Since $u_n \in C([0,+\infty); D(A_n)) \cap C^1((0,+\infty); D_{A_n}(1+\alpha/2,\infty))$ and $D(A_n) \subset C_b^1(\overline{\Omega}_n)$ with a continuous embedding (see Theorems B.2.2 and C.3.6(v)), we easily deduce that $Du_n \in C([0,+\infty)\times\overline{\Omega}_n) \cap C^{1,1}((0,+\infty)\times\overline{\Omega}_n)$. Moreover, according to Theorem C.1.4(ii), Du_n is twice continuously differentiable with respect to the space variables in $(0,+\infty)\times\Omega_n$. It follows that $v_n \in C([0,+\infty)\times\overline{\Omega}_n) \cap C^{1,1}((0,+\infty)\times\overline{\Omega}_n) \cap C^{0,2}((0,+\infty)\times\Omega_n)$ and $v_n(0,\cdot) = 0$.

We claim that, for any $T > 0$ arbitrarily fixed, a can be properly chosen such that the function v_n satisfies

$$\begin{cases} D_t v_n(t,x) - \mathcal{A} v_n(t,x) \le 0, & t \in (0,T],\, x \in \Omega_n, \\ \dfrac{\partial v_n}{\partial \nu}(t,x) \le 0, & t \in (0,T],\, x \in \partial\Omega_n. \end{cases}$$

Once the claim is proved, the classical maximum principle will yield $v_n \le |f|^2$ in $(0,T)\times\Omega_n$, and (12.1.14) will easily follow, with $C_T = a^{-1/2}$. In the sequel, to simplify the notation, we drop out the dependence on n, when there is no damage of confusion.

As a first step, we prove that

$$\frac{\partial v}{\partial \nu}(t,x) \le 0, \qquad t \in (0,T], \;\; x \in \partial\Omega_n. \tag{12.1.17}$$

This is the crucial point of the proof and it is the only part in which the convexity of Ω plays a crucial role. Since, by assumptions, $\partial u_n/\partial\nu$ identically vanishes on $\partial\Omega_n$ we have

$$\begin{aligned} \frac{\partial v}{\partial \nu}(t,x) &= 2u\frac{\partial u}{\partial \nu}(t,x) + 2at\sum_{i,j=1}^N D_{ij}^2 u(t,x) D_i u(t,x) \nu_j(x) \\ &= 2at\sum_{i,j=1}^N D_{ij}^2 u(t,x) D_i u(t,x)\nu_j(x), \end{aligned} \tag{12.1.18}$$

for any $t \in (0,T]$ and any $x \in \partial\Omega_n$. Now, we differentiate the relation

$$\frac{\partial u}{\partial \nu} = Du\cdot\nu = 0,$$

with respect to x_j, obtaining

$$\sum_{i=1}^{N} D_{ij}u(t,x)\nu_i(x) + \sum_{i=1}^{N} D_i u(t,x) D_j \nu_i(x) = 0. \tag{12.1.19}$$

Multiplying by $D_j u(t,x)$ both the sides of (12.1.19) and summing with respect to $j = 1, \ldots, N$, we get

$$\sum_{i,j=1}^{N} D_{ij}u(t,x)D_j u(t,x)\nu_i(x) + \sum_{i=1}^{N} D_i u(t,x)D_j u(t,x)D_j\nu_i(x) = 0. \tag{12.1.20}$$

From (12.1.18) and (12.1.20), we get

$$\frac{\partial v}{\partial \nu}(t,x) = -2at \sum_{i,j=1}^{N} D_i\nu_j(x)D_i u(t,x)D_j u(t,x), \qquad t \in (0,T], \;\; x \in \partial\Omega_n.$$

Since Ω_n is convex, $D\nu(x)$ is a positive definite matrix for any $x \in \Omega_n$ and (12.1.17) follows.

Let us now show that if a is suitably small, then $D_t v - \mathcal{A}v \leq 0$ in $(0,T]\times\Omega_n$. With some computations one can see that v satisfies the equation

$$D_t v - \mathcal{A}v = a|Du|^2 - 2\sum_{i,j=1}^{N} q_{ij}D_i u D_j u + g_1 + g_2,$$

for any $t \in (0,T]$ and any $x \in \Omega_n$, where

$$\begin{aligned} g_1(t,x) = {}& 2at \sum_{i,j=1}^{N} D_i b_j(x) D_i u(t,x) D_j u(t,x) + atc(x)|Du(t,x)|^2 \\ & +2atu(t,x)\sum_{i=1}^{N} D_i c(x) D_i u(t,x) + c(x)|u(t,x)|^2, \end{aligned}$$

$$\begin{aligned} g_2(t,x) = {}& -2at \sum_{i,j,h=1}^{N} q_{ij}(x) D_{ih}u(t,x) D_{jh}u(t,x) \\ & +2at \sum_{i,j,h=1}^{N} D_h q_{ij}(x) D_h u(t,x) D_{ij}u(t,x). \end{aligned}$$

Using (12.1.5) we get

$$\begin{aligned} 2atu\sum_{i=1}^{N} D_i c D_i u &\leq 2at|u||Du||Dc| \\ &\leq 2a\gamma t(1-c)|u||Du| \\ &\leq a\frac{\gamma}{2\varepsilon}t(1-c)u^2 + 2a\gamma\varepsilon t(1-c)|Du|^2, \end{aligned}$$

for any $\varepsilon > 0$. Recalling that $c \le -1$ and using (12.1.3), it follows that

$$g_1 \le \left(1 - a\frac{\gamma}{\varepsilon}t\right)c\,u^2 - at(2\beta - 1 + 2\gamma\varepsilon)c\,|Du|^2 + 2at(k_0 + \gamma\varepsilon)|Du|^2.$$

Since $\beta < 1/2$ we can choose $\varepsilon > 0$ such that $2\beta - 1 + 2\gamma\varepsilon < 0$. Therefore,

$$g_1 \le \left(1 - \frac{a\gamma}{\varepsilon}t\right)c\,u^2 + 2at(k_0 + \gamma\varepsilon)|Du|^2, \tag{12.1.21}$$

in $(0, T] \times \Omega_n$.

Now, we consider the function g_2. By (12.1.2) we have

$$\begin{aligned}\sum_{i,j,h=1}^{N} D_h q_{ij} D_h u D_{ij} u &\le q_0 \kappa |Du| \sum_{i,j=1}^{N} |D_{ij} u| \\ &\le q_0 \kappa N |Du||D^2 u| \\ &\le \frac{q_0^2 N^2}{4\varepsilon'}\kappa |Du|^2 + \varepsilon' \kappa |D^2 u|^2, \end{aligned}\tag{12.1.22}$$

for any $\varepsilon' > 0$. Then, using (12.1.1), we get

$$g_2 \le a\frac{q_0^2 N^2}{2\varepsilon'} t\kappa\,|Du|^2 + 2at(\varepsilon' - 1)\kappa\,|D^2 u|^2,$$

in $(0, T] \times \Omega_n$. Choosing $\varepsilon' < 1$ we obtain

$$g_2 \le a\frac{q_0^2 N^2}{2\varepsilon'} t\kappa |Du|^2, \tag{12.1.23}$$

in $(0, T] \times \Omega_n$. The estimates (12.1.21) and (12.1.23) imply that

$$\begin{aligned} D_t v - \mathcal{A} v \le{}& \left(1 - a\frac{\gamma}{\varepsilon}T\right)c\,u^2 \\ &+ \left\{a + 2aT(k_0 + \gamma\varepsilon)^+ + \left(a\frac{q_0^2 N^2}{2\varepsilon'}T - 2\right)\kappa\right\}|Du|^2, \end{aligned}$$

in $(0, T] \times \Omega_n$. It is now clear that there exists a sufficiently small value of $a = a(T) > 0$, independent of n, such that $D_t v - \mathcal{A} v \le 0$ in $(0, T] \times \Omega_n$.

Now, we briefly prove the estimate (12.1.15). It suffices to show that it holds for any $f \in D(A_n)$ (see (12.1.9)). Indeed, $D(A_n)$ is dense in $C^1_\nu(\overline{\Omega}_n)$. This is a classical result, and it can be checked adapting the proof of the forthcoming Lemma 13.1.10.

So, let us check (12.1.15) for any $f \in D(A_n)$. For this purpose, we introduce the function $w_n : [0, +\infty) \times \overline{\Omega}$ defined by

$$w_n(t, x) = |u_n(t, x)|^2 + a\,|Du_n(t, x)|^2, \qquad t \ge 0, \quad x \in \overline{\Omega}_n.$$

The above arguments show that $w_n \in C([0,T]\times\overline{\Omega}_n)$ for any $T>0$. Moreover, it belongs to $C^{1,2}((0,T]\times\Omega)$. Straightforward computations show that the function $w=w_n$ satisfies

$$D_t w(t,x) - \mathcal{A}w(t,x) = -2\sum_{i,j=1}^{N} q_{ij}(x)D_i u(t,x)D_j u(t,x) \\ +h_1(t,x)+h_2(t,x),$$

for any $t\in(0,T]$ and any $x\in\Omega_n$, where $h=h_1+h_2$ and h_1 and h_2 are defined as g_1 and g_2 with the coefficient $a\cdot t$ being everywhere replaced with a. Moreover, it satisfies $\partial w/\partial\nu = 0$ on $\partial\Omega_n$ and $w(0,\cdot)=f^2+a|Df|^2$. Arguing as above, we can show that for a suitably small value of $a>0$, independent of n and T, we have $h\le 0$. Thus, the classical maximum principle yields

$$w(t,x) \le e^{c_0 t}\sup_{x\in\Omega_n} w(0,x) \le e^{c_0 t}(\|f\|_\infty^2 + a\,\|Df\|_\infty^2), \qquad t>0,\ x\in\Omega_n,$$

and (12.1.15) follows with $C_1 = a^{-1/2}\vee 1$. ∎

Now, we prove a maximum principle for bounded classical solutions to the Neumann parabolic problem in Ω.

Theorem 12.1.5 *If $z\in C([0,T]\times\overline{\Omega})\cap C^{0,1}((0,T]\times\overline{\Omega})\cap C^{1,2}((0,T]\times\Omega)$ is bounded and satisfies*

$$\begin{cases} D_t z(t,x) - \mathcal{A}z(t,x) \le 0, & t\in(0,T],\quad x\in\Omega,\\ \dfrac{\partial z}{\partial\nu}(t,x)\le 0, & t\in(0,T],\quad x\in\partial\Omega,\\ z(0,x)\le 0, & x\in\Omega,\end{cases}$$

then

$$z(t,x)\le 0, \qquad t\in[0,T],\quad x\in\overline{\Omega}. \tag{12.1.24}$$

In particular, the Cauchy-Neumann problem (12.0.1) *admits at most one classical solution which is bounded and continuous in* $[0,T]\times\overline{\Omega}$. *Moreover, it satisfies* $\|u(t,\cdot)\|_\infty \le e^{c_0 t}\|f\|_\infty$, *where* c_0 *is given by* (12.1.4).

Proof. Let φ be as in (12.1.6). We can suppose that $\varphi\ge 0$ and $(\mathcal{A}-\lambda)\varphi\le 0$; otherwise we replace φ with $\varphi + C$, for a suitable constant $C>0$, and take a larger λ if needed.

Denote by $v:[0,T]\times\overline{\Omega}\to\mathbb{R}$ the function given by $v(t,x)=e^{-\lambda t}z(t,x)$ for any $t\in[0,T]$ and any $x\in\overline{\Omega}$, and consider the sequence $\{v_n\}$ defined by

$$v_n(t,x) = v(t,x) - \frac{1}{n}\varphi(x), \qquad t\in[0,T],\quad x\in\Omega.$$

We have

$$\begin{cases} D_t v_n(t,x) - (\mathcal{A}-\lambda)v_n(t,x) \le 0, & t\in(0,T],\quad x\in\Omega,\\ \dfrac{\partial v_n}{\partial\nu}(t,x)\le 0, & t\in(0,T],\quad x\in\partial\Omega,\\ v_n(0,x)\le -\dfrac{1}{n}\inf_{\Omega}\varphi, & x\in\Omega. \end{cases}$$

For any $n\in\mathbb{N}$ the function v_n attains its maximum in $[0,T]\times\overline{\Omega}$ at some point (t_n,x_n). Moreover, we have

$$\lim_{n\to+\infty} v_n(t_n,x_n) = \lim_{n\to+\infty}\sup_{[0,T]\times\overline{\Omega}} v_n = \sup_{[0,T]\times\overline{\Omega}} v. \tag{12.1.25}$$

According to the strong maximum principle (see [60, Theorem 2.14]), we cannot have $t_n>0$ and $x_n\in\partial\Omega$. If $(t_n,x_n)\in(0,T]\times\Omega$ we have

$$D_t v_n(t_n,x_n)\ge 0,\qquad \mathcal{A}v_n(t_n,x_n)-c(x_n)v_n(t_n,x_n)\le 0,$$

and, therefore,

$$(\lambda-c(x_n))\sup_{[0,T]\times\overline{\Omega}} v_n = (\lambda-c(x_n))v_n(t_n,x_n)\le(\lambda+D_t-\mathcal{A})v_n(t_n,x_n)\le 0,$$

which yields

$$\sup_{[0,T]\times\overline{\Omega}} v_n\le 0.$$

If instead $t_n=0$ we have

$$v_n(t_n,x_n)\le -\frac{1}{n}\inf_{\Omega}\varphi.$$

Thus letting n go to $+\infty$ and taking (12.1.25) into account, we obtain (12.1.24). ∎

Remark 12.1.6 It is worth noticing that in the proof of the quoted theorem, we never used the assumption that Ω is convex.

We can now conclude the proof of Theorem 12.1.3.

Proof of Theorem 12.1.3 (*continued*). To prove that u is a classical solution to the problem (12.0.1), it remains to show that u is continuous at $t=0$.

The proof is similar to that of Theorem 2.2.1 and it strongly relies on the gradient estimate (12.1.14). Hence we limit ourselves to sketching it.

For any arbitrarily fixed $x_0\in\overline{\Omega}$, we consider two open neighborhoods $U_1\subset U_0$ of x_0 such that $\Omega_0:=U_0\cap\Omega$ is sufficiently smooth. Then, we introduce a smooth function $\vartheta\in C^\infty(\overline{\Omega_0})$ such that $\vartheta\equiv 0$ in a neighborhood of $\Omega\cap\partial U_0$, $\vartheta\equiv 1$ in $U_1\cap\Omega$ and $\partial\vartheta/\partial\nu=0$ in $U_0\cap\partial\Omega$. Finally, for any $k\in\mathbb{N}$ such that

$\Omega_0 \subset \Omega_{n_k}$, we introduce the function $v_{n_k} = \vartheta T_{n_k}(\cdot)f$. Such a function solves the equation

$$D_t v_{n_k}(t,x) - \mathcal{A} v_{n_k}(t,x) = \psi_{n_k}(t,x), \qquad t>0,\ x\in\Omega_0,$$

where

$$\psi_{n_k}(t,x) = -(T_{n_k}(t)f)(x)((\mathcal{A}-c)\vartheta)(x) - 2\sum_{i,j=1}^N q_{ij}(x)(D_i T_{n_k}(t)f)(x)D_j\vartheta(x),$$

for any $t>0$ and any $x\in\Omega_0$, and, due to the choice of the function ϑ, its normal derivative vanishes on $\partial\Omega_0$.

From (12.1.14), we immediately see that

$$\|\psi_{n_k}(t,\cdot)\|_\infty \le \frac{C}{\sqrt{t}}, \qquad t\in(0,T), \tag{12.1.26}$$

for any $k\in\mathbb{N}$, and some positive constant $C=C(T)$, independent of k.

We now represent v_{n_k} through the variation-of-constants formula

$$v_{n_k}(t,\cdot) = T_0(t)(\vartheta f) + \int_0^t T_0(t-s)\psi_{n_k}(s,\cdot)ds, \qquad t>0,$$

where $\{T_0(t)\}$ is the strongly continuous analytic semigroup generated by the realization of the operator $\mathcal{A}$, with homogeneous Neumann boundary conditions, in $C(\overline{\Omega}_0)$ (see Theorem C.3.6(v)). Using the estimates (12.1.11) and (12.1.26) it is easy to show that

$$\begin{aligned}|(T_{n_k}(t))f(x) - f(x_0)| &= |v_{n_k}(t,x) - f(x_0)|\\ &\le |(T_0(t)(\vartheta f))(x) - f(x_0)| + C_1\int_0^t s^{-\frac{1}{2}}e^{c_0(t-s)}ds,\end{aligned}$$

for any $x\in\Omega_1$ and some positive constant C_1. Letting k go to $+\infty$, we easily see that u is continuous at $(0,x_0)$.

In conclusion, we have proved that $u\in C([0,+\infty)\times\overline{\Omega})\cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\overline{\Omega})$ and it is a solution of the Cauchy-Neumann problem (12.0.1). Moreover, since u is bounded in $[0,T]\times\Omega$ for any $T>0$ (see (12.1.13)), by Theorem 12.1.5, u is the unique bounded classical solution to the problem (12.0.1).

Now, we prove that the whole sequence $T_n(\cdot)f$ converges to u in $C^{1,2}(F)$ for any compact set $F\subset(0,+\infty)\times\overline{\Omega}$. For this purpose, we observe that, applying all the above arguments to any subsequence $\{T_{n_h}(\cdot)f\}$, we can show that there exists a subsequence $\{T_{n_{h_k}}(\cdot)f\}$ converging to u in $C^{1,2}(F)$ for any compact set F. This, of course, implies that the whole sequence $\{T_n(\cdot)f\}$ converges to u in $C^{1,2}(F)$, and we are done.

Now, if we set $(T(t)f)(x) := u(t,x)$ for any $t\in(0,+\infty)$, any $x\in\overline{\Omega}$ and any $f\in C_b(\overline{\Omega})$, where u is the solution to the problem (12.0.1) satisfying the initial condition $u(0,\cdot) = f$, it is immediate to check that $\{T(t)\}$ is a semigroup of the linear operators in $C_b(\overline{\Omega})$.

Finally, letting n go to $+\infty$ in (12.1.14), we deduce that $T(t)f$ satisfies (12.1.12) in $(0,T)$. Then, we can use the semigroup law to extend it to all the positive t (we refer the reader to the proof of Theorem 6.1.7 for the details). ∎

Our purpose now consists in showing that the more f is smooth, the more the behaviour near $t=0$ of the function $DT(t)f$ is good. More precisely, we want to show that

$$\|DT(t)f\|_\infty \le Ce^{c_0\frac{t}{2}}\|f\|_{C^1_b(\overline{\Omega})}, \qquad t>0, \tag{12.1.27}$$

for any $f \in C^1_\nu(\overline{\Omega})$. Unfortunately, it is not immediate to deduce (12.1.27) from (12.1.15), since this latter estimate has been proved for the functions f whose normal derivative vanish on $\partial\Omega_n$. To overcome such a difficulty, we use an approximation argument: given $f \in C^1_\nu(\overline{\Omega})$ we approximate it by a sequence $\{f_n\} \subset C^1_b(\overline{\Omega})$ for which (12.1.15) holds for any $n \in \mathbb{N}$. Then, to conclude that (12.1.27) holds, we need to show that $DT(t)f_n$ converges to $DT(t)f$ as n tends to $+\infty$ for any $t>0$, at least pointwise. This convergence result is proved in the next proposition which shows us that $\{T(t)\}$ satisfies the same continuity properties as the semigroup defined in Chapter 2.

Proposition 12.1.7 *If $\{f_n\} \subset C_b(\overline{\Omega})$ is a bounded sequence converging pointwise in Ω to a function $f \in C_b(\overline{\Omega})$, then $T(\cdot)f_n$ converges, as n tends to $+\infty$, to $T(\cdot)f$ in $C^{1,2}([\varepsilon,T]\times\overline{\Omega}')$ for any $0<\varepsilon<T$ and any bounded set $\Omega' \subset \Omega$.*

Further, if $\{f_n\}$ converges to f uniformly on compact subsets of $\overline{\Omega}$, then $T(\cdot)f_n$ converges to $T(\cdot)f$ uniformly in $[0,T]\times\overline{\Omega}'$ for any $T>0$ and any bounded set $\Omega' \subset \Omega$.

Finally $T(t)f$ can be represented in the form

$$(T(t)f)(x) = \int_\Omega f(y)p(t,x;dy), \quad t>0,\ x\in\overline{\Omega}, \tag{12.1.28}$$

where $p(t,x;dy)$ is a positive finite Borel measure on Ω. In the case when $c\equiv 0$, $p(t,x;dy)$ is a probability measure for any $t>0$ and any $x\in\overline{\Omega}$.

Proof. Throughout the proof, without loss of generality, we assume that $f\equiv 0$. To prove the first part of the proposition it suffices to adapt the arguments in the proof of Theorem 12.1.3. So, we limit ourselves to sketching the proof, pointing out the main differences.

Let $\{f_n\}$ be a bounded sequence in $C_b(\overline{\Omega})$ converging pointwise to zero in Ω, and set $u_n = T(\cdot)f_n$. Applying the Schauder estimate (C.1.19) and the maximum principle to the sequence $\{u_n\}$, we can extract a subsequence $\{u_{n_k}\}$ converging in $C^{1,2}([\varepsilon,T]\times\overline{\Omega}')$ to a function $u \in C^{1,2}((0,+\infty)\times\overline{\Omega})$, for any $0 < \varepsilon < T$ and any bounded set $\Omega' \subset \Omega$. Moreover, the function u is a bounded solution of the equation $D_t u - \mathcal{A}u = 0$ in $(0,+\infty) \times \Omega$, and its normal derivative vanishes on the boundary of Ω. To conclude that $u \equiv 0$, by Theorem 12.1.5 it suffices to show that u is continuous up to $t = 0$ and that $u(0,\cdot) = 0$. This can be obtained by a localization argument. So, we fix $x_0 \in \overline{\Omega}$. Moreover, we consider (as in the proof of Theorem 12.1.3) two open neighborhoods $U_1 \subset U_0$ of x_0 such that $\Omega_0 := U_0 \cap \Omega$ is sufficiently smooth. Further, we introduce a smooth function $\vartheta \in C^\infty(\overline{\Omega}_0)$ such that $\vartheta \equiv 0$ in a neighborhood of $\Omega \cap \partial U_0$, $\vartheta \equiv 1$ in $U_1 \cap \Omega$ and $\partial\vartheta/\partial\nu = 0$ in $U_0 \cap \partial\Omega$. Finally, we set $v_{n_k} = \vartheta\, u_{n_k}$. Then, we can write

$$v_{n_k}(t,\cdot) = T_0(t)(\vartheta f_{n_k}) + \int_0^t T_0(t-s)\psi_{n_k}(s,\cdot)ds, \qquad t > 0,$$

where $\{T_0(t)\}$ is the semigroup generated by the realization of $\mathcal{A}$ in $C(\overline{\Omega}_0)$ with homogeneous Neumann boundary conditions (see Theorem C.3.6(v)) and

$$\psi_{n_k} = -u_{n_k}(\mathcal{A} - c)\vartheta - 2\sum_{i,j=1}^N q_{ij} D_i u_{n_k} D_j \vartheta.$$

Using the gradient estimate (12.1.12) and the boundedness of $\{f_{n_k}\}$ it follows that

$$|v_{n_k}(t,x)| \le |(T_0(t)\vartheta f_{n_k})(x)| + C\sqrt{t}, \qquad t \in [0,T],\ x \in \overline{\Omega}_0, \quad k \in \mathbb{N}, \tag{12.1.29}$$

where $C > 0$ is a constant independent of $k \in \mathbb{N}$. We now claim that $T_0(t)(\vartheta f_{n_k})$ vanishes as k tends to 0 for any $t \in [0,T]$. Once the claim is proved, letting k go to $+\infty$ in (12.1.29), we will get $|u(t,x)| \le C\sqrt{t}$ and, consequently, u will turn out to be continuous at $(0,x_0)$, where it will vanish. To prove the claim, we take advantage of the L^p-theory. We observe that, for any $p \in (N,+\infty)$, the semigroup $\{T_0(t)\}$ extends to an analytic semigroup in $L^p(\Omega_0)$ (see Theorem C.3.6(iii)), and, by the Sobolev embedding theorem (see [2, Theorem 5.4]), the domain of its generator A_p is continuously embedded in $C(\overline{\Omega}_0)$. Since ϑf_{n_k} converges to zero in $L^p(\Omega_0)$ and $\|T_0(t)(\vartheta f_{n_k})\|_{D(A_p)} \le Ct^{-1}\|f_{n_k}\|_{L^p}$ for some positive constant C, independent of k, we deduce that $T_0(t)(\vartheta f_{n_k})$ converges to 0 in $D(A_p)$ and, consequently, uniformly in $\overline{\Omega}$, as k tends to $+\infty$. So far, we have proved that the subsequence u_{n_k} converges to zero in $C^{1,2}([\varepsilon,T] \times \overline{\Omega}')$ for any $0 < \varepsilon < T$ and any bounded $\Omega' \subset \Omega$. As in the proof of Theorem 12.1.3 we can now prove that the whole sequence $\{u_n\}$ converges to zero in $C^{1,2}([\varepsilon,T] \times \overline{\Omega}')$ for any $0 < \varepsilon < T$ and any bounded set $\Omega' \subset \Omega$, as stated.

Suppose now that $\{f_n\}$ converges to zero uniformly on compact subsets of $\overline{\Omega}$. By (12.1.29), with v_n instead of v_{n_k}, we get

$$|u_n(t,x)| \le \|T_0(t)(\vartheta f_n)\|_\infty + C\sqrt{t} \le e^{c_0 T}\|\vartheta f_n\|_\infty + C\sqrt{t}, \quad t \in [0,T],\ x \in \overline{\Omega}_1,$$

for any $n \in \mathbb{N}$, where $C > 0$ does not depend on $n \in \mathbb{N}$. Therefore, for any $\varepsilon > 0$, we have

$$\|u_n\|_{C([0,T]\times\overline{\Omega}_1)} \le e^{\varepsilon c_0}\|\vartheta f_n\|_\infty + C\sqrt{\varepsilon} + \|u_n\|_{C([\varepsilon,T]\times\overline{\Omega}_1)}.$$

Since u_n and ϑf_n vanish as n tends to $+\infty$, respectively in $[\varepsilon, T]\times\overline{\Omega}$ and in $\overline{\Omega}$, we get

$$\limsup_{n\to\infty}\|u_n\|_{C([0,T]\times\overline{\Omega}_1)} \le C\sqrt{\varepsilon},$$

which, of course, implies that u_n converges to zero uniformly in $[0,T]\times\overline{\Omega}_1$. Since Ω_1 is arbitrary, the conclusion follows.

We can now prove (12.1.28). By the Riesz representation theorem, for any $x \in \overline{\Omega}$, there exists a positive finite Borel measure $p(t,x;dy)$ in Ω such that (12.1.28) holds for any $f \in C_0(\Omega)$. Now, if $f \in C_b(\overline{\Omega})$, we consider a bounded sequence $\{f_n\} \subset C_0(\Omega)$ which converges to f uniformly on the compact sets of Ω. Writing (12.1.28) with f_n instead of f and letting n go to $+\infty$, we obtain the desired formula, by dominated convergence.

To conclude the proof, we observe that, if $c \equiv 0$, then $T(t)\mathbb{1} \equiv \mathbb{1}$ and, consequently, $\int_\Omega p(t,x;dy) = 1$ for any $t > 0$ and any $x \in \overline{\Omega}$. This implies that $p(t,x;dy)$ are all probability measures. ∎

We can now prove (12.1.27).

Proposition 12.1.8 *Under Hypotheses* 12.1.1*, there exists a positive constant C such that* (12.1.27) *holds for any $f \in C^1_\nu(\overline{\Omega})$.*

Proof. To prove (12.1.27) we fix $f \in C^1_\nu(\overline{\Omega})$ and, for any $k \in \mathbb{N}$, we consider a function $\vartheta_k \in C^1_c(\overline{\Omega})$ such that

$$0 \le \vartheta_k \le 1, \quad \|D\vartheta_k\|_\infty \le L, \quad \vartheta_k \equiv 1 \ \text{ in } \Omega_k, \quad \frac{\partial\vartheta_k}{\partial\nu} = 0 \ \text{ on } \partial\Omega,$$

where $L > 0$ is a constant independent of $k \in \mathbb{N}$ and Ω_k is as in (12.1.7). If n_0 is such that $\mathrm{supp}(\vartheta_k) \subset \overline{\Omega}_{n_0}$, then, for any $n > n_0$, the function $f_k = \vartheta_k f$ belongs to $C^1_\nu(\overline{\Omega}_n)$. Thus, $T_n(t)f_k$ satisfies the estimate (12.1.15). Since, according to Theorem 12.1.3, $T_n(t)f_k$ tends to $T(t)f_k$ in $C^1(\overline{\Omega}_n)$ for any $n \in \mathbb{N}$, letting n go to $+\infty$, we get

$$|(DT(t)f_k)(x)| \le C_1 e^{\frac{c_0}{2}t}\|f_k\|_{C^1_b(\overline{\Omega})} \le C_1 e^{\frac{c_0}{2}t}(1+L)\|f\|_{C^1_b(\overline{\Omega})},$$

for any $t > 0$ and $x \in \Omega$. Now, using Proposition 12.1.7, we obtain that $DT(t)f_n(x)$ converges to $(DT(t)f)(x)$ for any $t > 0$ and any $x \in \Omega$. Therefore, (12.1.27) follows with $C = C_1(1+L)$. ∎

12.2 Some consequences of the uniform gradient estimates

In this section we show some useful consequences of the uniform gradient estimates proved in Section 12.1. To begin with, we observe that, using (12.1.28), it is easy to see that the semigroup can be extended to the space $B_b(\Omega)$ of all the bounded Borel functions $f : \Omega \to \mathbb{R}$. We still denote by $\{T(t)\}$ such a semigroup. Moreover, as the following proposition shows, the so obtained semigroup is irreducible (i.e., $(T(t)\chi_U)(x) > 0$ for any $t > 0$, any $x \in \Omega$ and any open set $U \subset \Omega$) and it is strong Feller (i.e., it maps $B_b(\Omega)$ into $C_b(\overline{\Omega})$).

Proposition 12.2.1 *The semigroup $\{T(t)\}$ is irreducible and strong Feller.*

Proof. To begin with, let us prove that $\{T(t)\}$ is irreducible. Fix an open set $U \subset \Omega$. We are going to prove that $(T(t)\chi_U)(x) > 0$ for any $t > 0$ and any $x \in \overline{\Omega}$. For this purpose, we approximate χ_U by a sequence of nonnegative continuous functions f_n converging monotonically to χ_U and such that f_1 does not identically vanish in Ω. Then, according to the representation formula (12.1.28), $(T(t)f_n)(x)$ converges monotonically to $(T(t)\chi_U)(x)$ as n tends to $+\infty$, for any $t > 0$ and any $x \in \overline{\Omega}$. To conclude that $T(\cdot)\chi_U > 0$, it suffices to show that $T(t)f_1 > 0$ for any $t > 0$. By contradiction, let us suppose that there exist $t_0 > 0$ and $x_0 \in \Omega$ such that $(T(t_0)f_1)(x_0) = 0$. The nonnegativity of the semigroup $\{T(t)\}$ immediately implies that (t_0, x_0) is a minimum point of the function $T(\cdot)f_1$. Therefore, according to Proposition C.2.3(iii), $T(\cdot)f_1 \equiv 0$ for any $t \in (0, t_0]$ and any $x \in \Omega'$, where Ω' is any bounded open subset of Ω, which contains the point x_0. In particular, taking $t = 0$, we get $f_1 \equiv 0$ in Ω' and hence $f_1 \equiv 0$ in Ω, which, of course, cannot be the case.

Similarly, if $(T(t_0)f_1)(x_0) = 0$ at some point $(t_0, x_0) \in (0, +\infty) \times \partial\Omega$, we can find out a bounded open set Ω' of class $C^{2+\alpha}$ such that $x_0 \in \partial\Omega$ and $\Omega' \cap (x_0 + B(r)) \subset \Omega \cap (x_0 + B(r))$ for some $r > 0$. Then, according to Proposition D.0.5, Ω' satisfies the interior sphere condition, so that $\partial T(t_0)f/\partial\nu > 0$ at $x = x_0$ (see Proposition C.2.3(iii)), which, of course, cannot be the case.

To prove that the semigroup $\{T(t)\}$ is strong Feller, we adapt the technique of [37, Theorem 3.2], which strongly relies on the gradient estimate (12.0.4). For this purpose, we fix $f \in B_b(\Omega)$ and introduce a bounded sequence $\{f_n\} \in C_b(\overline{\Omega})$ converging pointwise to f. By (12.0.4), for any $t > 0$ and any $x_0 \in \overline{\Omega}$, we can determine a constant $C = C(t)$, such that

$$\|DT(t)f_n\|_\infty \le C, \qquad n \in \mathbb{N},$$

that, due to the convexity of Ω, implies that $T(t)f_n$ is a Lipschitz continuous function, uniformly with respect to $n \in \mathbb{N}$. Letting n go to $+\infty$, we deduce that $T(t)f$ is Lipschitz continuous as well. This completes the proof. ∎

Remark 12.2.2 In fact, in the proof of Proposition 12.2.1 we have shown that if $f \in C_b(\overline{\Omega})$ is a nonnegative function which does not identically vanish in Ω, then $T(t)f > 0$ in $\overline{\Omega}$ for any $t > 0$.

We now show that, if $f \in C^1_\nu(\overline{\Omega})$, then the gradient of $T(\cdot)f$ is continuous at $t = 0$.

Proposition 12.2.3 *If $f \in C^1_\nu(\overline{\Omega})$ then*

$$\lim_{t\to 0^+} \|T(t)f - f\|_{C^1_b(\overline{\Omega'})} = 0, \tag{12.2.1}$$

for any open bounded set $\Omega' \subset \Omega$, i.e., $DT(\cdot)f$ is continuous up to $\{0\} \times \overline{\Omega}$.

Proof. The proof is similar to that of Theorem 12.1.3. We fix $x_0 \in \overline{\Omega}$ and introduce two open neighborhoods U_0, U_1 of x_0 with $U_1 \subset U_0$. Moreover, we set $\Omega_0 = U_0 \cap \Omega$, $\Omega_1 = U_1 \cap \Omega$ and we assume that Ω_0 is sufficiently smooth. Finally, we denote by $\vartheta \in C^\infty(\overline{\Omega}_0)$ any function vanishing in a neighborhood of $\Omega \cap \partial U_0$ and such that $\vartheta \equiv 1$ in Ω_1, $\partial\vartheta/\partial\nu \equiv 0$ on $\partial\Omega \cap U_0$.

The same arguments as in the proof of Theorem 12.1.3 show that

$$v_n(t,\cdot) := \vartheta\, T(\cdot)f = T_0(t)(\vartheta f) + \int_0^t T_0(t-s)\psi(s,\cdot)ds,$$

where $\{T_0(t)\}$ is the semigroup generated by the realization of the operator $\mathcal{A}$ in $C(\overline{\Omega}_0)$ with homogeneous Neumann boundary conditions, and

$$\psi = -(T(\cdot)f)(\mathcal{A} - c)\vartheta - 2\sum_{i,j=1}^N q_{ij}(D_i T(\cdot)f)D_j\vartheta.$$

According to Proposition 12.1.8, for any $T > 0$ there exists a constant $C_1 > 0$ such that

$$\|\psi(t)\|_\infty \le C_1\|f\|_{C^1_\nu(\overline{\Omega})}, \qquad t \in (0,T]. \tag{12.2.2}$$

Moreover, since $\{u \in C^1(\overline{\Omega}_0) :\ \partial u/\partial\nu = 0 \text{ on } \partial\Omega_0\}$ belongs to the class $J_{1/2}$ between $C_b(\overline{\Omega}_0)$ and the domain of $\{T_0(t)\}$ (see Theorem C.3.6(v)), we easily deduce that

$$\|DT_0(t)g\|_\infty \le \frac{C_2}{\sqrt{t}}\|g\|_{C(\overline{\Omega}_0)} \le \frac{C_2}{\sqrt{t}}\|g\|_\infty, \qquad t \in (0,T], \tag{12.2.3}$$

for any $g \in C_b(\overline{\Omega})$, where $C_2 > 0$ is a suitable constant. Now, from (12.2.2) and (12.2.3) we get

$$\begin{aligned}
&|Dv(t,x) - Df(x_0)| \\
&= \left|(DT_0(t)(\vartheta f))(x) + \int_0^t (DT_0(t-s)\psi(s,\cdot))(x)ds - Df(x_0)\right| \\
&\le |DT_0(t)(\vartheta f)(x) - Df(x_0)| + C_1\|f\|_{C^1_b(\overline{\Omega})}\int_0^t \frac{C_2}{\sqrt{t-s}}ds,
\end{aligned}$$

for any $t \in (0,T)$ and any $x, x_0 \in U_0$. Since $DT_0(\cdot)(\vartheta f)$ is continuous at $(0, x_0)$, then Dv is continuous at $(0, x_0)$ as well. Finally, since $v = T(\cdot)f$ in Ω_1, we conclude that $DT(t)f$ is continuous at $(0, x_0)$, and (12.2.1) easily follows. ■

We now use the previous gradient estimates to characterize the weak generator $\widehat{A}$ of the semigroup $\{T(t)\}$ and to solve the elliptic problem

$$\begin{cases} \lambda u - \mathcal{A}u, & \text{in } \Omega,\\[1ex] \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{12.2.4}$$

when $f \in C_b(\overline{\Omega})$.

As in the case of the Dirichlet problem in the whole $\mathbb{R}^N$, the family of bounded operators $\{R(\lambda)\}$ defined by

$$(R(\lambda)f)(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)dt, \qquad x \in \overline{\Omega},$$

for any $f \in C_b(\overline{\Omega})$ and any $\lambda > c_0$, is the resolvent family associated with some closed operator $\widehat{A} : D(\widehat{A}) \subset C_b(\overline{\Omega}) \to C_b(\overline{\Omega})$. So from now on, we write $R(\lambda, \widehat{A})$ instead of $R(\lambda)$. Adapting the proofs in Section 2.3, it can be shown that the weak generator $(\widehat{A}, D(\widehat{A}))$ is given by

$$D(\widehat{A}) = \Bigg\{ f \in C_b(\overline{\Omega}) : \sup_{t\in(0,1)} \frac{\|T(t)f - f\|_\infty}{t} < +\infty \text{ and } \exists g \in C_b(\overline{\Omega}) \text{ s.t. } \lim_{t\to 0^+} \frac{(T(t)f)(x) - f(x)}{t} = g(x), \text{ for any } x \in \overline{\Omega} \Bigg\},$$

$$\widehat{A}f(x) = \lim_{t\to 0^+} \frac{(T(t)f)(x) - f(x)}{t}, \quad x \in \overline{\Omega}, \qquad f \in D(\widehat{A}).$$

Our aim now is to show that $D(\widehat{A}) = D(\mathcal{A})$ and $\widehat{A} = \mathcal{A}$, where

$$D(\mathcal{A}) = \Bigg\{ u \in C_b(\overline{\Omega}) \cap \bigcap_{1\le p<+\infty} W^{2,p}(\Omega \cap B(R)) \text{ for any } R > 0 : \mathcal{A}u \in C_b(\overline{\Omega}), \ \frac{\partial u}{\partial \nu}(x) = 0, \text{ for any } x \in \partial\Omega \Bigg\}. \tag{12.2.5}$$

For this purpose, we first prove the following maximum principle for the elliptic problem (12.2.4).

Proposition 12.2.4 *Let $u \in C_b(\overline{\Omega}) \cap W^{2,p}(\Omega \cap B(R))$, for any $R > 0$ and any $p \in [1, +\infty)$, be such that $\mathcal{A}u \in C_b(\overline{\Omega})$ and*

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) \le 0, & x \in \Omega,\\[1ex] \dfrac{\partial u}{\partial \nu}(x) \le 0, & x \in \partial\Omega, \end{cases}$$

for some $\lambda \geq \lambda_0$*, where* λ_0 *is as in* (12.1.6)*. Then* $u \leq 0$*. In particular, for any* $\lambda \geq \lambda_0$ *there exists at most one solution* $u \in D(\mathcal{A})$ *to the problem* (12.2.4)*.*

Proof. As in the proof of Theorem 12.1.5, we may assume that $\varphi \geq 0$ and $\mathcal{A}\varphi - \lambda_0\varphi \leq 0$. We introduce the sequence

$$u_n(x) = u(x) - \frac{1}{n}\varphi(x), \qquad x \in \Omega,$$

and we note that

$$\begin{cases} \lambda u_n(x) - \mathcal{A}u_n(x) \leq 0, & x \in \Omega, \\ \dfrac{\partial u_n}{\partial \nu}(x) \leq 0, & x \in \partial\Omega. \end{cases} \tag{12.2.6}$$

If we prove that $u_n \leq 0$ for any $n \in \mathbb{N}$, the conclusion will follow letting n go to $+\infty$. By contradiction, suppose that u_n has a maximum point at $x_n \in \overline{\Omega}$. Adapting the second proof of Lemma 4.1.2 to this situation, we can show that if $x_n \in \Omega$, then $((\mathcal{A} - c)u_n)(x_n) \leq 0$. Thus, by (12.2.6) we have $(\lambda - c(x_n))u_n(x_n) \leq 0$ and, hence, $u_n(x_n) \leq 0$.

Suppose now that $x_n \in \partial\Omega$ is such that $u_n(x_n) > 0$ and $u_n(x) < u_n(x_n)$ for any $x \in \Omega$. These assumptions will lead us to a contradiction since they will imply that $(\partial u_n/\partial\nu)(x_n) > 0$.

Let $y + B(r) \subset \Omega$ be such that $(y + \overline{B}(r)) \cap \partial\Omega = \{x_n\}$ and assume that $u_n > 0$ in $y + B(r)$. Moreover, fix $\alpha > 0$ such that the function $z : \mathbb{R}^N \to \mathbb{R}$, defined by $z(x) = e^{-\alpha|x-y|^2} - e^{-\alpha r^2}$ for any $x \in \mathbb{R}^N$, satisfies $\mathcal{A}z > 0$ in $D = \{x \in \mathbb{R}^N : r/2 < |x - y| < r\}$. Then, set $w_n = u_n + \varepsilon z$, where $\varepsilon > 0$ is such that

$$w_n(x) < u_n(x_n), \qquad x \in y + \partial B(r/2). \tag{12.2.7}$$

A straightforward computation shows that

$$\mathcal{A}w_n(x) = \mathcal{A}u_n(x) + \epsilon\mathcal{A}z(x) > \lambda u_n(x) > 0, \qquad x \in D. \tag{12.2.8}$$

Let $\hat{x}_n$ be the maximum point of w_n in $\overline{D}$. $\hat{x}_n$ belongs to ∂D (this is a classical result that can be obtained arguing as in the proof of Lemma 4.1.2) and, due to (12.2.7), it belongs to $y + \partial B(r)$. Since $z \equiv 0$ on $y + \partial B(r)$ and x_n is a maximum point for the function u_n in Ω, it follows that $w_n(\hat{x}_n) = w_n(x_n)$. Therefore, x_n is a maximum point of w_n in $\overline{D}$ as well. As a consequence,

$$\frac{\partial w_n}{\partial \nu}(x_n) = \frac{\partial u_n}{\partial \nu}(x_n) + \varepsilon\frac{\partial z}{\partial \nu}(x_n) \geq 0.$$

Since $\partial z/\partial\nu(x_n) < 0$, it follows that $(\partial u_n/\partial\nu)(x_n) > 0$. ∎

Proposition 12.2.5 *For any* $f \in C_b(\overline{\Omega})$ *and any* $\lambda > c_0$ *(see* (12.1.4)*), the function* $u = R(\lambda, \widehat{A})f$ *belongs to* $D(\mathcal{A})$ *and solves the problem* (12.2.4)*. Moreover,* $D(\widehat{A}) = D(\mathcal{A})$ *and* $\widehat{A}v = \mathcal{A}v$*, for any* $v \in D(\mathcal{A})$*. Finally,* $D(\mathcal{A}) \subset C^1_\nu(\overline{\Omega})$

with a continuous embedding and, for any $\omega > c_0$, there exists a positive constant M_ω such that

$$\|Du\|_\infty \leq M_\omega \, \|u\|_\infty^{\frac{1}{2}} \, \|(\mathcal{A} - \omega)\, u\|_\infty^{\frac{1}{2}}, \tag{12.2.9}$$

for any $u \in D(\mathcal{A})$.

Proof. Let $f \in C_b(\overline{\Omega})$ and, for any $\lambda > c_0$, set $u = R(\lambda, \widehat{A})f$ and $u_n = R(\lambda, A_n)f$ ($n \in \mathbb{N}$), where A_n is the generator of the semigroup $\{T_n(t)\}$ (see (12.1.8), (12.1.9)). Of course,

$$u_n(x) = \int_0^{+\infty} e^{-\lambda t}(T_n(t)f)(x)dt, \quad x \in \overline{\Omega}_n. \tag{12.2.10}$$

Therefore, for any $\lambda > c_0$ and any $n \in \mathbb{N}$, it holds that

$$\|u_n\|_\infty \leq \frac{1}{\lambda - c_0}\|f\|_\infty, \qquad \|\mathcal{A}u_n\|_\infty \leq \left(1 + \frac{\lambda}{\lambda - c_0}\right)\|f\|_\infty.$$

Since $T_n(\cdot)f$ tends to $T(\cdot)f$ locally uniformly in $(0, +\infty) \times \Omega$ (see Theorem 12.1.3), from (12.2.10) taking (12.1.11) into account, we immediately see that

$$\lim_{n \to +\infty} u_n = u,$$

pointwise in $\overline{\Omega}$ and in $L^p(\Omega_k)$, for any $k \in \mathbb{N}$. Furthermore, by the L^p-estimates in Theorem C.1.2, we have

$$\|u_m - u_n\|_{W^{2,p}(\Omega_k)} \leq c(p,k)\|u_m - u_n\|_{L^p(\Omega_{k+1})}, \qquad m, n > k,$$

for any $p \in (1, +\infty)$ and any $k \in \mathbb{N}$, where $c(p,k) > 0$ is a constant independent of m, n. Therefore, u_n converges to u in $W^{2,p}(\Omega_k)$, for any $k \in \mathbb{N}$ and any $p \in [1, +\infty)$. Hence, $u \in W^{2,p}(\Omega \cap B(R))$, for any $R > 0$ and any $p \in [1, +\infty)$. The Sobolev embedding theorems (see [2, Theorem 5.4]) imply that u_n converges to u also in $C^1(\overline{\Omega}_k)$ for any $k \in \mathbb{N}$, and, therefore, $\partial u/\partial \nu = 0$ on $\partial\Omega$. Finally, since $\lambda u_n - \mathcal{A}u_n = f$ for any $n \in \mathbb{N}$, letting n go to $+\infty$, it follows that $\lambda u - \mathcal{A}u = f$ in Ω. Therefore, u belongs to $D(\mathcal{A})$ and solves the problem (12.2.4). Since $R(\lambda, \widehat{A})$ is surjective from $C_b(\overline{\Omega})$ onto $D(\widehat{A})$, we deduce that $D(\widehat{A}) \subset D(\mathcal{A})$ and $\widehat{A} = \mathcal{A}$ in $D(\widehat{A})$.

Conversely, let $u \in D(\mathcal{A})$ and set $f = \lambda u - \mathcal{A}u \in C_b(\overline{\Omega})$, where $\lambda = 2\max(\lambda_0, c_0)$, and c_0, λ_0 are given, respectively, by (12.1.4) and (12.1.6). By the above results, the function $v = R(\lambda, \widehat{A})f$ is a bounded solution of the problem (12.2.4) as well. Proposition 12.2.4 yields that $u \equiv v$, implying, in particular, that $u \in D(\widehat{A})$.

To complete the proof, let us prove (12.2.9). For this purpose, we fix $u \in D(\mathcal{A})$, $\omega > c_0$ and $\lambda > 0$. Then, we set

$$u(x) = (R(\lambda + \omega, \widehat{A})f)(x) = \int_0^{+\infty} e^{-(\lambda+\omega)\, t}(T(t)f)(x)\, dt, \qquad x \in \overline{\Omega}, \tag{12.2.11}$$

where $f = (\lambda+\omega)\,u - \mathcal{A}u$. By virtue of the estimate (12.1.12), we may differentiate under the integral sign in (12.2.11) obtaining

$$Du(x) = \int_0^{+\infty} e^{-(\lambda+\omega)\,t}(DT(t)f)(x)\,dt, \qquad x\in\overline{\Omega}$$

and, then,

$$|Du(x)| \le C_\omega \|f\|_\infty \int_0^{+\infty} \frac{e^{-\lambda\,t}}{\sqrt{t}}dt = \frac{C_\omega\sqrt{\pi}}{\sqrt{\lambda}}\|f\|_\infty, \qquad x\in\Omega,$$

where $C_\omega > 0$ is a constant. Therefore,

$$\|Du\|_\infty \le C_\omega\sqrt{\pi}\left(\sqrt{\lambda}\|u\|_\infty + \frac{\|(\mathcal{A}-\omega)\,u\|_\infty}{\sqrt{\lambda}}\right),$$

and, taking the minimum over $\lambda > 0$, we get (12.2.9). ■

12.3 Pointwise gradient estimates and their consequences

In this section we prove some pointwise estimates for the gradient of $T(t)f$, similar to those proved in Chapter 7. We assume that $c\equiv 0$. Indeed, as it is immediately seen, some of such estimates fail if $T(t)\mathbb{1}\not\equiv\mathbb{1}$ (take $f\equiv\mathbb{1}$ in (12.3.2)) which is the case if $c\not\equiv 0$.

Proposition 12.3.1 *Assume that the coefficient c of the operator $\mathcal{A}$ identically vanishes in Ω and that Hypotheses* 12.1.1 *are satisfied with the condition* (12.1.2) *being replaced by the following one:*

$$\sum_{i,j=1}^N\left(\sum_{h=1}^N D_h q_{ij}(x)\xi_h\right)^2 \le q_0\kappa(x)|\xi|^2, \qquad x\in\Omega,\ \xi\in\mathbb{R}^N. \tag{12.3.1}$$

Then, for any $p>1$ and any $f\in C^1_\nu(\overline{\Omega})$, we have

$$|(DT(t)f)(x)|^p \le e^{\sigma_p t}(T(t)(|Df|^p))(x), \qquad t\ge 0,\ x\in\overline{\Omega}, \tag{12.3.2}$$

where $\sigma_p = pk_0 + \frac{p}{4}q_0$, if $p\ge 2$ and $\sigma_p = pk_0 + \frac{p}{4(p-1)}q_0$, if $1<p<2$.

Further, if $q_{ij}\equiv\delta_{ij}$ for any $i,j=1,\dots,N$, then (12.3.2) *can be extended also to the case when $p=1$, setting $\sigma_p = k_0$ for any $p\in[1,+\infty)$.*

Proof. The proof is similar to and even simpler than those of Theorems 7.1.2 and 7.3.1. Hence, we just sketch it in the case when $p \in (1,2]$ and the diffusion coefficients q_{ij} depend on x. For this purpose, we fix $\delta > 0$ and introduce the function $w : [0,+\infty) \times \overline{\Omega} \to \mathbb{R}$ defined by

$$w(t,x) = \left(|Du(t,x)|^2 + \delta\right)^{\frac{p}{2}}, \qquad t > 0, \quad x \in \overline{\Omega},$$

where $u = T(\cdot)f$. By Propositions 12.1.8, 12.2.3 and Theorems 12.1.3, C.1.4, $w \in C_b([0,+\infty) \times \overline{\Omega}) \cap C^{0,1}((0,+\infty) \times \overline{\Omega}) \cap C^{1,2}((0,+\infty) \times \Omega)$ and it satisfies the equation

$$D_t w(t,x) - \mathcal{A}w(t,x) = f_1(t,x) + f_2(t,x), \quad t > 0, \ x \in \Omega,$$

where

$$\begin{aligned} f_1 = {} & p\left(|Du|^2 + \delta\right)^{\frac{p}{2}-1} \\ & \times\left(\sum_{i,j,h=1}^N D_h q_{ij} D_h u D_{ij} u + \sum_{i,j=1}^N D_i b_j D_i u D_j u - \mathrm{Tr}(QD^2uD^2u)\right), \end{aligned} \tag{12.3.3}$$

$$f_2 = p(2-p)\left(|Du|^2 + \delta\right)^{\frac{p}{2}-2} \langle QD^2uDu, D^2uDu\rangle.$$

Taking (12.1.3) (with $c \equiv 0$) and (12.3.1) into account, we obtain that

$$\begin{aligned} f_1 \le {} & p\left(|Du|^2 + \delta\right)^{\frac{p-2}{2}} \\ & \times\left(\varepsilon\kappa|D^2u|^2 + \frac{q_0}{4\varepsilon}|Du|^2 + k_0|Du|^2 - \mathrm{Tr}(QD^2uD^2u)\right), \end{aligned}$$

for any $\varepsilon > 0$. On the other hand, applying (7.1.4) with $|\alpha| = |\beta| = 1$, we obtain

$$\langle QD^2uDu, D^2uDu\rangle \le |Du|^2 \mathrm{Tr}(QD^2uD^2u).$$

Then, choosing $\varepsilon = p-1$, we get

$$f_1 + f_2 \le \sigma_p\left(|Du|^2 + \delta\right)^{\frac{p-2}{2}} |Du|^2 = \sigma_p w - \delta\sigma_p\left(|Du|^2 + \delta\right)^{\frac{p-2}{2}},$$

which yields

$$D_t w - \mathcal{A}w \le \sigma_p w - (\sigma_p \wedge 0)\delta^{\frac{p}{2}}.$$

Therefore, by a comparison argument we see that

$$w(t,\cdot) \le e^{\sigma_p t} T(t)\left((|Df|^2 + \delta)^{\frac{p}{2}}\right), \qquad t > 0,$$

if $\sigma_p \geq 0$, and

$$w(t,\cdot) \leq e^{\sigma_p t} T(t)\big((|Df|^2+\delta)^{\frac{p}{2}} - \delta^{\frac{p}{2}}\big) + \delta^{\frac{p}{2}}, \qquad t>0,$$

if $\sigma_p < 0$. Then, letting δ go to 0^+, we get the assertion from Proposition 12.1.7. ∎

As it is immediately seen, the assumption (12.3.1) is rather restrictive, since it does not allow us to consider diffusion coefficients of polynomial type. As the following proposition shows, this trouble may be overcome assuming a dissipativity condition of negative type on the drift term, in order to balance the growth at infinity of the derivatives of the q_{ij}'s $(i,j=1,\ldots,N)$.

Proposition 12.3.2 *Assume that the coefficient c of the operator $\mathcal{A}$ identically vanishes in Ω. Moreover, assume that Hypotheses* 12.1.1 *but* (12.1.2) *are satisfied with k_0 being replaced by a negative function $k : \Omega \to \mathbb{R}$, and that there exist two constants $C>0$ and $\alpha \in (0,1]$ such that*

$$|Dq_{ij}(x)| \leq C(\kappa(x))^{\alpha}, \qquad x\in\Omega,\ \ i,j=1,\ldots,N$$

and

$$\widetilde{\sigma}_p := \sup_{x\in\Omega}\left(\frac{C^2N^2}{4p_0(p_0-1)\kappa_0^{1-\alpha}}(\kappa(x))^{\alpha} + k(x)\right) < +\infty. \tag{12.3.4}$$

for some $p_0 \in (1,2)$. Then, for any $p \geq p_0$,

$$|(DT(t)f)(x)|^p \leq e^{\widetilde{\sigma}_p t}(T(t)(|Df|^p))(x), \qquad t\geq 0,\ \ x\in\overline{\Omega}. \tag{12.3.5}$$

Proof. It is similar to that of Proposition 12.3.1. The only difference is in the estimate of term $\sum_{i,j,h=1}^N D_h q_{ij} D_i u D_{jh} u$ which now should be estimated as follows:

$$\left|\sum_{i,j,h=1}^N D_h q_{ij} D_h u D_{ij} u\right| \leq C\kappa^{\alpha}\left(\frac{N^2}{4\varepsilon}|Du|^2 + \varepsilon|D^2u|^2\right),$$

for any $\varepsilon > 0$ and, then, suitably choosing ε in order to make the coefficient in front of $|D^2u|^2$ in the expression of f_1+f_2 vanish. ∎

We can now use the gradient estimates in Propositions 12.3.1 and 12.3.2 to prove a second type of pointwise gradient estimates.

Proposition 12.3.3 *Under the assumptions of Proposition* 12.3.1, *for any $f\in C_b(\overline{\Omega})$ we have*

$$|(DT(t)f)(x)|^p \leq \left(\frac{\sigma_2}{2\kappa_0(1-e^{-\sigma_2 t})}\right)^{\frac{p}{2}}(T(t)(|f|^p))(x), \qquad t>0,\ \ x\in\overline{\Omega}, \tag{12.3.6}$$

for any $p \geq 2$, *and*

$$|(DT(t)f)(x)|^p \leq \frac{c_p\sigma_p t^{1-\frac{p}{2}}}{\kappa_0(1-e^{-\sigma_p t})}(T(t)(|f|^p))(x), \qquad t>0, \ x \in \overline{\Omega}, \quad (12.3.7)$$

for any $1 < p < 2$, *where* $c_p = (p(p-1))^{-p/2}$ *and* σ_p *is given by Proposition* 12.3.1. *When* $\sigma_p = 0$ *in* (12.3.6) *and* (12.3.7), *we replace* $\sigma_p/(1-e^{-\sigma_p t})$ *with* $1/t$.

The same results hold true under the assumptions of Proposition 12.3.2, *provided we take* $p \geq p_0$ *and replace everywhere* σ_p *with* $\widetilde{\sigma}_p$.

Proof. Without loss of generality, we limit ourselves to proving the assertion under the assumptions of Proposition 12.3.1.

We first prove (12.3.6). For any $\delta > 0$ and any $n \in \mathbb{N}$, we introduce the function $\Phi_n : (0,t) \to C_b(\overline{\Omega})$ defined by

$$\Phi_n(s) = T_n(s)\Big((|T_n(t-s)f|^2+\delta)^{\frac{p}{2}}\Big),$$

where $\{T_n(t)\}$ is, as usual, the semigroup introduced before Theorem 12.1.3. Moreover, we set $g_n(s) = (|T_n(t-s)f|^2+\delta)^{p/2}$ for any $s \in (0,t)$. As it is immediately seen, $g_n(s) \in D(A_n)$ for any $s \in (0,t)$ (see (12.1.9)). This implies that $A_n\Phi_n = T_n(\cdot)A_n g_n$. Hence, taking the positivity of the semigroup $\{T_n(t)\}$ into account, we get

$$\begin{aligned}
\Phi_n'(s) &= T_n(s)\left(A_n g_n(s) - p(g(s))^{1-\frac{2}{p}}T_n(t-s)f\, A_nT_n(t-s)f\right)\\
&= T_n(s)\Big(p(g_n(s))^{1-\frac{4}{p}}\Big((p-1)|T_n(t-s)f|^2+\delta\Big)\\
&\qquad\qquad \times\langle QDT_n(t-s)f, DT_n(t-s)f\rangle\Big)\\
&\geq p(p-1)\kappa_0\\
&\qquad \times T_n(s)\left((g_n(s))^{1-\frac{4}{p}}|DT_n(t-s)f|^2\left(|T_n(t-s)f|^2+\frac{\delta}{p-1}\right)\right)\\
&\geq p(p-1)\kappa_0 T_n(s)\left((g_n(s))^{1-\frac{2}{p}}|DT_n(t-s)f|^2\right). \qquad (12.3.8)
\end{aligned}$$

Now, we fix $x \in \overline{\Omega}$ and $\varepsilon \in (0,t/2)$. Integrating (12.3.8) with respect to s from ε to $t-\varepsilon$, we get

$$\begin{aligned}
&\left(T_n(t-\varepsilon)\left((|T_n(\varepsilon)f|^2+\delta)^{\frac{p}{2}}\right)\right)(x)\\
&\geq p(p-1)\kappa_0\int_\varepsilon^{t-\varepsilon}\left(T_n(s)\left((g_n(s))^{1-\frac{2}{p}}|DT_n(t-s)f|^2\right)\right)(x)ds.
\end{aligned}$$

Since $\{T_n(t)\}$ is a strongly continuous semigroup, then, for any $x \in \overline{\Omega}$, the function $\left(T_n(t-\varepsilon)\left\{(|T_n(\varepsilon)f|^2+\delta)^{\frac{p}{2}}\right\}\right)(x)$ tends to $\left(T_n(t)\left\{(|f|^2+\delta)^{\frac{p}{2}}\right\}\right)(x)$

as ε goes to 0. Therefore, from the monotone convergence theorem, we get

$$\begin{aligned}&\left(T_n(t)\left((|f|^2+\delta)^{\frac{p}{2}}\right)\right)(x)\\ &\geq p(p-1)\kappa_0\int_0^t\left(T_n(s)\left((g_n(s))^{1-\frac{2}{p}}|DT_n(t-s)f|^2\right)\right)(x)ds,\end{aligned}\qquad(12.3.9)$$

for any $x\in\Omega$. Now, arguing as in the proof of Proposition 12.3.1, it easily follows that for any $n\in\mathbb{N}$ the semigroup $\{T_n(t)\}$ satisfies

$$|(DT_n(t)f)(x)|^p\leq e^{\sigma_p t}(T_n(t)(|Df|^p))(x),\qquad t\geq 0,\ x\in\overline{\Omega},$$

where σ_p is the same as in (12.3.2). Hence, using the Hölder inequality, we can write

$$\begin{aligned}|DT_n(t)f|^p&\leq e^{\sigma_p s}T_n(s)\left(|DT_n(t-s)f|^p\right)\\ &=e^{\sigma_p s}T_n(s)\Big((g_n(s))^{\frac{p}{2}-1}|DT_n(t-s)f|^p\,(g_n(s))^{1-\frac{p}{2}}\Big)\\ &\leq e^{\sigma_p s}\left\{T_n(s)\left((g_n(s))^{1-\frac{2}{p}}|DT_n(t-s)f|^2\right)\right\}^{\frac{p}{2}}\{T_n(s)g_n(s)\}^{1-\frac{p}{2}}.\end{aligned}$$

Then, using the Young and Jensen inequalities and recalling that $(a+b)^{p/2}\leq a^{p/2}+b^{p/2}$ for any $a,b>0$, we get

$$\begin{aligned}|DT_n(t)f|^p\leq e^{\sigma_p s}\bigg\{&\frac{p}{2}\varepsilon^{\frac{2}{p}}T_n(s)\left((g_n(s))^{1-\frac{2}{p}}|DT_n(t-s)f|^2\right)\\ &+\left(1-\frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}\left(T_n(t)(|f|^p)+\delta^{\frac{p}{2}}\right)\bigg\}.\end{aligned}$$

Integrating from 0 to t and taking (12.3.9) into account gives

$$\begin{aligned}\frac{1-e^{-\sigma_p t}}{\sigma_p}|DT_n(t)f|^p\leq&\frac{\varepsilon^{\frac{2}{p}}}{2(p-1)\kappa_0}T_n(t)\left((|f|^2+\delta)^{\frac{p}{2}}\right)\\ &+t\left(1-\frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}\left(T_n(t)(|f|^p)+\delta^{\frac{p}{2}}\right),\end{aligned}$$

where, if $\sigma_p=0$, we replace the term $(1-e^{-\sigma_p t})/\sigma_p$ with t. Letting δ go to 0 and, then, minimizing with respect to $\varepsilon\in(0,+\infty)$, we deduce (12.3.6) with $\{T(t)\}$ being replaced with the semigroup $\{T_n(t)\}$. To get (12.3.6) it is now sufficient to take the limit as n goes to $+\infty$, recalling that $T_n(\cdot)f$ converges to $T(\cdot)f$ in $C^{1,2}(D)$ for any compact set $D\subset(0,+\infty)\times\overline{\Omega}$ (see the last part of the proof of Theorem 12.1.3).

To prove (12.3.7) it suffices to apply the Jensen inequality and (12.3.6) with $p=2$. Indeed, since $p(t,x;dy)$ are probability measures for any $t>0$, $x\in\overline{\Omega}$, it follows that $(T(t)(|f|^2))^{p/2} \le T(t)(|f|^p)$. Therefore,

$$
\begin{aligned}
|(DT(t)f)(x)|^p &= \left(|(DT(t)f)(x)|^2\right)^{\frac{p}{2}} \\
&\le \left(\frac{\sigma_2}{2\kappa_0(1-e^{-\sigma_2 t})}(T(t)(f^2))(x)\right)^{\frac{p}{2}} \\
&\le \left(\frac{\sigma_2}{2\kappa_0(1-e^{-\sigma_2 t})}\right)^{\frac{p}{2}} (T(t)(|f|^p))(x)
\end{aligned}
$$

and (12.3.7) follows.

Corollary 12.3.4 *For any $f \in C_b(\overline{\Omega})$ it holds that*

$$
\|DT(t)f\|_\infty \le \left(\frac{\overline{\sigma}_2}{2\kappa_0(1-e^{-\overline{\sigma}_2 t})}\right)^{\frac{1}{2}} \|f\|_\infty, \qquad t>0,
$$

if $\overline{\sigma}_2 \neq 0$, and

$$
\|DT(t)f\|_\infty \le \frac{1}{\sqrt{2\kappa_0 t}}\|f\|_\infty, \qquad t>0,
$$

if $\overline{\sigma}_2 = 0$. Here, $\overline{\sigma}_2 = \sigma_2$ under the assumptions of Proposition 12.3.1*, whereas $\overline{\sigma}_2 = \widetilde{\sigma}_2$ under the assumptions of Proposition* 12.3.2*.*

Corollary 12.3.4 gives uniform gradient estimates sharper than those of Theorem 12.1.3. Indeed, it shows that we can take $\omega = 0$ in (12.1.12). Moreover, under the assumptions of Proposition 12.3.1 it gives an explicit description of the constant C, in terms of q_0 and k_0 (see (12.1.3) and (12.3.1)). Finally, in the case when $\sigma_2 < 0$, (12.3.4) shows that the sup-norm of the gradient of $T(t)f$ vanishes exponentially as t tends to $+\infty$.

Another interesting consequence of the gradient estimate in Proposition 12.3.3 is shown in the following proposition which provides us a Liouville type theorem. We omit the proof since it can be obtained arguing as in the proof of Theorem 7.2.5.

Proposition 12.3.5 *Suppose that Proposition* 12.3.3 *holds with $\overline{\sigma}_2 \le 0$ (where $\overline{\sigma}_2$ is as in Corollary* 12.3.4*). If $f \in D(\mathcal{A})$ is such that $\mathcal{A}f = 0$, then f is constant.*

To conclude this section we show that under the same assumptions as in Section 12.1, we can prove estimates similar to (12.3.2) even if a bit weaker. In any case, they can allow to improve the uniform gradient estimate (12.1.12). In fact, they show that as t tends to $+\infty$ the sup-norm of $DT(t)f$ stays bounded.

Proposition 12.3.6 *Under Hypotheses* 12.1.1*, there exists a constant $C_p > 0$ such that*

$$
|(DT(t)f)(x)|^{\frac{p}{2}} \le C_p T(t)\left((f^2+|Df|^2)^{\frac{p}{2}}\right)(x), \qquad t>0, \quad x \in \overline{\Omega}, \tag{12.3.10}
$$

for any $f \in C^1_\nu(\overline{\Omega})$.

Proof. Since the proof is similar to that of Proposition 12.3.1, we limit ourselves to sketching it, pointing out the main differences. In the case when $p \in (1,2]$ and $f \in C^{2+\alpha}_c(\overline{\Omega})$, we introduce, for any $\delta > 0$, the function w defined by

$$w(t,x) = \left(a|u(t,x)|^2 + |Du(t,x)|^2 + \delta\right)^{\frac{p}{2}}, \qquad t > 0, \quad x \in \Omega.$$

Here $a > 0$ is a real parameter to be fixed later on. By virtue of Propositions 12.1.8 and 12.2.3, the function w is bounded and continuous in $[0,+\infty) \times \overline{\Omega}$. Moreover, by Theorem C.1.4, w is differentiable once with respect to the time variable and twice with respect to the space variables in $(0,+\infty) \times \Omega$, and it solves the differential equation $D_t w - \mathcal{A}w = f_1 + f_2$, where now

$$\begin{aligned} f_1 = p\, w^{1-\frac{2}{p}} \Bigg(& \sum_{i,j,h=1}^N D_h q_{ij} D_h u D_{ij} u + \sum_{i,j=1}^N D_i b_j D_i u D_j u \\ & -a\langle QDu, Du\rangle - \mathrm{Tr}(QD^2uD^2u) \Bigg), \end{aligned} \tag{12.3.11}$$

$$f_2 = p(2-p) w^{1-\frac{4}{p}} \langle Q(auDu + D^2uDu), auDu + D^2uDu\rangle.$$

Moreover, $\partial w/\partial\nu \le 0$ in $(0,+\infty) \times \partial\Omega$. Using the Cauchy-Schwarz inequality twice, first for the inner product induced by the matrix Q and, then, for the Euclidean one, we get

$$\begin{aligned} & \langle Q(auDu + D^2uDu), auDu + D^2uDu\rangle \\ \le & \left[\left(\mathrm{Tr}(QD^2uD^2u)\right)^{\frac{1}{2}} |Du| + a|u| \left(\langle QDu, Du\rangle\right)^{\frac{1}{2}} \right]^2 \\ \le & \left(a|u|^2 + |Du|^2\right) \left\{ \mathrm{Tr}(QD^2uD^2u) + a\langle QDu, Du\rangle \right\}. \end{aligned}$$

Hence,

$$f_2 \le p(2-p)(au^2 + |Du|^2)^{\frac{p}{2}-1} \left\{ \mathrm{Tr}(QD^2uD^2u) + a\langle QDu, Du\rangle \right\}.$$

Therefore, we get

$$\begin{aligned} D_t w - \mathcal{A}w = p\, w^{1-\frac{2}{p}} \Bigg\{ & \sum_{i,j,h=1}^N D_h q_{ij} D_h u D_{ij} u + \sum_{i,j=1}^N D_i b_j D_i u D_j u \\ & -(p-1)\left[a\langle QDu, Du\rangle + \mathrm{Tr}(QD^2uD^2u)\right] \Bigg\}. \end{aligned} \tag{12.3.12}$$

The first term in the right-hand side of (12.3.12) is immediately estimated by (12.1.22) (where we take $\varepsilon' = (p-1)$), whereas, to estimate the remaining ones, it suffices to use the dissipativity condition (12.1.3) and the ellipticity condition (12.1.1) as in the proof of Proposition 12.1.4. So, finally, we get

$$D_t w - \mathcal{A}w \le p\, w^{1-\frac{2}{p}} \left(K_p + \frac{k_0}{\kappa_0} - a(p-1) \right) \kappa\, |Du|^2,$$

where K_p is a suitable positive constant. It is now clear that we can choose a sufficiently large such that $D_t w - \mathcal{A}w \le 0$ in $(0,+\infty) \times \Omega$. Hence, from the maximum principle in Theorem 12.1.5 we get

$$(a|u(t,x)|^2 + |Du(t,x)|^2 + \delta)^{\frac{p}{2}} \le \left(T(t) \left((af^2 + |Df|^2 + \delta)^{\frac{p}{2}} \right) \right)(x),$$

for any $t > 0$ and any $x \in \Omega$. Letting δ go to 0, the assertion follows. ∎

Remark 12.3.7 Repeating verbatim the proof of Proposition 12.3.3, it can be easily checked that, for any $n \in \mathbb{N}$, the semigroup $\{T_n(t)\}$, introduced in the proof of Theorem 12.1.3, satisfies (12.3.10) with the same constant C_p.

Theorem 12.3.8 *Under Hypotheses* 12.1.1*, for any* $p \in (1,+\infty)$ *and any* $f \in C_b(\overline{\Omega})$*, we have*

$$|(DT(t)f)(x)|^p \le \left(\frac{1}{p(p-1)\kappa_0 t} + 1 \right)^{\frac{p}{2}} (T(t)(|f|^p))(x), \tag{12.3.13}$$

for any $t > 0$*. In particular, the function* $DT(\cdot)f$ *is bounded in* $(a,+\infty) \times \Omega$ *for any* $a > 0$*.*

Proof. It can be obtained arguing as in the proof of Proposition 12.3.3. The only difference is that now, using (12.3.10) with $T(t)$ replaced with $T_n(t)$, we obtain

$$\begin{aligned} |DT_n(t)f|^p \le C_p \bigg\{ & \frac{p}{2} \varepsilon^{\frac{2}{p}} T_n(s) \left[(g_n(s))^{1-\frac{2}{p}} (|T_n(t-s)f|^2 + |DT_n(t-s)f|^2) \right] \\ & + \left(1 - \frac{p}{2}\right) \varepsilon^{\frac{2}{p-2}} \left(T_n(t)|f|^p + \delta^{\frac{p}{2}} \right) \bigg\}. \end{aligned}$$

Using the Jensen inequality as in the last part of the proof of Proposition 12.3.3, we obtain

$$\begin{aligned} T_n(s) \left((g_n(s))^{1-\frac{2}{p}} |T_n(t-s)f|^2 \right) &\le T_n(s) g_n(s) \\ &\le T_n(s) \left(T_n(t-s)(|f|^p) + \delta^{\frac{p}{2}} \right) \\ &= T_n(t)(|f|^p) + \delta^{\frac{p}{2}}. \end{aligned}$$

Hence,

$$|DT_n(t)f|^p \le C_p\bigg\{\frac{p}{2}\varepsilon^{\frac{2}{p}}T_n(s)\left[(g_n(s))^{1-\frac{2}{p}}|DT_n(t-s)f|^2\right]$$
$$+\left(1-\frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}\left(T_n(t)(|f|^p)+\delta^{\frac{p}{2}}\right)$$
$$+\frac{p}{2}\varepsilon^{\frac{2}{p}}(T_n(t)(|f|^p)+\delta^{\frac{p}{2}})\bigg\},$$

and now (12.3.13) follows arguing as in the proof of Proposition 12.3.3. ■

12.4 The invariant measure of the semigroup

In this section we briefly generalize some results of Chapter 8 to the case of the invariant measure associated with the semigroup $\{T(t)\}$ considered in this chapter. To simplify the notation, throughout this section we simply write L^p_μ and $W^{k,p}_\mu$ ($k \in \mathbb{N}$, $p \in [1,+\infty)$) instead of $L^p(\Omega,\mu)$ and $W^{k,p}(\Omega,\mu)$. Moreover, we denote by $\|\cdot\|_p$ the usual norm of L^p_μ.

In view of the extension of the gradient estimates in Section 12.3 to this setting, we assume that c identically vanishes in $\overline{\Omega}$.

To begin with let us prove the following theorem.

Theorem 12.4.1 *There exists at most one invariant measure of* $\{T(t)\}$*. Such an invariant measure is absolutely continuous with respect to the Lebesgue measure and its density* ϱ *is a positive function which belongs to* $W^{1,p}_{\rm loc}(\Omega)$ *for any* $p \in (1,+\infty)$*. In particular,* ϱ *is continuous in* Ω *(not necessarily bounded).*

Proof. We split the proof into three steps.

Step 1. Here, we assume that μ is an invariant measure of $\{T(t)\}$. Repeating with slight changes the same arguments as in the proof of Theorem 8.2.2, we can easily show that μ admits a density $\varrho \in W^{1,p}_{\rm loc}(\Omega)$ (see also [21, Section 2] for more details). Moreover, using the Harnack inequality (see [142, Corollary 5.3]), we can show that

$$\sup_{x\in K}\varrho(x) \le C\inf_{x\in K}\varrho(x), \tag{12.4.1}$$

for any compact set K and some positive constant $C = C(K)$. Of course, the estimate (12.4.1) and the continuity of ϱ in Ω imply that ϱ is everywhere positive.

To conclude this step of the proof, let us show that μ is absolutely continuous with respect to the Lebesgue measure. For this purpose, let $\{\Omega_n\}$ be a sequence of bounded sets such that $\Omega_n \subset \Omega_{n+1}$ for any $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. By the above results,

$$\mu(\Omega_n) = \int_{\Omega_n} \varrho dx, \qquad n \in \mathbb{N}.$$

The Fatou lemma now implies that

$$1 = \mu(\Omega) = \int_{\Omega} \varrho dx.$$

Therefore, ϱ is everywhere positive in Ω and it belongs to $L^1(\Omega)$, i.e., μ is absolutely continuous with respect to the Lebesgue measure on the σ-algebra of the Borel set of Ω.

Step 2. We now prove that there exists at most a unique invariant measure of $\{T(t)\}$. For this purpose, we begin by observing that the semigroup $\{T(t)\}$ is ergodic, i.e.,

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t T(s)f ds = \int_{\Omega} f d\mu, \qquad f \in L^2_\mu. \tag{12.4.2}$$

To prove the formula (12.4.2), one can argue as in the proof of Proposition 8.1.11 with minor changes. Indeed, Proposition 8.1.13 holds also in this situation as it can be easily seen repeating step by step the same proof (note in particular that, using the formula (12.1.28), one can show that, if $\{f_n\}$ is a bounded sequence of bounded and continuous functions converging to a function $f \in B_b(\mathbb{R}^N)$ pointwise, then $T(\cdot)f_n$ converges to $T(\cdot)f$ pointwise in $[0, +\infty) \times \overline{\Omega}$ as n tends to $+\infty$). We need to modify the proof of Proposition 8.1.11 only to show that $\chi_A \in C = \{f \in L^2_\mu : T(t)f = f \ \mu\text{–a.e. for any } t > 0\}$ if and only if $\mu(A) = 0$ or $\mu(A) = 1$. For this purpose, suppose that $\chi_A \in C$ and $\mu(A) > 0$. Then, $T(1)\chi_A = \chi_A$ μ-almost everywhere in Ω. Therefore, $T(1)\chi_A > 0$ μ-almost everywhere in A. Since the semigroup is strong Feller (see Proposition 12.2.1), then $T(1)\chi_A \in C_b(\overline{\Omega})$. Moreover, according to Remark 12.2.2, the function $T(2)\chi_A$ is everywhere positive in Ω. Therefore, the equality $T(2)\chi_A = \chi_A$ can hold μ-almost everywhere in Ω if and only if $\mu(\Omega \setminus A) = 0$ that is if and only if $\mu(A) = 1$.

Step 3. We can now conclude the proof. Suppose that μ_1 and μ_2 are two different invariant measures of $\{T(t)\}$. Then, according to the Chacon-Ornstein theorem (see [124, Chapter 3, Section 8]), for any Borel set A we can determine two sets M_1 and M_2, with $\mu_j(M_j) = 1$ $(j = 1, 2)$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \int_0^n (T(s)\chi_A)(x) ds = \int_{\Omega} \chi_A d\mu_j = \mu_j(A), \qquad x \in M_j, \quad j = 1, 2. \tag{12.4.3}$$

Let us observe that $M_1 \cap M_2 \neq \varnothing$. Indeed, since the densities ϱ_1 and ϱ_2 of μ_1 and μ_2 (with respect to the Lebesgue measure) are positive in Ω, then, for any compact set $K \subset \Omega$, the restriction μ_j^K of the measure μ_j to K $(j = 1, 2)$ is equivalent to the Lebesgue measure (restricted to the Borel sets of K). Fix a compact set K such that $\mu_j(M_j \cap K) > 0$ $(j = 1, 2)$ and suppose, by contradiction, that $M_1 \cap M_2 = \varnothing$. Since $\mu_1(\Omega \setminus M_1) = 0$, then $\mu_1^K(M_1 \cap K) = \mu_1^K(K)$. Moreover,

$$\mu_1^K((M_1 \cup M_2) \cap K) = \mu_1^K(M_1 \cap K) + \mu_1^K(M_2 \cap K) \leq \mu_1^K(K) = \mu_1^K(M_1 \cap K).$$

Hence, $\mu_1^K(M_2 \cap K) = 0$. Since, μ_1^K and μ_2^K are equivalent to the Lebesgue measure, then $\mu_2^K(M_2 \cap K) = 0$, which, of course, cannot be the case.

Now, taking $x \in M_1 \cap M_2$, from (12.4.3), we easily deduce that $\mu_1(A) = \mu_2(A)$ and the arbitrariness of A allows us to conclude that $\mu_1 = \mu_2$. ∎

The following theorem gives a precise description of the behaviour of the function $T(t)f$ as t tends to $+\infty$, for any $f \in L^2_\mu$. For the proof, we refer the reader to [42, Theorems 3.4.2 & 4.2.1].

Theorem 12.4.2 *For any $f \in L^p_\mu$ we have*

$$\lim_{t \to +\infty} \|T(t)f - \overline{f}\|_p = 0,$$

where $\overline{f} = \int_\Omega f d\mu$.

As in the case when $\Omega = \mathbb{R}^N$ the main existence result of an invariant measure of $\{T(t)\}$ is the Khas'minskii theorem.

Theorem 12.4.3 (Khas'minskii) *Suppose that there exists a function $\varphi \in C^2(\overline{\Omega})$ such that*

$$\varphi \geq 0, \qquad \lim_{|x| \to +\infty} \mathcal{A}\varphi(x) = -\infty, \qquad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Then, the semigroup $\{T(t)\}$ admits an invariant measure μ.

Proof. It can be obtained arguing as in the proof of Theorem 8.1.20, adapting to our situation the proof of Theorems 8.1.18 and 8.1.19. ∎

From now on, we always assume that the semigroup $\{T(t)\}$ admits an invariant measure. In such a situation, as the following proposition shows, the semigroup can be extended to a strongly continuous semigroup of contractions defined in $L^p(\Omega)$ for any $p \in [1, +\infty)$. We will still denote by $\{T(t)\}$ such a semigroup.

Proposition 12.4.4 *Let μ be the invariant measure of $\{T(t)\}$. Then, the semigroup can be extended to a strongly continuous semigroup of contractions in L^p_μ for any $p \in [1, +\infty)$. Moreover, the set $D(\mathcal{A})$ (see (12.2.5)) is a core of the infinitesimal generator L_p of the semigroup in L^p_μ.*

Proof. It can be obtained arguing as in the proofs of Propositions 8.1.8 and 8.1.9. ■

We now use the gradient estimates in Section 12.3 to derive some gradient estimates for functions in L^p_μ.

Proposition 12.4.5 *Under the hypotheses of Proposition 12.3.3, $T(t)$ maps L^p_μ into $W^{1,p}_\mu$, for any $t > 0$. Moreover, under the assumptions of Proposition 12.3.1,*

$$\|DT(t)f\|_p \le \begin{cases} \left(\dfrac{c_p\sigma_p t^{1-\frac{p}{2}}}{\kappa_0(1-e^{-\sigma_p t})}\right)^{\frac{1}{p}} \|f\|_p, & p \in (1,2), \\ \left(\dfrac{\sigma_2}{2\kappa_0(1-e^{-\sigma_2 t})}\right)^{\frac{1}{2}} \|f\|_p, & p \ge 2, \end{cases} \tag{12.4.4}$$

for any $t > 0$, where c_p and σ_p are the same as in the quoted proposition. When $\sigma_p = 0$ in (12.3.6) and (12.3.7), we replace $\sigma_p/(1-e^{-\sigma_p t})$ with $1/t$.

Similarly, under the assumptions of Proposition 12.3.2 the same estimate holds for any $p \ge p_0$, provided we replace everywhere σ_p with $\widetilde{\sigma}_p$, which is given by (12.3.4).

Proof. It follows arguing as in the proof of Proposition 8.3.2, taking the pointwise gradient estimates in Proposition 12.3.3 into account. ■

As the following proposition shows, the estimates (12.4.4) can be used to give a partial characterization of the domain of the infinitesimal generator of the semigroup $\{T(t)\}$ in L^p_μ.

Proposition 12.4.6 *Let $p > 1$, under the assumptions of Proposition 12.3.1 and $p \ge p_0$, under the assumptions of Proposition 12.3.2. Then, $D(L_p) \subset W^{1,p}_\mu$ and, for any $\omega > 0$, there exists a positive constant $M_{\omega,p}$ such that*

$$\|Df\|_p \le M_{\omega,p}\|f\|_p^{\frac{1}{2}}\|(L_p-\omega)f\|_p^{\frac{1}{2}}, \qquad f \in D(L_p). \tag{12.4.5}$$

Moreover, under the assumptions of Proposition 12.3.1, the estimate (12.4.5) holds true also with $\omega = 0$, if $\sigma_p < 0$. Similarly, under the assumptions of Proposition 12.3.2, the estimate (12.4.5) holds true also with $\omega = 0$, if $\widetilde{\sigma}_p < 0$.

Proof. The proof is similar to that of Proposition 8.3.3. So we skip the details. ■

To conclude this section, we consider the case when the operator $\mathcal{A}$ is given by

$$\mathcal{A}\varphi = \Delta\varphi - \langle DU, D\varphi\rangle, \tag{12.4.6}$$

on smooth functions, under the following hypothesis on U:

Hypothesis 12.4.7 *The function U belongs to $C^2(\mathbb{R}^N)$,*

$$\lim_{|x|\to+\infty} U(x) = +\infty$$

and

$$\sum_{i,j=1}^{N} D_{ij}U(x)\xi_i\xi_j \geq 0, \qquad x \in \mathbb{R}^N,\ \ \xi \in \mathbb{R}^N.$$

Let us denote by μ the measure defined by

$$\mu(dx) = K^{-1}e^{-U(x)}dx, \qquad K = \int_\Omega e^{-U(x)}dx. \tag{12.4.7}$$

Lemma 12.4.8 *The measure μ in* (12.4.7) *is the invariant measure of* $\{T(t)\}$.

Proof. Let us fix $f \in C_b(\overline{\Omega})$ and denote by $\{T_n(t)\}$ ($n \in \mathbb{N}$) the approximating semigroup defined in Section 12.1. We know that the function $T_n(t)f$ belongs to $C^2(\overline{\Omega}_n)$ for any $t > 0$. Moreover, for any $t > 0$, $\partial T_n(t)f/\partial\nu = 0$ on $\partial\Omega_n$ and $T_n(t)f$ converges pointwise to $T(t)f$ as n tends to $+\infty$. An integration by parts shows that

$$\int_{\Omega_n} \mathcal{A}T_n(t)f d\mu = 0,$$

for any $t > 0$. Therefore,

$$\frac{d}{dt}\int_{\Omega_n} T_n(t)f d\mu = \int_\Omega \mathcal{A}T_n(t)f d\mu = 0, \qquad t > 0,$$

which implies that

$$\int_{\Omega_n} T_n(t)f d\mu = \int_{\Omega_n} f d\mu, \qquad t > 0,\ \ n \in \mathbb{N}.$$

Letting n go to $+\infty$, from the dominated convergence theorem we deduce that

$$\int_\Omega T(t)f d\mu = \int_\Omega f d\mu, \qquad t > 0,$$

which, of course, implies that μ is the invariant measure of $\{T(t)\}$. ∎

As it has been proved by G. Da Prato and A. Lunardi in [40], a precise characterization of the domain of the infinitesimal generator of $\{T(t)\}$ in L^2_μ is available, when $\mathcal{A}$ is given by (12.4.6). As in Chapter 8, we simply write L instead of L_2.

Theorem 12.4.9 *Under Hypothesis* 12.4.7*, the resolvent set* $\rho(L)$ *contains the halfline* $(0,+\infty)$ *and*

$$D(L)=\left\{u\in W^{2,2}_\mu:\ \langle DU,Du\rangle\in L^2_\mu,\ \frac{\partial u}{\partial\nu}=0 \text{ on } \partial\Omega\right\}. \tag{12.4.8}$$

Moreover,

$$\|D^kR(\lambda,L)\|_{L(L^2_\mu)}\le 2^k\lambda^{\frac{k}{2}-1},\qquad k=0,1,2. \tag{12.4.9}$$

Proof. We split the proof into two steps.

Step 1. Here, we prove that, for any $f\in L^2_\mu$ and any $\lambda>0$, the Neumann problem

$$\begin{cases}\lambda u(x)-\mathcal{A}u(x)=f(x), & x\in\Omega,\\ \dfrac{\partial u}{\partial\nu}(x)=0, & x\in\partial\Omega,\end{cases} \tag{12.4.10}$$

admits a unique solution u belonging to the space defined by the right-hand side of (12.4.8) and

$$\|D^ku\|_2\le 2^k\lambda^{\frac{k}{2}-1}\|f\|_2. \tag{12.4.11}$$

Let us first prove the uniqueness part. For this purpose, let u be a solution to the problem (12.4.10) (corresponding to $f\equiv 0$) with the claimed regularity properties. An integration by parts shows that

$$\lambda\int_\Omega u^2d\mu=\lambda\int_\Omega(\mathcal{A}u)ud\mu=-\frac{1}{2}\int_\Omega|Du|^2d\mu,$$

and, of course, this implies that $u=0$.

To prove the existence part, we approximate the operator $\mathcal{A}$ with the sequence of operators $\{\mathcal{A}_n\}$ defined as the operator $\mathcal{A}$, with U being replaced by the function U_n given by

$$U_n(x)=U(x)+\frac{n}{2}(\mathrm{dist}(x,\Omega))^2,\qquad x\in\mathbb{R}^N,\ \ n\in\mathbb{N}.$$

As it is immediately seen, the function U_n tends to $+\infty$ as $|x|$ tends to $+\infty$. Moreover, since Ω is convex, U_n is convex in $\mathbb{R}^N$. Therefore, according to Remark 8.4.4, the realization $L^{(n)}$ in $L^2_{\mu_n}$ of the operator $\mathcal{A}_n$ with domain $D(L^{(n)})=\{u\in W^{2,2}(\mathbb{R}^N,\mu_n):\langle DU_n,Du\rangle\in L^2(\mathbb{R}^N,\mu_n)\}$ generates a strongly continuous semigroup in $L^2(\mathbb{R}^N,\mu_n)$, where

$$\mu_n(dx)=K_n^{-1}e^{-U_n(x)}dx,\qquad K_n=\int_{\mathbb{R}^N}e^{-U_n(x)}dx.$$

Let us extend the function f to the whole of $\mathbb{R}^N$ by zero and let us still denote by f such an extended function. For any $\lambda>0$, the equation

$$\lambda u-\mathcal{A}_nu=f$$

admits a unique solution $u \in D(L^{(n)})$ which satisfies

$$\|D^k u_n\|_{L^2(\mathbb{R}^N,\mu_n)} \le 2^k \lambda^{\frac{k}{2}-1} \|f\|_{L^2(\mathbb{R}^N,\mu_n)}, \qquad k = 0, 1, 2; \tag{12.4.12}$$

see Theorem 8.4.2 and Corollary 8.4.3.

We now use a compactness argument to prove that, up to a subsequence, u_n converges to a solution of the problem (12.4.10) which belongs to $W^{2,2}_\mu$ and satisfies $\langle DU, Du\rangle \in L^2_\mu$. For this purpose, we begin by observing that

$$\|f\|_{L^2(\mathbb{R}^N,\mu_n)} = \left(\frac{1}{K_n}\int_\Omega f^2 e^{-U} dx\right)^{\frac{1}{2}} = \left(\frac{\int_\Omega e^{-U} dx}{\int_{\mathbb{R}^N} e^{-U_n} dx}\right)^{\frac{1}{2}} \|f\|_2.$$

Therefore, the sequence $\{\|f\|_{L^2(\mathbb{R}^N,\mu_n)}\}$ is bounded. From (12.4.12) we now deduce that the restriction of the functions u_n to Ω gives rise to a bounded sequence in $W^{2,2}_\mu$. Indeed, $\|D^k u\|_{L^2(\Omega,\mu)} \le \|D^k u\|_{L^2(\Omega,\mu_n)}$. Hence, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ which converges weakly in $W^{2,2}_\mu$ and strongly in $W^{3/2,2}(\Omega \cap B(R))$, to a function $u \in W^{2,2}_\mu$, for any $R > 0$. Moreover, we can assume that u_n converges to u pointwise a.e. in Ω. This implies that the function u solves the differential equation in (12.4.10). Since u and $\Delta u \in L^2_\mu$, then the function $\langle DU, Du\rangle$ belongs to L^2_μ. Further, (12.4.9) follows by virtue of (12.4.12).

Finally, let us prove that the normal derivative of u identically vanishes on $\partial\Omega$. Using the formula (8.1.29), we can show that

$$\begin{aligned}
\int_{\mathbb{R}^N} \langle Du_n, D\psi\rangle d\mu &= -\int_{\mathbb{R}^N} \mathcal{A}_n u_n \psi d\mu_n \\
&= -\int_{\mathbb{R}^N\setminus\Omega} (\mathcal{A}_n u_n)\psi d\mu_n - \int_\Omega (\mathcal{A}_n u_n)\psi d\mu_n \\
&= -\lambda \int_{\mathbb{R}^N\setminus\Omega} u_n \psi d\mu_n - \int_\Omega (\mathcal{A}_n u_n)\psi d\mu_n \\
&= -\lambda \int_{\mathbb{R}^N\setminus\Omega} u_n \psi d\mu_n + \int_\Omega \langle Du_n, D\psi\rangle d\mu_n \\
&\quad -\frac{1}{K_n}\int_{\partial\Omega} \frac{\partial u_n}{\partial\nu}\psi e^{-U_n} d\sigma,
\end{aligned}$$

for any $\psi \in C^\infty_c(\mathbb{R}^N)$, where $d\sigma$ is the surface measure related to $\partial\Omega$. Therefore,

$$\frac{1}{K_n}\int_{\partial\Omega} \frac{\partial u_n}{\partial\nu}\psi e^{-U_n} d\sigma = -\int_{\mathbb{R}^N\setminus\Omega} \langle Du_n, D\psi\rangle d\mu_n - \lambda\int_{\mathbb{R}^N\setminus\Omega} u_n \psi d\mu_n, \tag{12.4.13}$$

for any $\psi \in C^\infty_c(\mathbb{R}^N)$. Let us observe that the right-hand side of (12.4.13) vanishes as n tends to $+\infty$. Indeed, the sequence $\{u_n\}$ is bounded in $W^{2,2}_\mu$

and, as it is easily seen, the $W^{1,2}(\mathbb{R}^N\setminus\Omega,\mu_n)$-norm of ψ goes to 0 as n tends to $+\infty$. As far as the left-hand side in (12.4.13) is concerned, we observe that, since u_n converges to u in $W^{3/2,2}(\Omega\cap\operatorname{supp}\psi)$, then $\partial u_n/\partial\nu$ tends to $\partial u/\partial\nu$ in $L^2(\partial\Omega\cap\operatorname{supp}\psi)$ as n tends to $+\infty$. So, we have

$$\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\psi e^{-U}d\sigma=\lim_{n\to+\infty}\int_{\partial\Omega}\frac{\partial u_n}{\partial\nu}\psi e^{-U_n}d\sigma$$

from which we deduce that $\partial u/\partial\nu=0$ on $\partial\Omega$.

Finally, the formula (12.4.9) follows from (12.4.12), letting n go to $+\infty$.

Step 2. Denote by $\widehat{L}$ the realization of the operator $\mathcal{A}$ in L^2_μ with domain given by the right-hand side of (12.4.8).

Since the operator $\tilde{L}$ is dissipative and self-adjoint in L^2_μ, according to Theorem B.2.9, it generates a strongly continuous analytic semigroup $\{S(t)\}$ in L^2_μ. To conclude the proof, we need to show that $T(t)\equiv S(t)$ for any $t>0$. For this purpose, we begin by observing that $\tilde{L}$ and L coincide in $C^\infty_c(\Omega)$. To see it, it suffices to observe that $C^\infty_c(\Omega)$ is properly contained in the space defined by the right-hand side of (12.4.8). Moreover, it is also contained in $D(L)$. Indeed, according to Proposition 12.4.4, the space defined by (12.2.5) is a core of L.

Now, fix $f\in C^\infty_c(\Omega)$ and $t>0$. Moreover, let $\psi:[0,t]\to L^2_\mu$ be the function defined by $\psi(s)=S(t-s)T(s)f$ for any $s\in[0,t]$. Since $\{S(t)\}$ and $\{T(t)\}$ are both strongly continuous semigroups and $\{T(t)\}$ is also analytic, then the function ψ is continuously differentiable in $[0,t)$ with values in L^2_μ and

$$\begin{aligned}\psi'(s)&=-S(t-s)T(s)Lf+S(t-s)\widetilde{L}T(s)f\\&=-S(t-s)T(s)Lf+S(t-s)T(s)\widetilde{L}f\\&=0.\end{aligned}$$

Therefore, $\psi(s)=\psi(0)$ for any $s\in(0,t)$. Since ψ is continuous up to $s=t$, we deduce that $\psi(t)=\psi(0)$ or, equivalently, that $T(t)f\equiv S(t)f$. This completes the proof. ∎

To conclude this section we prove the Poincaré inequality.

Theorem 12.4.10 *Suppose that $\mathcal{A}$ is given by* (12.4.6) *with $U\in C^2(\mathbb{R}^N)$ satisfying*

$$\sum_{i,j=1}^N D_{ij}U(x)\xi_i\xi_j\ge d_0|\xi|^2,\qquad x\in\Omega,\ \ \xi\in\mathbb{R}^N,$$

for some negative constant d_0. Then,

$$\int_{\mathbb{R}^N}|f-\overline{f}|^2d\mu\le-\frac{1}{d_0}\int_{\mathbb{R}^N}|Df|^2d\mu,\qquad f\in W^{1,2}_\mu. \tag{12.4.14}$$

Moreover,

$$\|T(t)f - \overline{f}\|_2 \le e^{\frac{t}{d_0}} \|f - \overline{f}\|_2, \qquad t > 0,$$

for any $f \in L^2_\mu$. Finally, we have a spectral gap for L, namely

$$\sigma(L) \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \ \mathrm{Re}\ \lambda \le 1/d_0\}.$$

Proof. The Poincaré inequality can be proved arguing as in the proof of Theorem 8.6.3, taking Theorem 12.4.2 into account and observing that according to Proposition 12.3.1,

$$|(DT(t)f)(x)|^2 \le e^{2d_0 t}(T(t)(|Df|^2))(x), \qquad t > 0, \ \ x \in \Omega.$$

The second and the last part of the theorem can be proved repeating the same arguments as in the proof of Proposition 8.6.4. Hence, we omit the details. ∎

Chapter 13

The Cauchy-Neumann problem: the nonconvex case

13.0 Introduction

In this chapter we consider the case when Ω is a nonconvex unbounded open subset of $\mathbb{R}^N$, uniformly of class $C^{2+\alpha}$, for some $\alpha \in (0,1)$, and we show that, under somewhat heavier assumptions on the coefficients of the operator

$$\mathcal{A}\varphi(x) = \sum_{i,j=1}^{N} q_{ij}(x)D_{ij}\varphi(x) + \sum_{j=1}^{N} b_j(x)D_i\varphi(x) + c(x)\varphi(x), \qquad x \in \Omega, \tag{13.0.1}$$

we can recover the results proved in Chapter 12. In the convex case the keystone to prove the existence of a classical solution to the problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x), & t>0,\ x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, & t>0,\ x \in \partial\Omega, \\ u(0,x) = f(x), & x \in \overline{\Omega} \end{cases} \tag{13.0.2}$$

(i.e., a function $u \in C([0,+\infty)\times\overline{\Omega}) \cap C^{0,1}((0,+\infty)\times\overline{\Omega}) \cap C^{1,2}((0,+\infty)\times\Omega)$ which solves the problem (13.0.2) pointwise) was the *a priori* estimate (12.0.4) for the classical solution u_n to the Cauchy problem (12.1.10) in bounded domains. Such an estimate was proved by using the Bernstein method. The convexity of Ω played a crucial role since it allowed us to approximate Ω by convex bounded domains Ω_n, and the convexity of Ω_n was essential to apply the Bernstein method. Indeed, if F is a smooth convex open set, then any smooth function w defined in F, with normal derivative identically vanishing on ∂F, satisfies $\partial|Dw|^2/\partial\nu \le 0$ on ∂F (see Proposition 12.1.4). In the case when Ω is not convex, there is no a priori information on the sign of the normal derivative of $|Du_n|^2$ on $\partial\Omega$ and so the technique of Section 12.1 cannot be applied. To overcome such a difficulty, here we do not approximate the domain but we rather deal directly with the problem in the whole of Ω. Taking the geometry of Ω into account, we introduce a suitable positive function m and we prove that, if $\partial u/\partial\nu = 0$ on $\partial\Omega$, then $\partial(m|Du|^2)/\partial\nu \le 0$ on $\partial\Omega$. Once we have this preliminary result, we apply the Bernstein method to the function

$v := u^2 + atm|Du|^2$, u being the classical solution to the problem (13.0.2), and we get (12.0.4) for the function u. To apply such a technique we need to assume that u is sufficiently smooth in order to be able to differentiate v once with respect to time and twice with respect to the time variables in $(0, +\infty) \times \Omega$.

Next, we focus our attention on showing that the Cauchy-Neumann problem (13.0.2) actually admits a (unique) classical solution which satisfies the smoothness assumptions that we needed to apply the Bernstein method.

To guarantee the uniqueness of the bounded classical solution to the problem (13.0.2), we assume as in the previous Chapters, the existence of a Lyapunov function for the operator $\mathcal{A}$, which allows us to prove a maximum principle.

To prove the existence of a solution to the problem (13.0.2) with the claimed regularity, we adapt to this situation the technique of Chapter 11. For this purpose, we consider auxiliary problems in $L^p(\Omega)$, which involve suitable new operators $\mathcal{A}_\varepsilon$ ($\varepsilon > 0$) and we show that, for p large enough and f smooth enough, there exists a unique bounded classical solution u_ε to the problem (13.0.2) with $\mathcal{A}$ replaced with $\mathcal{A}_\varepsilon$. The regularity of the functions u_ε and the suitable choice of the operators $\mathcal{A}_\varepsilon$ allow us to apply the *a priori* gradient estimate proved before to the functions u_ε ($\varepsilon > 0$) obtaining that

$$\|Du_\varepsilon(t, \cdot)\|_\infty \le \frac{C_T}{\sqrt{t}} \|f\|_\infty, \qquad t \in (0, T], \tag{13.0.3}$$

for any $T > 0$ and some positive constant C_T, depending on T, but being independent of ε. At this point, we show that there exists a sequence $\{u_{\varepsilon_n}\}$ which converges to a solution u of the problem (13.0.2). Letting n go to $+\infty$ in (13.0.3), we find that u satisfies (13.0.3) as well.

A standard approximation argument yields the gradient estimate in the general case when f is bounded and continuous. Finally, the semigroup rule can be used to show that, for any $\omega > 0$, there exists a positive constant $C = C(\omega)$ such that

$$\|Du(t, \cdot)\|_\infty \le \frac{C}{\sqrt{t}} e^{\omega t} \|f\|_\infty, \qquad t > 0. \tag{13.0.4}$$

This procedure works if we assume, among other assumptions, that the diffusion coefficients are bounded in Ω with their first-order derivatives. In particular, this assumption is essential when we deal with the auxiliary problem in $L^p(\Omega)$. However, if Ω is an exterior domain, we can allow the diffusion coefficients to be unbounded in Ω. In such a situation, we approximate the operator $\mathcal{A}$ by a sequence of elliptic operators $\{\mathcal{A}_n\}$ with bounded diffusion coefficients to which we can apply the previous results, getting the existence of a unique bounded classical solution u_n satisfying the gradient estimate (13.0.4). Then, letting n go to $+\infty$, we finally see that u_n converges to the (unique) solution to the problem (13.0.2), which, of course, satisfies (12.0.4).

As a consequence of the results so far proved we can define, as usual, a semigroup $\{T(t)\}$ of linear bounded operators in the space $C_b(\overline{\Omega})$. We show that the results in Proposition 12.1.7 as well as all the results in Subsection 12.2 can be extended to the case when Ω is not convex. Concerning the pointwise estimate, we confine ourselves to the case when $c \equiv 0$. In the case when the diffusion coefficients are bounded we show that the semigroup $\{T(t)\}$ satisfies (12.0.7) with $e^{k_0 p t}$ being replaced with $C_p e^{M_p t}$ for some constants $C_p > 0$ and $M_p \in \mathbb{R}$. In the case when Ω is an exterior domain, we are able to recover the estimate (12.0.7), by assuming new conditions on the coefficients, which are slightly more restrictive than those used to prove the uniform gradient estimate. But we still allow the q_{ij}'s to be unbounded. On the other hand, still when Ω is an exterior domain, we can prove an estimate similar to (12.3.10), under the same set of hypotheses used to get (12.0.4).

Next, we derive another type of pointwise gradient estimates that, in some cases, allow us to improve the estimates on the behaviour of the sup-norm of $DT(t)f$ when t approaches infinity and $f \in C_b(\overline{\Omega})$, as well as to prove a Liouville type theorem for the operator $\mathcal{A}$.

Finally, in Section 13.4, we briefly generalize to this situation some results of Chapter 12.

The results of this chapter have been proved in [16].

13.1 The case of bounded diffusion coefficients

We begin this section by recalling that, if Ω is uniformly of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$, then the distance function

$$x \mapsto d(x) = \mathrm{dist}(x, \partial\Omega),$$

belongs to $C_b^2(\Gamma(\rho))$ for some $\rho > 0$, where

$$\Gamma(\rho) := \{x \in \overline{\Omega} : d(x) < \rho\}. \tag{13.1.1}$$

Moreover,

$$Dd(x) = -\nu(\xi(x)), \tag{13.1.2}$$

where $\xi(x)$ is the unique point on $\partial\Omega$ such that $d(x,\partial\Omega) = |x - \xi(x)|$. In particular, formula (13.1.2) implies that $|Dd(x)| = 1$ for any $x \in \Gamma(\rho)$. We refer to Appendix D for the proof of these properties concerning the distance function.

Finally, according to Remark D.0.3,

$$M_0 := \inf_{x\in\partial\Omega} \left\{ \frac{\partial \nu}{\partial \tau}(x) \cdot \tau,\ |\tau| = 1,\ \tau \cdot \nu(x) = 0 \right\} > -\infty.$$

Therefore, we can define the constant ω_0 which will be used throughout this chapter, by setting

$$\omega_0 = \max\{0, -M_0\}. \tag{13.1.3}$$

We can now state the hypotheses that we always assume to be satisfied in this section.

Hypotheses 13.1.1 (i) Ω is an open nonconvex unbounded set uniformly of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$;

(ii) the coefficients q_{ij} and b_j $(i,j = 1,\ldots,N)$ and c belong to $C^{1+\alpha}_{\rm loc}(\overline{\Omega})$. Moreover, $q_{ij} \in C^1_b(\overline{\Omega})$, for any $i,j = 1,\ldots,N$, and

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \geq \kappa_0|\xi|^2, \qquad x \in \Omega, \quad \xi \in \mathbb{R}^N, \tag{13.1.4}$$

for some positive constant κ_0;

(iii) $c(x) \leq 0$ for any $x \in \overline{\Omega}$ and there exist two constants c_0 and γ such that

$$-c_0 := \sup_{x\in\overline{\Omega}} c(x), \qquad |Dc(x)| \leq \gamma(1 - c(x)), \quad x \in \Omega;$$

(iv) there exist $k_1, k_2 \in \mathbb{R}$ and $s_1, s_2 \geq 0$ such that

$$(a) \quad \sum_{i,j=1}^N D_ib_j(x)\xi_i\xi_j \leq (k_1 - s_1c(x))|\xi|^2, \qquad x \in \Omega, \quad \xi \in \mathbb{R}^N;$$

$$(b) \quad -\sum_{j=1}^N b_j(x)D_jd(x) \leq k_2 - s_2c(x), \qquad x \in \Gamma(\rho),$$

with

$$s_1 + \frac{\omega_0}{2}s_2 < \frac{1}{2}; \tag{13.1.5}$$

(v) there exist two constants $d_1, d_2 \geq 0$ such that

$$|b(x)| \leq d_1\exp(d_2|x|), \qquad x \in \Omega;$$

(vi) there exist a function $\varphi \in C^2(\overline{\Omega})$ and a constant $\lambda_0 > c_0$ such that

$$\lim_{|x|\to+\infty}\varphi(x) = +\infty, \qquad \sup_{x\in\Omega}(\mathcal{A} - \lambda_0)\varphi(x) < +\infty, \qquad \frac{\partial\varphi}{\partial\nu}(x) \geq 0, \quad x \in \partial\Omega.$$

The requirement that $q_{ij} \in C^1_b(\overline{\Omega})$ $(i,j = 1,\ldots,N)$ as well as Hypothesis 13.1.1(v) are essential in order to prove the existence of a bounded classical solution to the problem (13.0.2). The geometry of Ω is taken into account by Hypothesis 13.1.1(iv)(b) which requires that the growth of the component of the drift along the outward normal direction is balanced by the potential in a neighborhood of $\partial\Omega$.

13.1.1 *A priori* estimate

In this subsection we are devoted to prove the following *a priori* estimate

$$|Du(t,x)| \le C_T t^{-\frac{1-k}{2}} \|f\|_{C_b^k(\overline{\Omega})}, \qquad t \in (0,T],\ x \in \overline{\Omega},\ k = 0,1, \quad (13.1.6)$$

for any $T > 0$ and any $f \in C_b^k(\overline{\Omega})$ (satisfying $\partial f/\partial\nu = 0$ if $k = 1$), when u is a bounded classical solution to the problem (13.0.2) with the further property that $Du \in C_b([0,T] \times \overline{\Omega}) \cap C^{0,1}((0,T] \times \overline{\Omega}) \cap C^{1,2}((0,T] \times \Omega)$. We will deal with the existence of such a solution in the next subsection. In order to prove (13.1.6) we adapt to our situation the classical Bernstein method. For this purpose, we introduce the function $m : \overline{\Omega} \to \mathbb{R}$ defined by

$$m(x) = 1 + \omega_0 \psi(d(x)), \qquad x \in \overline{\Omega}, \qquad (13.1.7)$$

where ω_0 is given by (13.1.3) and $\psi \in C_b^2([0,+\infty))$ is any function such that

$$0 \le \psi \le \frac{\varepsilon_0}{2}, \quad \psi(0) = 0, \quad \psi(s) = \frac{\varepsilon_0}{2}, \quad s > \varepsilon_0, \qquad (13.1.8)$$

$$\psi'(0) = 1, \quad 0 \le \psi' \le 1, \quad -\frac{2}{\varepsilon_0} \le \psi'' \le 0, \qquad (13.1.9)$$

for some $0 < \varepsilon_0 < \rho$, being ρ given by (13.1.1). As it is readily seen,

$$1 \le m \le 1 + \frac{\omega_0 \varepsilon_0}{2}$$

$$m(x) = 1 + \frac{\omega_0 \varepsilon_0}{2}, \quad x \in \Omega \setminus \Gamma(\varepsilon_0), \quad m(x) = 1, \quad x \in \partial\Omega.$$

Let us prove the following fundamental lemma.

Lemma 13.1.2 *Let $u \in C^2(\overline{\Omega})$ be such that $\partial u/\partial\nu = 0$ on $\partial\Omega$. Then, the function $v := m|Du|^2$ satisfies*

$$\frac{\partial v}{\partial \nu}(x) \le 0, \quad x \in \partial\Omega.$$

Proof. Since $Du(x) \cdot \nu(x) = 0$ for any $x \in \partial\Omega$, differentiating along a unit tangent vector τ, we obtain

$$\frac{\partial}{\partial \tau}(Du(x) \cdot \nu(x)) = D^2 u(x)\tau \cdot \nu(x) + \frac{\partial \nu}{\partial \tau}(x) \cdot Du(x) = 0, \quad x \in \partial\Omega. \ (13.1.10)$$

We now observe that, from (13.1.2), (13.1.3) and (13.1.9), it follows that, if $x \in \partial\Omega$ and $\tau \cdot \nu(x) = 0, |\tau| = 1$, then

$$\begin{aligned}\frac{\partial m}{\partial \nu}(x) - 2\frac{\partial \nu}{\partial \tau}(x) \cdot \tau &= \omega_0\, \psi'(0) Dd(x) \cdot \nu(x) - 2\frac{\partial \nu}{\partial \tau}(x) \cdot \tau \\ &= -\omega_0 - 2\frac{\partial \nu}{\partial \tau}(x) \cdot \tau \le 0. \qquad (13.1.11)\end{aligned}$$

Therefore, if we choose, for any $x \in \partial\Omega$,

$$\tau = \tau(x) = m(x)|Du(x)|^{-1}Du(x) = |Du(x)|^{-1}Du(x),$$

from (13.1.10) and (13.1.11) we get

$$\begin{aligned}\frac{\partial v}{\partial \nu}(x) &= \frac{\partial m}{\partial \nu}(x)|Du(x)|^2 + 2m(x)D^2u(x)Du(x)\cdot\nu(x)\\ &= |Du(x)|^2\left\{\frac{\partial m}{\partial \nu}(x) - 2\frac{\partial \nu}{\partial \tau}(x)\cdot\tau\right\} \le 0, \qquad x \in \partial\Omega.\end{aligned}$$

■

We can now prove the *a priori* estimate (13.1.6) requiring, for the time being, a further regularity property to the function u, that will be removed later.

Proposition 13.1.3 *Assume that Hypotheses* 13.1.1 *are satisfied and let* $u \in C([0,T]\times\overline{\Omega})\cap C^{0,1}((0,T]\times\overline{\Omega})\cap C^{1,2}((0,T]\times\Omega)$ *be a bounded solution of the Cauchy-Neumann problem* (13.0.2)*. Further, assume that the function* Du *is bounded in* $[0,T]\times\overline{\Omega}$ *and it belongs to* $C^{0,1}((0,T]\times\overline{\Omega})\cap C^{1,2}((0,T]\times\Omega)$*, for any* $T>0$*. Then, there exists a constant* $C>0$ *such that* (13.1.6) *holds. In particular, if* $c\equiv 0$*, then* (13.1.6)*, with* $k=1$*, holds with a constant being independent of* T*.*

Proof. We begin by proving (13.1.6) with $k=0$. For this purpose, let us define the function

$$v(t,x) = |u(t,x)|^2 + a\,t\,m(x)|Du(t,x)|^2\ , \quad t\ge 0,\ x\in\overline{\Omega},$$

where $a>0$ is a parameter that will be chosen later. By the assumptions, $v\in C_b([0,T]\times\overline{\Omega})\cap C^{0,1}((0,T]\times\overline{\Omega})\cap C^{1,2}((0,T]\times\Omega)$ and $v(0,\cdot)=f^2$.

We claim that there exists a suitable value of $a>0$, independent of f,t,x, such that

$$\begin{cases} D_t v(t,x) - \mathcal{A}v(t,x) \le Mv, & t\in(0,T],\ \ x\in\Omega,\\[1ex] \dfrac{\partial v}{\partial \nu}(t,x) \le 0 & t\in(0,T],\ \ x\in\partial\Omega,\end{cases} \tag{13.1.12}$$

for some constant $M\ge 0$, having the same dependence as a. Then, by the maximum principle in Theorem 12.1.5, applied to the function $e^{-Mt}v$, we get

$$v(t,x) \le e^{MT}\sup_{x\in\overline{\Omega}} v(0,x) = e^{MT}\|f\|_\infty^2, \qquad t\in[0,T],\ x\in\overline{\Omega},$$

which yields (13.1.6), with $k=0$ and $C_T = e^{MT/2}a^{-1/2}$, since $m(x)\ge 1$ for any $x\in\overline{\Omega}$.

According to Lemma 13.1.2, $\partial v/\partial \nu \leq 0$ on $\partial\Omega$. Thus we have only to prove that v satisfies the differential inequality in (13.1.12). A straightforward computation shows that v solves the following equation:

$$\begin{aligned} D_t v - \mathcal{A}v = {} & am|Du|^2 + cu^2 - 2\langle QDu, Du\rangle \\ & +2at\Big\{ - m\mathrm{Tr}\,(QD^2uD^2u) + \frac{1}{2}mc|Du|^2 \\ & \qquad + m\sum_{i,j,h=1}^{N} D_h q_{ij} D_h u D_{ij} u + m\sum_{i,j=1}^{N} D_i b_j D_i u D_j u \\ & \qquad + mu\langle Dc, Du\rangle - \frac{\omega_0}{2}\psi''(d)\langle QDd, Dd\rangle |Du|^2 \\ & \qquad - \frac{\omega_0}{2}\psi'(d)(\mathcal{A}d - cd)|Du|^2 - 2\omega_0\psi'(d)\langle QDd, D^2uDu\rangle \Big\}. \end{aligned} \tag{13.1.13}$$

Let us estimate each term in the right-hand side of (13.1.13). By the ellipticity condition (13.1.4) we have

$$-m\,\mathrm{Tr}\,(QD^2uD^2u) \leq -m\kappa_0|D^2u|^2. \tag{13.1.14}$$

Using the boundedness of Dq_{ij} $(i, j = 1, \ldots, N)$ and the Hölder and Young inequalities, we can write

$$\begin{aligned} m\sum_{i,j,h=1}^{N} D_h q_{ij} D_h u D_{ij} u & \leq m\bigg(\sum_{i,j=1}^{N} |\langle Dq_{ij}, Du\rangle|^2\bigg)^{\frac{1}{2}} \bigg(\sum_{i,j=1}^{N} |D_{ij}u|^2\bigg)^{\frac{1}{2}} \\ & \leq \frac{1}{4\delta}\|DQ\|_\infty m\,|Du|^2 + \delta\|DQ\|_\infty\, m\,|D^2u|^2, \end{aligned} \tag{13.1.15}$$

for any $\delta > 0$. From Hypothesis 13.1.1(iv)(a) we get

$$m\sum_{i,j=1}^{N} D_i b_j D_i u D_j u \leq k_1 m|Du|^2 - s_1 mc|Du|^2 \tag{13.1.16}$$

and, from Hypothesis 13.1.1(iii), it follows that, for arbitrary $\delta > 0$,

$$mu\langle Dc, Du\rangle \leq \gamma m(1-c)|u||Du| \leq \gamma\delta m(1-c)|Du|^2 + \frac{\gamma}{4\delta}m(1-c)u^2. \tag{13.1.17}$$

Using (13.1.9) and the Hölder inequality yields

$$-\frac{\omega_0}{2}\psi''(d)\langle QDd, Dd\rangle|Du|^2 \leq \frac{\omega_0}{\varepsilon_0}\|Q\|_\infty|Du|^2. \tag{13.1.18}$$

Analogously, by Hypothesis 13.1.1(iv)(b) and (13.1.8) we have

$$-\frac{\omega_0}{2}\psi'(d)(\mathcal{A}d - cd)|Du|^2 = -\frac{\omega_0}{2}\psi'(d)\big(\mathrm{Tr}(QD^2d) + \langle b, Dd\rangle\big)|Du|^2$$

$$\leq \frac{\omega_0}{2}\left(\|Q\|_\infty\|D^2d\|_\infty + k_2\right)|Du|^2 - \frac{\omega_0}{2}s_2c|Du|^2. \tag{13.1.19}$$

Finally

$$-2\omega_0\psi'(d)\langle QDd, D^2uDu\rangle \leq 2\omega_0\|Q\|_\infty|D^2u|\,|Du|$$

$$\leq \frac{\omega_0}{2\delta}\|Q\|_\infty|Du|^2 + 2\delta\omega_0\|Q\|_\infty|D^2u|^2. \tag{13.1.20}$$

Now, combining the estimates (13.1.14)-(13.1.20) and using the fact that $m \geq 1$, whenever it is necessary, we get

$$\begin{aligned}D_tv - \mathcal{A}v \leq{}& (a\,m - 2\kappa_0)|Du|^2 + cu^2\\ &+2at\bigg\{(\delta\|DQ\|_\infty + 2\delta\omega_0\|Q\|_\infty - \kappa_0)\,m|D^2u|^2\\ &-\left(s_1 + \gamma\delta + \frac{\omega_0}{2}s_2 - \frac{1}{2}\right)mc|Du|^2\\ &+\bigg[\frac{\|DQ\|_\infty}{4\delta} + k_1 + \gamma\delta + \omega_0\|Q\|_\infty\left(\frac{1}{\varepsilon_0} + \|D^2d\|_\infty\right)\\ &+\frac{k_2\omega_0}{2} + \frac{\omega_0}{2\delta}\|Q\|_\infty\bigg]m\,|Du|^2 + \frac{\gamma}{4\delta}m\,(1-c)u^2\bigg\}.\end{aligned}$$

By (13.1.5) we can choose δ small enough to have

$$\|DQ\|_\infty\delta + 2\omega_0\delta\|Q\|_\infty - \kappa_0 \leq 0, \qquad s_1 + \gamma\delta + \frac{\omega_0}{2}s_2 - \frac{1}{2} \leq 0.$$

Therefore, we obtain

$$D_tv - \mathcal{A}v \leq (a\,m - 2\kappa_0)|Du|^2 + cu^2 + 2aT\left\{Km\,|Du|^2 + \frac{\gamma}{4\delta}(1-c)m\,u^2\right\},$$

where we have set

$$K := \frac{\|DQ\|_\infty}{4\delta} + k_1 + \gamma\delta + \omega_0\|Q\|_\infty\left(\frac{1}{\varepsilon_0} + \|D^2d\|_\infty\right) + \frac{k_2\omega_0}{2} + \frac{\omega_0}{2\delta}\|Q\|_\infty.$$

Since $m \leq 1 + \omega_0\varepsilon_0/2$, it follows that

$$\begin{aligned}D_tv - \mathcal{A}v \leq{}& \left[\left(1 + \frac{\omega_0\varepsilon_0}{2}\right)a(1 + 2\,TK) - 2\kappa_0\right]|Du|^2\\ &-\left[-1 + a\,T\left(1 + \frac{\omega_0\varepsilon_0}{2}\right)\frac{\gamma}{2\delta}\right]cu^2 + a\,T\left(1 + \frac{\omega_0\varepsilon_0}{2}\right)\frac{\gamma}{2\delta}u^2.\end{aligned}$$

Now, it is clear that there exists a sufficiently small value $a > 0$, which is independent of f, t, x, such that v satisfies the differential inequality in (13.1.12) with $M = a\,T\gamma(2 + \omega_0\varepsilon_0)/(4\delta)$.

In order to prove (13.1.6) with $k = 1$, it suffices to replace the function v with $u^2 + am|Du|^2$ and to argue as above. ■

13.1.2 Existence of the solution to problem (13.0.2) and uniform gradient estimates

The aim of this subsection is to prove the existence of the bounded classical solution u to the problem (13.0.2) and to show that u satisfies the estimate (13.1.6). The crucial point consists in proving the statement in the case when f is smooth. Indeed, the general case then will follow by approximating $f \in C_b(\overline{\Omega})$ with a sequence of smooth functions. For this purpose, by adapting the technique of Subsection 11.3.2, we approximate the operator $\mathcal{A}$ by new operators $\mathcal{A}_\varepsilon = \mathcal{A} + c_\varepsilon$, where the auxiliary potential c_ε is chosen in such a way that $\mathcal{A}_\varepsilon$ generates a strongly continuous analytic semigroup $\{T_\varepsilon(t)\}$ in $L^p(\Omega)$ for any $p \in [2, +\infty)$. We prove that, for any $\varepsilon > 0$, the function $u_\varepsilon = T_\varepsilon(\cdot)f$ solves the problem (13.0.2), with $\mathcal{A}_\varepsilon$ replacing $\mathcal{A}$ and, for p large enough, it enjoys the regularity properties of Proposition 13.1.3. Hence, it verifies (13.1.6). Taking the limit as ε goes to 0^+ allows us to prove the existence of a classical solution of (13.0.2) as well as of (13.1.6) for such a solution.

Generation of analytic semigroups in L^p-spaces

For notational convenience, throughout this subsection, we denote by $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$ the usual norms of the spaces $L^p(\Omega)$ and $W^{k,p}(\Omega)$ ($k \in \mathbb{N}$), respectively.

We assume more restrictive conditions on the coefficients of the operator $\mathcal{A}$ in (13.0.1). More precisely, in this subsection we always assume the following hypotheses.

Hypotheses 13.1.4 (i) the coefficients q_{ij} and b_j ($i, j = 1, \ldots, N$) and c belong to $C^{1+\alpha}_{\rm loc}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Moreover $q_{ij} \in C^1_b(\overline{\Omega})$, for any $i, j = 1, \ldots, N$, and

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \geq \kappa_0|\xi|^2, \qquad x \in \Omega, \quad \xi \in \mathbb{R}^N,$$

for some positive constant κ_0;

(ii) there exist $\beta > 0$ and $\sigma \in (0, \sqrt{2\kappa_0})$ such that

$$(i)\;\; c(x) \leq -1, \qquad (ii)\;\; |Dc(x)| \leq \beta|c(x)|, \qquad (iii)\;\; |b(x)| \leq \sigma|c(x)|^{\frac{1}{2}}, \tag{13.1.21}$$

for any $x \in \Omega$.

For any $p \in [2,+\infty)$, we introduce the Banach space

$$D_p = \left\{ u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega, \ \ cu \in L^p(\Omega) \right\}, \tag{13.1.22}$$

which is endowed with the norm

$$\|u\|_{D_p} = \|u\|_{2,p} + \|c\,u\|_p, \qquad u \in D_p,$$

and the subspace

$$D = \left\{ u \in C_c^2(\overline{\Omega}) \ : \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

As the following lemma shows D is dense in D_p for any $p \in [2,+\infty)$.

Lemma 13.1.5 *For any $p \in [2,+\infty)$, D is dense in D_p.*

Proof. Let U_n and g_n be as in Definition D.0.1 and let $\{x_i + B(R)\}_{i\in\mathbb{N}}$ be a countable family of balls with the following properties:

(i) the closure of $x_i + B(2R)$ is contained in Ω, for any $i \in \mathbb{N}$;

(ii) $\cup_{i\in\mathbb{N}}(x_i + B(R))$ covers $\Omega \setminus \Omega_\varepsilon$, where $\Omega_\varepsilon = \{x \in \mathbb{R}^N : d(x) < \varepsilon\}$ and $\varepsilon > 0$ is given by the condition (ii) of Definition D.0.1;

(iii) there exists $s \in \mathbb{N}$ such that at most s among the balls $\{x_i + B(2R)\}$ overlap.

We now introduce a covering $\{\Omega_k\}_{k\in\mathbb{N}}$ of $\overline{\Omega}$ defined as follows:

$$\Omega_{2k} = x_k + B(R), \qquad \Omega_{2k+1} = V_k = g_k^{-1}(B(1/2)),$$

and with this covering we associate a partition of the unity $\{\vartheta_k\}$ such that

(i) the C^2-norms of the functions ϑ_k are bounded by a constant independent of k;

(ii) ϑ_{2k+1} satisfies the Neumann condition $\dfrac{\partial \vartheta_{2k+1}}{\partial \nu} = 0$ on $\Omega_{2k+1} \cap \partial\Omega$.

Now, let $u \in W^{2,p}(\Omega)$ be such that $\partial u/\partial \nu = 0$ on $\partial\Omega$ and $cu \in L^p(\Omega)$, and fix $\varepsilon > 0$. We claim that there exists a function $\varphi \in C^2(\overline{\Omega})$ with compact support in $\overline{\Omega}$, such that $\partial \varphi/\partial \nu = 0$ on $\partial\Omega$ and

$$\|u - \varphi\|_{W^{2,p}(\Omega)} + \|cu - c\varphi\|_{L^p(\Omega)} < \varepsilon. \tag{13.1.23}$$

To prove the claim, for any $k \in \mathbb{N}$ we set $u_k = \vartheta_k u$, $a_{2k} = \|c\|_{L^\infty(x_k+B(2R))}$, $a_{2k+1} = \|c\|_{L^\infty(V_k\cap\Omega)}$. We observe that

$$u(x) = \sum_{k=1}^{+\infty} u_k(x), \qquad x \in \Omega,$$

and that $a_k \geq 1$ since, by assumptions, $c \leq -1$. At this point, the idea is to approximate each function u_k by regular functions. Consider first the case when k is even. Then, $u_k \in W^{2,p}(\mathbb{R}^N)$ with $\operatorname{supp} u_k \subseteq x_{k/2} + B(R)$. By classical results, there exists a function $v_k \in C^2(\mathbb{R}^N)$ with $\operatorname{supp} v_k \subseteq x_{k/2} + B(2R)$ such that

$$\|u_k - v_k\|_{W^{2,p}(\mathbb{R}^N)} = \|u_k - v_k\|_{W^{2,p}(x_{k/2}+B(2R))} < \frac{\varepsilon}{4^k a_k}. \tag{13.1.24}$$

It follows that

$$\|cu_k - cv_k\|_{L^p(x_{k/2}+B(2R))} \leq a_k\|u_k - v_k\|_{L^p(x_{k/2}+B(2R))} < \frac{\varepsilon}{4^k}. \tag{13.1.25}$$

Now, assume that k is odd and consider the function $\tilde{u}_k$ defined by $\tilde{u}_k(y) = u_k(g_n^{-1}(y))$ for any $y \in B^+(1/2) = \{z \in B(1/2) \mid z_N > 0\}$. Since the change of variables is of class C^2, $\tilde{u}_k \in W^{2,p}(B^+(1/2))$ and, if $y \in B(1/2)$ with $y_N = 0$, then, taking Remark D.0.2 into account, we get

$$\begin{aligned}\frac{\partial \tilde{u}_k}{\partial y_N}(y) &= \langle D\tilde{u}_k(y), e_N\rangle = \langle (Du_k)(g_n^{-1}(y)), \mathrm{Jac}(g_n^{-1})(y)e_N\rangle \\ &= -\frac{1}{\alpha_n(g_n^{-1}(y))}\langle (Du_k)(g_n^{-1}(y)), \nu(g_n^{-1}(y))\rangle = 0,\end{aligned}$$

since $\partial u_k/\partial\nu = u\partial\vartheta_k/\partial\nu + \vartheta_k\partial u/\partial\nu = 0$, thanks to the choice of ϑ_k. It follows that the extension of $\tilde{u}_k$ with value zero in $\mathbb{R}^N_+ \setminus B^+(1/2)$ belongs to $W^{2,p}(\mathbb{R}^N_+)$ and its normal derivative identically vanishes on $\{y_N = 0\}$. A standard technique based on a truncation argument and convolution with functions which are even with respect to the last variable allows us to construct, for any $\varepsilon' > 0$, a new function $\tilde{v}_k \in C^2(\mathbb{R}^N_+)$ with $\operatorname{supp} \tilde{v}_k \subseteq B(1)$, $\partial\tilde{v}_k/\partial y_N = 0$, if $y \in B(1)$ and $y_N = 0$, such that

$$\|\tilde{v}_k - \tilde{u}_k\|_{W^{2,p}(B^+(1))} \leq \frac{\varepsilon'}{4^k a_k}.$$

It is immediate to check that the function $v_k = \tilde{v}_k(g_n(\cdot))$ belongs to $C^2(U_n \cap \overline{\Omega})$, it has support contained in U_n and its normal derivative vanishes on $U_n \cap \partial\Omega$. Since the C^2-norm of g_n can be taken independent of n (by the condition (iii) in Definition D.0.1), we can choose ε' sufficiently small such that

$$\|v_k - u_k\|_{W^{2,p}(U_n\cap\Omega)} < \frac{\varepsilon}{4^k}a_k, \qquad \|cv_k - cu_k\|_{L^p(U_n\cap\Omega)} < \frac{\varepsilon}{4^k}. \tag{13.1.26}$$

At this point, for any $n \in \mathbb{N}$, we set

$$v_n(x) = \sum_{k=1}^{n} v_k(x), \quad x \in \overline{\Omega}.$$

Of course, $v_n \in C_c^2(\overline{\Omega})$ and $\partial v_n/\partial \nu = 0$ on $\partial\Omega$. Moreover, using (13.1.24), (13.1.25) and (13.1.26), we obtain

$$\|u - v_n\|_{D_p} \le \frac{2}{3}\varepsilon + \left\| \left(1 - \sum_{k=1}^{n} \vartheta_k\right) u \right\|_{W^{2,p}(\Omega)} + \left\| \left(1 - \sum_{k=1}^{n} \vartheta_k\right) Vu \right\|_{L^p(\Omega)}. \tag{13.1.27}$$

As n goes to $+\infty$, the two last terms in the right-hand side of (13.1.27) vanish by dominated convergence. Therefore, choosing n sufficiently large, we get (13.1.23) with $\varphi = v_n$. ∎

Now, our purpose consists in showing that, under Hypotheses 13.1.4, the operator $\mathcal{A}$ with domain D_p is well defined and generates an analytic semigroup $\{T_p(t)\}$, and that, for any $f \in C_b(\overline{\Omega})$ and any $p > N$, the function $T_p(t)f$ is a classical solution to the problem (13.0.2) satisfying the assumptions of Proposition 13.1.3.
For this purpose, we are going to adapt the techniques in Subsection 11.3.2, which are based on integration by parts. We note that since the conormal vector associated with the operator Q needs not to coincide with the normal derivative on $\partial\Omega$, integrating by parts, we are led to some surface integrals which we would avoid. For this purpose, we show that we can replace the matrix Q with a new matrix Q' such that $\mathrm{Tr}(QD^2u) = \mathrm{Tr}(Q'D^2u)$ for any $u \in D_p$ and the conormal vector associated with Q' coincides with the normal vector on $\partial\Omega$. This allows us to eliminate the surface integrals. For this purpose, for any $x \in \Omega$ we introduce the matrix $S(x)$ defined by

$$S(x)y = \vartheta(x)\big\{\langle y, Q(x)Dd(x)\rangle Dd(x) - \langle y, Dd(x)\rangle Q(x)Dd(x)\big\}, \qquad y \in \mathbb{R}^N,$$

where $\vartheta \in C_b^2(\mathbb{R}^N)$ is such that $\vartheta = 1$ in $\Gamma(\rho/2)$ and it vanishes outside $\Gamma(\rho)$ (see (13.1.1)). Then, we set $Q'(x) = Q(x) + S(x)$. As the following lemma shows, Q' has all the claimed properties.

Lemma 13.1.6 *We have:*

(i) $q'_{ij} \in C_b^1(\overline{\Omega})$ $(i,j = 1,\dots,N)$ *and* $\sum_{i,j=1}^N q'_{ij}(x)\xi_i\xi_j \ge \kappa_0|\xi|^2$ *for any* $x \in \Omega$ *and any* $\xi \in \mathbb{R}^N$;

(ii) $\mathrm{Tr}\,(QA)(x) = \mathrm{Tr}\,(Q'A)(x)$, *for any symmetric matrix* A *and any* $x \in \overline{\Omega}$;

(iii) $Q'(x)\nu(x) = \langle \nu(x), Q(x)\nu(x)\rangle \nu(x)$, *for any* $x \in \partial\Omega$.

Moreover, for any $u \in D_p$ *and any* $v \in W^{1,q}(\Omega)$*, with* $\frac{1}{p}+\frac{1}{q}=1$*, we have*

$$\int_\Omega (A_0u)v\,dx = -\sum_{i,j=1}^N \int_\Omega vD_jq'_{ij}\,D_iu\,dx - \sum_{i,j=1}^N \int_\Omega q'_{ij}D_iu\,D_jv\,dx. \quad (13.1.28)$$

Proof. Since S is skew-symmetric, the properties (i) and (ii) follow immediately. Similarly, the property (iii) follows easily taking (13.1.2) into account. In order to prove (13.1.28), it suffices to write $A_0u = \mathrm{Tr}\,(Q'D^2u)$, and to integrate by parts using the property (iii). ■

For notational convenience, we now introduce the vector-valued function $b' : \overline{\Omega} \to \mathbb{R}^N$ defined by

$$b'_i(x) = \sum_{j=1}^N D_jq'_{ij}(x) - b_i(x), \qquad x \in \overline{\Omega}, \quad i = 1, \ldots, N.$$

Taking the assumption (13.1.21) into account, it is immediate to see that

$$|b'| \le |b| + \sqrt{N}\|DQ'\|_\infty \le \sigma|c|^{\frac{1}{2}} + \sqrt{N}\|DQ'\|_\infty. \qquad (13.1.29)$$

Proposition 13.1.7 *For any* $p \ge 2$*, the operator* A_p *is well defined in* D_p*. Moreover, there exists a positive constant* λ_0*, depending on* $\kappa_0, \sigma, p, N, \|Q\|_\infty$*,* $\|DQ\|_\infty, \gamma$*, such that*

$$\|\lambda u - A_p\|_p \ge (\lambda - \lambda_0)\|u\|_p, \qquad \lambda > \lambda_0, \quad u \in D_p. \qquad (13.1.30)$$

Moreover, for any $\lambda > \lambda_0$*, there exist two constants* $C_1, C_2 > 0$ (*depending also on* λ)*, such that*

$$C_1\|u\|_{D_p} \le \|\lambda u - Au\|_p \le C_2\|u\|_{D_p}, \quad u \in D_p. \qquad (13.1.31)$$

Proof. It can be obtained arguing as in the proof of Lemma 11.3.9 and Proposition 11.3.10, writing $\mathcal{A}u = \mathrm{Tr}(Q'D^2u) + \langle b, Du\rangle + cu$. Indeed, observe that, according to Lemma 13.1.6, the conormal derivative relevant to the operator $u \mapsto \mathrm{Tr}(Q'D^2u)$ coincides, up to a multiplication factor, with the normal derivative. Therefore, integrating by parts gives rise to the same integrals as in the proof of Proposition 11.3.10. ■

Next step consists in proving that, for any $p \in [2, +\infty)$, the operator A with domain D_p generates an analytic semigroup of linear operators in $L^p(\Omega)$. For this purpose, we approximate the operator $\mathcal{A}$ by operators with bounded coefficients.

Proposition 13.1.8 *For any* $p \ge 2$*,* (A_p, D_p) *generates a strongly continuous analytic semigroup* $\{T_p(t)\}$ *in* $L^p(\Omega)$*.*

Proof. The proof is close to that of Proposition 11.3.12, so that we limit ourselves to sketching it. To prove that A_p generates a strongly continuous analytic semigroup, we fix $\sigma' \in (\sigma, \sqrt{2\kappa_0})$ and replace the operator A_p with the operator $B_p = A_p - k$ defined in $D(A_p)$, choosing k sufficiently large such that

$$|b'(x)| \le \sigma' |c'(x)|^{\frac{1}{2}}, \qquad x \in \overline{\Omega},$$

where $c' = c - k$. This is possible since b' satisfies (13.1.29). Then, we approximate B_p by the operators $B_{p,\varepsilon} = \mathrm{Tr}(Q'D^2\,\cdot) + \langle b_\varepsilon, D\,\cdot\rangle + c'_\varepsilon$ where b_ε and c'_ε are the same as in (11.3.35), i.e.,

$$b_\varepsilon = \frac{b}{\sqrt{1-\varepsilon c}}, \qquad c'_\varepsilon = \frac{c}{1-\varepsilon c} + k.$$

Arguing as in the proof of Proposition 11.3.12 and taking Proposition 13.1.7 into account, we can show that, for any $\lambda > \lambda_0$ (where λ_0 is given by Proposition 13.1.7) and any $f \in L^p(\Omega)$, there exists a sequence of functions $u_{\varepsilon_n} \in W^{2,p}(\Omega)$, solutions to the equations $\lambda u_{\varepsilon_n} - B_{p,\varepsilon_n} u_{\varepsilon_n} = f$, such that $\partial u_{\varepsilon_n}/\partial\nu = 0$ on $\partial\Omega$ and $c_{\varepsilon_n} u_{\varepsilon_n} \in L^p(\Omega)$. Moreover, u_{ε_n} converges weakly in $W^{2,p}(\Omega)$ to a solution $u \in W^{2,p}(\Omega)$ of the equation $\lambda u - B_p u = f$ such that $cu \in L^p(\Omega)$. To conclude that $u \in D_p$ it suffices to show that $\partial u/\partial\nu = 0$ on $\partial\Omega$. By the classical interior L^p-estimates applied to the operator $\lambda - A_0$ (see Theorem C.1.1), for any pair of bounded and smooth open sets $\Omega' \subset \Omega'' \subset \Omega$, we have

$$\|u_{\varepsilon_n} - u_{\varepsilon_m}\|_{W^{2,p}(\Omega')} \le C(\|u_{\varepsilon_n} - u_{\varepsilon_m}\|_{L^p(\Omega'')} + \|f_{\varepsilon_n} - f_{\varepsilon_m}\|_{L^p(\Omega'')}),$$

where $f_{\varepsilon_n} = f + \langle b_{\varepsilon_n}, Du_{\varepsilon_n}\rangle + c'_{\varepsilon_n} u_{\varepsilon_n}$ and C is a suitable positive constant independent of n, m. Since u_{ε_n} and f_{ε_n} converge to u and $f + \langle b, Du\rangle + cu$, respectively, in $L^p(\Omega \cap B(R+1))$, u_{ε_n} converges to u in $W^{2,p}(\Omega')$. Therefore, by continuity $\partial u_{\varepsilon_n}/\partial\nu$ converges to $\partial u/\partial\nu$ on $\partial\Omega'$ and, consequently, $\partial u/\partial\nu = 0$ on $\partial\Omega'$ as well. From the arbitrariness of Ω', it follows that $\partial u/\partial\nu = 0$ on $\partial\Omega$. Hence, the operator $\lambda I - B_p$ is surjective for any $\lambda > \lambda_0$. Since it is quasi-dissipative, due to (13.1.30), and since it has domain dense in $L^p(\Omega)$, according to the Lumer-Phillips theorem (see Theorem B.1.7), the operator B_p generates a strongly continuous semigroup $\{R(t)\}$ in $L^p(\Omega)$.

To show that $\{R(t)\}$ is analytic, it suffices to show that there exists a positive constant ς such that, for any $u \in D_p$,

$$\left|\mathrm{Im}\int_\Omega u^*\, B_p u\, dx\right| \le \varsigma\left(-\mathrm{Re}\int_\Omega u^* B_p u dx\right), \tag{13.1.32}$$

where $u^* = |u|^{p-2}\bar{u}$. Indeed, (13.1.32) implies that the numerical range of B_p is contained in the sector $\{\lambda \in \mathbb{C} : |\mathrm{Im}\,\lambda| \le -\varsigma\,\mathrm{Re}\,\lambda\}$, and Theorem A.3.6 yields the analyticity of $\{R(t)\}$.

Integrating by parts, it is immediate to check that the real and imaginary parts of $\int_\Omega u^* B_p u\, dx$ are given, respectively, by (11.3.36) and (11.3.42), with q' instead of q, where q' is the bilinear form induced by the matrix Q'. Since Q and Q' differ only in a skew symmetric matrix S, it follows that

$$\mathrm{Re}(\bar{|}u|^2 q'(Du, Du)) = \mathrm{Re}(|u|^2 q(Du, Du))$$

and

$$\mathrm{Re}(\overline{u}^2 q'(Du, Du)) = \mathrm{Re}(\overline{u}^2 q(Du, Du)).$$

Hence, the real part of $\int_\Omega u^* B_p u\, dx$ can be estimated by

$$-\mathrm{Re}\int_\Omega u^* B_p u\, dx \geq \left(p - 1 - \frac{(\sigma')^2}{2\kappa_0}\right)F^2 + G^2 + \frac{1}{2}H^2, \tag{13.1.33}$$

where

$$\begin{aligned} F^2 &= \int_\Omega |u|^{p-4} q(\mathrm{Re}(\overline{u}Du), \mathrm{Re}(\overline{u}Du))dx, \\ G^2 &= \int_\Omega |u|^{p-4} q(\mathrm{Im}(\overline{u}Du), \mathrm{Im}(\overline{u}Du))dx, \\ H^2 &= \int_\Omega |c'||u|^p d\mu \end{aligned}$$

(see the proof of Proposition 11.3.12 for further details).

As far as the imaginary part of the integral is concerned, we observe that

$$\begin{aligned} \mathrm{Im}(|u|^2 q'(Du, D\overline{u})) &= q'(\mathrm{Re}(\overline{u}Du), \mathrm{Im}(\overline{u}Du)) + q'(\mathrm{Im}(\overline{u}Du), \mathrm{Re}(\overline{u}Du)) \\ &= -2s(\mathrm{Re}(\overline{u}Du), \mathrm{Im}(\overline{u}Du)), \end{aligned}$$

where s denotes the bilinear form induced by the matrix S. Moreover,

$$\begin{aligned} \mathrm{Im}(q'(\overline{u}Du, \overline{u}Du)) &= -q'(\mathrm{Re}(\overline{u}Du), \mathrm{Im}(\overline{u}Du)) + q'(\mathrm{Im}(\overline{u}Du), \mathrm{Re}(\overline{u}Du)) \\ &= 2q(\mathrm{Re}(\overline{u}Du), \mathrm{Im}(\overline{u}Du)). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{Im}\int_\Omega u^* B_p u\, dx = &-p\int_\Omega |u|^{p-4} s(\mathrm{Re}(\overline{u}Du), \mathrm{Im}(\overline{u}Du))dx \\ &+(p-2)\int_\Omega |u|^{p-4} q(\mathrm{Re}(\overline{u}Du), \mathrm{Im}(\overline{u}Du))dx \\ &-\mathrm{Im}\int_\Omega |u|^{p-2}\overline{u}\,\langle b', Du\rangle\, dx, \end{aligned}$$

that differs from (11.3.43) only in the integral containing the bilinear form s, that can be estimated by

$$\begin{aligned}
&\left|\int_\Omega |u|^{p-4}s(\mathrm{Re}(\overline{u}Du),\mathrm{Im}(\overline{u}Du))dx\right|\\
&\le \|S\|_\infty\int_\Omega |u|^{p-4}|\mathrm{Re}(\overline{u}Du)||\mathrm{Im}(\overline{u}Du)|dx\\
&\le \|S\|_\infty\left(\int_\Omega |u|^{p-4}|\mathrm{Re}(\overline{u}Du)|^2dx\right)^{\frac12}\left(\int_\Omega |u|^{p-4}|\mathrm{Im}(\overline{u}Du)|^2dx\right)^{\frac12}\\
&\le \frac{1}{\kappa_0}\|S\|_\infty F\,G. \qquad (13.1.34)
\end{aligned}$$

Using (11.3.44), we can write

$$\begin{aligned}
&\left|(p-2)\int_\Omega |u|^{p-4}q(\mathrm{Re}(\overline{u}Du),\mathrm{Im}(\overline{u}Du))dx - \mathrm{Im}\int_\Omega |u|^{p-2}\overline{u}\,\langle b',Du\rangle\,dx\right|\\
&\le (p-2)FG+\frac{\sigma'}{\sqrt{\kappa_0}}GH. \qquad (13.1.35)
\end{aligned}$$

Combining (13.1.34) and (13.1.35), we get

$$\left|\mathrm{Im}\int_\Omega u^*\,B_pu\,dx\right| \le \left(p-2+\frac{p}{\kappa_0}\|S\|_\infty\right)FG+\frac{\sigma'}{\sqrt{\kappa_0}}GH. \qquad (13.1.36)$$

From (13.1.33) and (13.1.36) we obtain (13.1.32) provided we choose $\varsigma\ge 1$ large enough. The proof is now complete. ■

We now study the relation between the semigroup $\{T_p(t)\}$ and the Cauchy-Neumann problem (13.0.2).

Proposition 13.1.9 *For any $p>N$ and $f\in C_c^{2+\alpha}(\overline{\Omega})$ such that $\partial f/\partial\nu=0$ on $\partial\Omega$, the function $u=T_p(\cdot)f$ is the unique bounded classical solution to the problem* (13.0.2). *Moreover, $u\in C^{1+\alpha/2,2+\alpha}_{\rm loc}([0,+\infty)\times\overline{\Omega})$ and Du belongs to $C_b([0,T]\times\overline{\Omega})\cap C^{1,2}((0,T]\times\Omega)$ for any $T>0$.*

Proof. Let f be as in the statement of the proposition. As it is easily seen, $f\in D_p$ (see (13.1.22)). Therefore, the density of D_p in $L^p(\Omega)$ implies that the function u is continuous from $[0,+\infty)$ in D_p (see Lemma 13.1.5, Theorem B.2.2 and Proposition B.2.5) and, hence, in $W^{2,p}(\Omega)$. Since $p>N$, according to the Sobolev embedding theorems (see [2, Theorem 5.4]), $u\in C([0,+\infty);C^1_b(\overline{\Omega}))$. In particular, u and Du are continuous functions in $[0,+\infty)\times\overline{\Omega}$ and they are bounded in $[0,T]\times\overline{\Omega}$ for any $T>0$.

Let us now prove that $u\in C^{1,2}((0,T]\times\Omega)$ for any $T>0$. For this purpose, we observe that, since $\{T_p(t)\}$ is an analytic semigroup, $u\in C^1([\varepsilon,T];D_p)$ for any $0<\varepsilon<T$. Using again the inclusion $D_p \subseteq W^{2,p}(\Omega)$ and the Sobolev embedding theorems, we find that $D_t u \in C([\varepsilon,T]\times\overline{\Omega})$ and, by the arbitrariness of ε, we have $D_t u \in C((0,T]\times\overline{\Omega})$.

In order to show that the second-order space derivatives of u are continuous in $(0,T]\times\Omega$, we consider two bounded open sets Ω_1 and Ω_2, contained in Ω such that $\mathrm{dist}(\Omega_1,\Omega\setminus\Omega_2)>0$. For any fixed $t,t_0\in[\varepsilon,T]$, the function $v=u(t,\cdot)-u(t_0,\cdot)$ solves the equation

$$\sum_{i,j=1}^{N} q_{ij}D_{ij}v = T_p(t)\mathcal{A}f - T_p(t_0)\mathcal{A}f - \langle b, Dv\rangle - cv := g(t,t_0,\cdot),$$

where the right-hand side belongs to $W^{1,p}(\Omega_2)$. By the classical L^p-regularity theory (see Theorem C.1.2), v belongs to $W^{3,p}(\Omega_1)$ and

$$\begin{aligned}
&\|u(t,\cdot)-u(t_0,\cdot)\|_{W^{3,p}(\Omega_1)}\\
&\le C_1\big(\|g(t,t_0,\cdot)\|_{W^{1,p}(\Omega_2)} + \|u(t,\cdot)-u(t_0,\cdot)\|_{W^{1,p}(\Omega_2)}\big)\\
&\le C_2\big(\|T(t)\mathcal{A}f - T(t_0)\mathcal{A}f\|_{W^{1,p}(\Omega)} + \|u(t,\cdot)-u(t_0,\cdot)\|_{W^{2,p}(\Omega)}\big),
\end{aligned}$$

for some positive constants C_1 and C_2, independent of t,t_0. Since the map $t\mapsto T(t)\mathcal{A}f$ is continuous in $(0,+\infty)$ with values in D_p, then, in particular, it is continuous in $(0,+\infty)$ with values in $W^{1,p}(\Omega)$. Since u is continuous in $(0,+\infty)$ with values in $W^{2,p}(\Omega)$, $u\in C((0,+\infty);W^{3,p}(\Omega_1))$. Therefore, using once more the Sobolev embedding theorems, we deduce that $u\in C([\varepsilon,T];C^2(\overline{\Omega}))$, which implies that $Du, D^2u \in C((0,+\infty)\times\overline{\Omega})$, by the arbitrariness of ε, T and Ω. Thus, we have established that $u\in C^{1,2}((0,+\infty)\times\overline{\Omega})$, and, consequently, that u is a bounded classical solution to the Cauchy-Neumann problem (13.0.2).

To show that $u\in C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}([0,+\infty)\times\overline{\Omega})$ and $Du\in C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((0,+\infty)\times\overline{\Omega})$, let $\{\Omega_n\}$ be a sequence of smooth and bounded open sets such that

$$(i)\ \Omega_n\subset\Omega_{n+1},\qquad n\in\mathbb{N},\qquad (ii)\ \partial\Omega\subset\bigcup_{n\in\mathbb{N}}\partial\Omega_n,\qquad (iii)\ \Omega=\bigcup_{n\in\mathbb{N}}\Omega_n.$$

Denote by $\{T_n(t)\}$ the analytic semigroup generated by the realization A_n of the operator $\mathcal{A}$ with homogeneous Neumann boundary conditions in $C(\overline{\Omega}_n)$ (see Theorem C.3.6(v)). As it is easily seen, if n is sufficiently large, then $f\in D_A(\alpha+1,\infty)=\{u\in C^{2+\alpha}(\overline{\Omega}_n) : \partial u/\partial\nu = 0 \text{ on } \partial\Omega_n\}$ (see Theorem C.3.6). This implies that for such values of n, the function $u_n=T_n(\cdot)f$ belongs to $C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}([0,+\infty)\times\overline{\Omega}_n)$ and $Du_n\in C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((0,+\infty)\times\Omega_n)$. Moreover, by the classical maximum principle, it follows that

$$\|u_n\|_\infty\le\|f\|_\infty,\qquad n\in\mathbb{N}. \tag{13.1.37}$$

Now, we fix $k\in\mathbb{N}$ such that $\mathrm{supp}\, f\subset\overline{\Omega}_k$, and two bounded smooth open subsets $\Omega'\subset\Omega''\subset\subset\Omega$, such that $\mathrm{dist}(\Omega',\Omega\setminus\Omega'')>0$. By the classical

Schauder estimates in Theorems C.1.4, C.1.5 and by (13.1.37), we deduce that there exist two constants $C_1 = C_1(k)$ and $C_2 = C_2(\varepsilon, \Omega', \Omega'')$ such that

$$\|u_n - u_m\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega}_k)} \leq C_1\|u_n - u_m\|_{C([0,T]\times\overline{\Omega}_{k+1})} \leq 2C_1\|f\|_\infty,$$

$$\|Du_n - Du_m\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T]\times\overline{\Omega}')} \leq C_2\|u_n - u_m\|_{C([0,T]\times\overline{\Omega}'')} \leq 2C_2\|f\|_\infty,$$

for any $n, m > k$. By a compactness argument, we can determine a subsequence $\{u_{n_k}\}$ converging to a function $v \in C^{1+\alpha/2,2+\alpha}_{\rm loc}([0,+\infty) \times \overline{\Omega})$ in $C^{1,2}(F)$, for any compact set $F \subset [0,+\infty) \times \overline{\Omega}$, such that Du_{n_k} converges to $Dv \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty) \times \Omega)$ in $C^{1,2}([\varepsilon, T] \times \Omega')$ for any $\varepsilon \in (0,T)$ and any $\Omega' \subset\subset \Omega$. As it is immediately seen, v is a classical solution to the problem (13.0.2) and, consequently, it coincides with u, by Theorem 12.1.5. We have so proved that u admits continuous third-order space derivatives in $(0,+\infty) \times \Omega$. This finishes the proof.

∎

Proof of the existence of the classical bounded solution to (13.0.2)

We are now in position to prove the existence of the solution to the problem (13.0.2) together with the gradient estimate (13.1.6). For this purpose, as it has been already pointed out at the beginning of the section, we approximate the operator $\mathcal{A}$ by operators satisfying the assumptions of Subsection 13.1.2, including the extra condition (13.1.4). To begin with, we prove the following approximation result which will be useful in the sequel.

Lemma 13.1.10 *Let $f \in C_b(\overline{\Omega})$. Then, there exists a sequence of functions $\{f_n\} \subset C^{2+\alpha}_c(\overline{\Omega})$ with the following properties:*

(i) $\|f_n\|_\infty \leq C$, *for some positive constant* C *and any* $n \in \mathbb{N}$;

(ii) $\dfrac{\partial f_n}{\partial \nu} = 0$ *on* $\partial\Omega$ *for any* $n \in \mathbb{N}$;

(iii) f_n *converges to* f *as* n *tends to* $+\infty$, *uniformly on compact sets of* $\overline{\Omega}$.

Moreover, let $f \in C^1_\nu(\overline{\Omega})$. Then, there exists a sequence of functions $\{f_n\} \subset C^{2+\alpha}_c(\overline{\Omega})$ with the following properties:

(i) $\partial f_n/\partial\nu = 0$ *on* $\partial\Omega$ *for any* $n \in \mathbb{N}$;

(ii) $\|f_n\|_{C^k_b(\overline{\Omega})} \leq C\|f\|_{C^k_b(\overline{\Omega})}$ $(k = 0, 1)$, *for any* $n \in \mathbb{N}$ *and some constant* $C > 0$, *independent of* n;

(iii) f_n *and* Df_n *converge to* f *and* Df, *respectively, as* n *tends to* $+\infty$, *uniformly on compact sets of* $\overline{\Omega}$.

Proof. We first prove the second part of the statement, since we need it to show the first one. In order to prove the second part of the statement, it is sufficient to repeat, step by step, the same construction as in the proof of Lemma 13.1.5 and to observe that $\sum_{k=1}^n \vartheta_k$ and $\sum_{k=1}^n D\vartheta_k$ converge as r tends to $+\infty$, respectively to $\mathbb{1}$ and 0, uniformly on any compact set Ω' of $\overline{\Omega}$. Moreover, we can also assume that the $C^{2+\alpha}$-norm of ϑ_k can be estimated by a constant independent of k.

We can now prove the first part of the lemma. For this purpose, we observe that if $\{T(t)\}$ denotes the semigroup associated with the Laplacian with homogeneous Neumann boundary condition in Ω, then, for any $n \in \mathbb{N}$, the function $g_n = T(1/n)f$ belongs to $C^2(\overline{\Omega})$, $\partial g_n/\partial\nu = 0$ on $\partial\Omega$ and $\|g_n\|_\infty \le \|f\|_\infty$. Moreover, g_n converges to f as n tends to $+\infty$, uniformly on compact sets of $\overline{\Omega}$. Now, applying the second part of the lemma to any function g_n, we can conclude the proof. ■

Theorem 13.1.11 *Under Hypotheses* 13.1.1*, for any* $f \in C_b(\overline{\Omega})$*, the Cauchy-Neumann problem* (13.0.2) *admits a unique bounded classical solution* u*. The function* u *belongs to* $C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\overline{\Omega})$ *and it satisfies the gradient estimate* (13.1.6) *with* $k = 0$*. If, in addition,* $f \in C^1_\nu(\overline{\Omega})$*, then the estimate* (13.1.6) *holds with* $k = 1$*.*

Proof. The uniqueness of the bounded classical solution to the problem (13.0.2) follows immediately from Theorem 12.1.5. So, let us prove the existence part.

In the case when $f \in C^{2+\alpha}_c(\overline{\Omega})$ the proof follows repeating the same arguments as in Step 1 of the proof of Theorem 11.3.4. Hence, it is omitted.

Let us now consider the case when $f \in C_b(\overline{\Omega})$. Let $\{f_n\}$ be a sequence of functions in $C^{2+\alpha}_c(\overline{\Omega})$ with $\partial f_n/\partial\nu = 0$ on $\partial\Omega$ and assume that f_n converges to f uniformly on compact sets of $\overline{\Omega}$ and it satisfies $\|f_n\|_\infty \le C\|f\|_\infty$ with C being independent of n (see Lemma 13.1.10). Denote by u_n the solution to the problem (13.0.2), with initial datum f_n. Arguing as in Step 2 of the proof of Theorem 11.3.4, we can easily show that, up to a subsequence, u_n converges in $C^{1,2}(F)$, for any compact set $F \subset (0,+\infty)\times\overline{\Omega}$, to a function $u \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\overline{\Omega})$ which solves the differential equation $D_t u - \mathcal{A}u = 0$ and such that $\partial u/\partial\nu = 0$ on $\partial\Omega$.

It remains to show that u is continuous at $(0, x_0)$ with value $f(x_0)$ for any $x_0 \in \overline{\Omega}$, in order to conclude that u is the bounded classical solution to (13.0.2). To this aim it suffices to argue as in the proof of Proposition 12.1.7, taking the gradient estimate (13.1.6) into account.

In the case when $f \in C^1_\nu(\overline{\Omega})$, we can assume that the sequence $\{f_n\}$ satisfies also $\|f_n\|_{C^1_b(\overline{\Omega})} \le C\|f\|_{C^1_b(\overline{\Omega})}$. Then, letting n go to $+\infty$ in the following estimate

$$|Du_n(t,x)| \le C_T\|f_n\|_{C^1_b(\overline{\Omega})} \le C'_T\|f\|_{C^1_b(\overline{\Omega})}, \qquad 0 < t \le T,\ x \in \overline{\Omega},$$

we get (13.1.6) with $k = 1$. ■

Remark 13.1.12 In the case when $f \in C_c^{2+\alpha}(\overline{\Omega})$, the Schauder estimates in Proposition 13.1.9, that we used to prove the convergence of the sequence u_n, show that Du_n is bounded in $[0,T] \times \overline{\Omega}$ for any $T > 0$ and it converges to Du in $[0,T] \times \overline{\Omega}'$ for any bounded open set $\Omega' \subset \Omega$. Therefore, the function Du is continuous in $[0,+\infty) \times \overline{\Omega}$ and it is bounded in $[0,T] \times \overline{\Omega}$ for any $T > 0$. If $c \equiv 0$, then Du is bounded in $[0,+\infty) \times \overline{\Omega}$, since the estimate (13.1.6) with $k = 1$ holds with $T = +\infty$ (see the end of the proof of Proposition 13.1.3).

Finally, by classical regularity results (see Theorem C.1.4), Du belongs to $C^{1,2}((0,+\infty) \times \Omega)$.

For any $f \in C_b(\overline{\Omega})$, any $t > 0$ and any $x \in \overline{\Omega}$, we set $(T(t)f)(x) = u(t,x)$, where, according to Theorem 13.1.11, u is the unique bounded classical solution to the problem (13.0.2). Then, $\{T(t)\}$ is a semigroup of linear operators in $C_b(\overline{\Omega})$, such that $\|T(t)\|_{L(C_b(\overline{\Omega}))} \le 1$ for any $t > 0$, as a consequence of Theorem 12.1.5. To conclude this section we state some remarkable properties of the semigroup $\{T(t)\}$, which we need in Subsection 13.3.

Theorem 13.1.13 *Let $\{f_n\} \subset C_b(\overline{\Omega})$ be a bounded sequence converging pointwise to some function $f \in C_b(\overline{\Omega})$ as n tends to $+\infty$. Then, for any $0 < \varepsilon < T < +\infty$ and any bounded set $\Omega' \subset \overline{\Omega}$, $T(\cdot)f_n$ converges to $T(\cdot)f$ in $C^{1,2}([\varepsilon,T] \times \Omega')$. Further, if f_n converges to f locally uniformly in $\overline{\Omega}$, then $T(\cdot)f_n$ converges to $T(\cdot)f$ uniformly in $[0,T] \times \Omega'$.*

As a consequence, for any $t > 0$ and $x \in \overline{\Omega}$ there exists a positive, Borel measure $p(t,x;\cdot)$ such that $p(t,x;\Omega) \le 1$ and

$$(T(t)f)(x) = \int_\Omega f(y)p(t,x;dy), \qquad f \in C_b(\overline{\Omega}). \tag{13.1.38}$$

Proof. It suffices to argue as in the proof of Proposition 12.1.7. ■

Let $\widehat{A}$ be the weak generator of the semigroup $\{T(t)\}$ defined in Section 12.2. The following proposition characterizes the operator $\widehat{A}$.

Proposition 13.1.14 *It holds that*

$$\begin{cases} D(\widehat{A}) = \Big\{u \in C_b(\overline{\Omega}) \cap \bigcap_{1 \le p < +\infty} W^{2,p}(\Omega \cap B(R)) \ \text{ for any } R > 0 : \\ \qquad\qquad \mathcal{A}u \in C_b(\overline{\Omega}), \ \ \dfrac{\partial u}{\partial \nu}(x) = 0 \ \text{ for any } x \in \partial\Omega\Big\}, \\ \widehat{A}u = \mathcal{A}u, \qquad u \in D(\widehat{A}). \end{cases} \tag{13.1.39}$$

In particular, $\mathcal{A}$ and $T(t)$ commute on $D(\widehat{A})$ for any $t > 0$.
Finally, $D(\widehat{A}) \subset C^1_\nu(\overline{\Omega})$ with a continuous embedding and, for any $\omega > c_0$, there exists a positive constant M_ω such that

$$\|Du\|_\infty \le M_\omega \, \|u\|_\infty^{\frac{1}{2}} \, \|(\mathcal{A} - \omega)\, u\|_\infty^{\frac{1}{2}}, \tag{13.1.40}$$

for any $u \in D(\widehat{A})$.

Proof. The proof is similar to that of Proposition 12.2.5. So we limit ourselves to sketching it, pointing out the main differences. To prove that, for any $f \in C_b(\overline{\Omega})$, the function

$$u(x) = R(\lambda, \widehat{A})f(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)dt, \qquad x \in \overline{\Omega}$$

belongs to the space defined in the right-hand side of (13.1.39) and it solves the elliptic equation $\lambda u - \mathcal{A}u = f$, we use an approximation argument which slightly differs from that used in the proof of the quoted proposition. We first consider the case when $f \in C^{2+\alpha}_c(\overline{\Omega})$ is such that $\partial f/\partial\nu = 0$ on $\partial\Omega$. For any $n \in \mathbb{N}$, we introduce the operator $A^{(n)}$ with bounded coefficients belonging to $C^\alpha(\Omega \cap B(R))$ for any $R > 0$, defined by

$$A^{(n)}u = \sum_{i,j=1}^N q_{ij} D_{ij}u + \sum_{i=1}^N b_i^{(n)} D_i u + c^{(n)}u,$$

for any $u \in D(A^{(n)})$ where

$$D(A^{(n)}) = \bigg\{ u \in \bigcap_{p\ge 1} W^{2,p}(\Omega \cap B(R)) \text{ for any } R > 0 : \\ u, A^{(n)}u \in C_b(\overline{\Omega}),\ \partial u/\partial\nu = 0 \text{ on } \partial\Omega \bigg\}.$$

We assume that $b_i^{(n)} \equiv b_i$ and $c^{(n)} \equiv c$ in $\Omega \cap B(n)$ and $c^{(n)} \le 0$ in $\overline{\Omega}$. According to Proposition C.3.3 (with $c_0 = 0$) and Theorem C.3.6(v), we deduce that the operator $A^{(n)}$ generates an analytic semigroup of contractions $\{T_n(t)\}$ in $C_b(\overline{\Omega})$. Arguing as in Step 1 of the proof of Theorem 11.3.4, it turns out that $T_n(t)f$ converges to $T(t)f$ pointwise in $\overline{\Omega}$. Therefore, by dominated convergence, the sequence $R(\lambda, A^{(n)})f = \int_0^{+\infty} e^{-\lambda t}T_n(t)f dt$ converges to $u = R(\lambda, \widehat{A})f$ pointwise in $\overline{\Omega}$ and in $L^p(\Omega')$ for any bounded subset Ω' of Ω and any $p \in [1, +\infty)$. Now arguing as in the proof of Proposition 12.2.5, we can show that $u \in D(A)$.

In the general case, when $f \in C_b(\overline{\Omega})$, we consider a sequence of functions $\{f_n\} \subset C^{2+\alpha}_c(\overline{\Omega})$, bounded in $C_b(\overline{\Omega})$, with $\partial f_n/\partial\nu = 0$ on $\partial\Omega$, and converging to f pointwise in $\overline{\Omega}$ (see Lemma 13.1.10). Using the representation formula

(13.1.38), we can show that $(T(t)f_n)(x)$ converges to $(T(t)f)(x)$ for any $t > 0$ and any $x \in \overline{\Omega}$. Since $\|T(t)f_n\|_\infty \le \|f_n\|_\infty \le C$, C being a positive constant independent of n, the dominated convergence theorem implies that $R(\lambda, \widehat{A})f_n$ tends to $R(\lambda, \widehat{A})f$ in $L^p(\Omega \cap B(R))$ for any $R > 0$ and any $p \in [1, +\infty)$. Now, arguing as in the previous step, we can show that $R(\lambda, \widehat{A})f$ belongs to the space defined by the right-hand side of (13.1.39) and solves the differential equation $\lambda u - \mathcal{A}u = f$. This completes the first part of the proof which shows that the realization A of the operator $\mathcal{A}$ with domain given by the right-hand side of (13.1.39) is an extension of $\widehat{A}$.

To show that $D(A) \subset D(\widehat{A})$ and to prove (13.1.40) it suffices to argue as in the proof of Proposition 12.2.5. For this purpose, we remark that Proposition 12.2.4, which is the main ingredient to prove the inclusion $D(A) \subset D(\widehat{A})$, holds also when Ω is nonconvex. ■

13.2 The case when Ω is an exterior domain

In this section we deal with the case when $\Omega \subset \mathbb{R}^N$ is an open subset of $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus \Omega$ is bounded and has uniformly $C^{2+\alpha}$-smooth boundary. Our aim consists in extending the gradient estimate (13.1.6) to some cases in which the diffusion coefficients q_{ij} $(i, j = 1, \ldots, N)$ are unbounded in Ω. Let us state the hypotheses that we always assume in this section.

Hypotheses 13.2.1 (i) Hypotheses 13.1.1(i), 13.1.1(v), 13.1.1(vi) are satisfied;

(ii) the functions $q_{ij} \equiv q_{ji}$, b_j and c $(i, = 1, \ldots, N)$ enjoy the same regularity assumptions as in Hypothesis 13.1.1(ii);

(iii) there exist constants $\beta_0 > 0$ and $r \in [0, \sqrt{2}/N)$ and a function $\kappa : \mathbb{R}^N \to \mathbb{R}$ such that

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \ge \kappa(x)|\xi|^2, \qquad x \in \Omega, \quad \xi \in \mathbb{R}^N, \quad 0 < \kappa_0 := \inf_{x\in\overline{\Omega}} \kappa(x);$$

$$|Dq_{ij}(x)| \le \beta_0\kappa(x) + r|c(x)|^{\frac{1}{2}}(\kappa(x))^{\frac{1}{2}}, \qquad x \in \Omega, \quad i, j = 1, \ldots, N;$$

(iv) there exists a positive constant k_1 such that

$$\sum_{i,j=1}^N D_i b_j(x)\xi_i\xi_j \le k_1|\xi|^2, \qquad x \in \Omega, \quad \xi \in \mathbb{R}^N.$$

Set $\Omega_\rho = \Omega \setminus \Gamma(\rho)$, where $\Gamma(\rho)$ has been defined in (13.1.1) and fix, now and for the rest of this section, a real number $\overline{R} > 0$ such that

$$\mathbb{R}^N \setminus \Omega_\rho \subseteq B(\overline{R}).$$

For any $n \in \mathbb{N}$ such that $n > \overline{R}$, let us introduce a nondecreasing odd function $\psi_n \in C_b^2(\mathbb{R})$ such that

(i) $\psi_n(x) = x$ for any $0 \le x \le n$;

(ii) $\psi_n(x) = n + \frac{1}{2}$ for any $x \ge n+1$;

(iii) $0 \le \psi_n'(x) \le 1$ for any $x \in \mathbb{R}$.

Then, we set $\Psi_n(x) = (\psi_n(x_1), \ldots, \psi_n(x_N))$ for any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, and we define the functions $q_{ij}^{(n)}$, $c^{(n)}$ by

$$q_{ij}^{(n)}(x) = q_{ij}(\Psi_n(x)), \qquad c^{(n)}(x) = c(\Psi_n(x)), \qquad x \in \Omega.$$

Finally, we introduce the operator $\mathcal{A}^{(n)}$ defined on smooth functions by

$$\mathcal{A}^{(n)}u(x) = \sum_{i,j=1}^N q_{ij}^{(n)}(x) D_{ij}u(x) + \sum_{i=1}^N b_i(x) D_i u(x) + c^{(n)}(x)u(x), \qquad x \in \Omega. \tag{13.2.1}$$

We observe that, by construction, $q_{ij}^{(n)}, c^{(n)}$ are bounded in $\overline{\Omega}$ together with their first-order derivatives. Moreover, $q_{ij}^{(n)}, c^{(n)} \in C^{1+\alpha}(\Omega \cap B(R))$, for any $R > 0$ and they satisfy Hypotheses 13.1.1(iii) and 13.2.1(ii), with $\kappa(x)$ replaced with $\kappa^{(n)}(x) = \nu(\Psi_n(x))$ and with the same constants κ_0, β_0, r. We note that the functions $b_i^{(n)}(x) = b_i(\Psi_n(x))$, $i = 1, \ldots, N$, do not preserve Hypothesis 13.2.1(ii), in general. This is the reason why we do not approximate the drift.

We assume that $\mathcal{A}^{(n)}$, $n > \overline{R}$, satisfies the following additional hypothesis.

Hypothesis 13.2.2 There exist a positive function $\varphi_n \in C^2(\overline{\Omega})$ and a positive constant $\lambda_0^{(n)} > 0$ such that

$$\lim_{|x| \to +\infty} \varphi_n(x) = +\infty, \qquad \sup_\Omega \left(\mathcal{A}^{(n)}\varphi_n - \lambda_0^{(n)}\varphi_n \right) < +\infty,$$

$$\frac{\partial \varphi_n}{\partial \nu}(x) \ge 0, \quad x \in \partial\Omega.$$

Remark 13.2.3 A case in which both the Hypotheses 13.1.1(vi) and 13.2.2 are satisfied is when there exist two constants $\lambda_0 > 0$ and $C \in \mathbb{R}$ such that

$$2\,\mathrm{Tr}(Q(x)) + 2\langle F(x), x\rangle + (c(x) - \lambda_0)(1 + |x|^2) \le C,$$

$$2\langle F(x), x\rangle - \lambda_0(1+|x|^2) \le C,$$

in a neighborhood of ∞. Indeed, in such a case the function φ defined by

$$\varphi(x) = \psi(x) + (1-\psi(x))(1+|x|^2), \qquad x \in \overline{\Omega},$$

where $\psi \in C^2_c(B(\overline{R}+1))$ is such that $\psi = 1$ in $B(\overline{R})$, is a Lyapunov function for both the operators $\mathcal{A}^n$ $(n \in \mathbb{N})$ and $\mathcal{A}$.

At this point, by applying to the operator $\mathcal{A}^{(n)}$, $n > \overline{R}$, the results of the previous section and, in particular, Theorem 13.1.11, we deduce that, for any $f \in C_b(\overline{\Omega})$, the parabolic problem

$$\begin{cases} D_t u(t,x) - \mathcal{A}^{(n)} u(t,x) = 0, & t>0,\ x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, & t>0,\ x \in \partial\Omega, \\ u(0,x) = f(x), & x \in \Omega, \end{cases} \tag{13.2.2}$$

admits a unique bounded classical solution $u_n \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\overline{\Omega})$ and for any $T \in (0,+\infty)$ there exists $C^{(n)} > 0$ such that

$$|Du_n(t,x)| \le C^{(n)} t^{-\frac{1-k}{2}} \|f\|_{C^k_b(\overline{\Omega})}, \qquad t \in (0,T],\quad x \in \overline{\Omega},\quad k = 0,1,$$

for any $f \in C^k_b(\overline{\Omega})$, satisfying $\partial f/\partial\nu = 0$ on $\partial\Omega$ if $k=1$. We observe that the dependence of $C^{(n)}$ on n relies on the fact that, according to Proposition 13.1.3, such a constant depends on the C^1-norms of the diffusion coefficients $q^{(n)}_{ij}$ $(i,j = 1,\ldots,N)$, which in fact depend on n and blow up as n tends to $+\infty$. Now, we want to show that, in the framework of this section, it is possible to modify a little bit the computations in the proof of Proposition 13.1.3 in order to make the constant C in (13.1.6), independent of n. Once this step is done, a standard approximation argument will prove that the sequence $\{u_n\}$ converges to the unique bounded classical solution to the problem (13.0.2) which satisfies (13.1.6).

In order to carry out our program, we observe that, by construction, $q^{(n)}_{ij}(x) = q_{ij}(x)$ $(i,j = 1,\ldots,N)$ and $c^{(n)}(x) = V(x)$ if $n > \overline{R}$ and $x \in \Gamma(\rho)$. Hence, we have

$$\begin{aligned} &(i) \quad -\Big((\mathcal{A}^{(n)} - c^{(n)})d\Big)(x) = -\Big((\mathcal{A}-c)d\Big)(x) \le M_1, \quad x \in \Gamma(\rho),\ n > \overline{R}, \\ &(ii) \quad |Q^{(n)}(x)Dd(x)| = |Q(x)Dd(x)| \le M_2, \quad x \in \Gamma(\rho),\ n > \overline{R}, \end{aligned} \tag{13.2.3}$$

where $M_1 \in \mathbb{R}$ and $M_2 \ge 0$ depend on the sup-norm of q_{ij}, b_i, c $(i,j = 1,\ldots,N)$ on the compact set $\overline{\Gamma(\rho)}$. We use such conditions in the next proposition.

Proposition 13.2.4 *Let u be the bounded classical solution to the parabolic problem* (13.2.2)*, where $\mathcal{A}^{(n)}$ is defined by* (13.2.1)*. Then, u verifies* (13.1.6)*, with a constant C_T independent of n.*

Proof. We limit ourselves to proving the statement in the case when u satisfies the regularity assumptions of Proposition 13.1.3. The general case then will follow arguing as in Theorem 13.1.11. Moreover, since the proof we provide is similar to that of Proposition 13.1.3, we just sketch it, pointing out the main differences in the case when $k=0$ in (13.1.6).

Fix $n\in\mathbb{N}$ and introduce the function $v:[0,+\infty)\times\overline{\Omega}\to\mathbb{R}$ defined by

$$v(t,x)=|u(t,x)|^2+a\,t\,m(x)|Du(t,x)|^2,\qquad t\geq 0,\;\; x\in\overline{\Omega}.$$

Since v satisfies the boundary condition $\partial v/\partial\nu\leq 0$ on $\partial\Omega$, we have only to show that $D_tv-\mathcal{A}^{(n)}v\leq Mv$, for some $a,M>0$ independent of n. Recalling that $D_tv-\mathcal{A}^{(n)}v$ is given by (13.1.13) with $\mathcal{A}$ replaced with $\mathcal{A}^{(n)}$ and observing that $q_{ij}^{(n)},c^{(n)}$ satisfy Hypothesis 13.2.1(iii), with κ being replaced with $\kappa^{(n)}$, we get

$$-m\,\mathrm{Tr}\,(Q^{(n)}D^2uD^2u)\leq -m\kappa^{(n)}|D^2u|^2, \tag{13.2.4}$$

$$\begin{aligned}m\sum_{i,j,k=1}^N D_kq_{ij}^{(n)}D_kuD_{ij}u&\leq Nm\left(\beta_0\kappa^{(n)}+r|c^{(n)}|^{\frac12}\left(\kappa^{(n)}\right)^{\frac12}\right)|Du||D^2u|\\&\leq\frac{N^2\beta_0^2}{4\varepsilon}m\,\kappa^{(n)}\,|Du|^2+\varepsilon m\,\kappa^{(n)}\,|D^2u|^2\\&\quad-\tilde\varepsilon\,N^2r^2m\,c^{(n)}\,|Du|^2+\frac{1}{4\tilde\varepsilon}m\kappa^{(n)}|D^2u|^2,\end{aligned} \tag{13.2.5}$$

for any $\varepsilon,\tilde\varepsilon\in\;(0,1)$. From Hypotheses 13.2.1(iv) it follows that

$$m\sum_{i,j=1}^N D_ib_jD_iuD_ju\leq k_1m\,|Du|^2, \tag{13.2.6}$$

and from Hypothesis 13.1.1(iii), which is satisfied also by $c^{(n)}$ with the same constant γ, we get, for arbitrary $\delta>0$,

$$m\,u\langle Dc^{(n)},Du\rangle\leq\gamma\delta m\,(1-c^{(n)})|Du|^2+\frac{\gamma}{4\delta}m\,(1-c^{(n)})u^2.$$

Using (13.1.9) and (13.2.3)(ii) it follows that

$$-\frac{\omega_0}{2}\psi''(d(\cdot))\sum_{i,j=1}^N q_{ij}^{(n)}D_id(\cdot)D_jd(\cdot)|Du|^2\leq\frac{\omega_0}{\varepsilon_0}M_2|Du|^2. \tag{13.2.7}$$

Analogously, by (13.1.8) and (13.2.3)(i) we obtain

$$-\frac{\omega_0}{2}\psi'(d(\cdot))(\mathcal{A}^{(n)}d(\cdot)-c^{(n)}d(\cdot))|Du|^2\leq\frac{\omega_0}{2}M_1|Du|^2. \tag{13.2.8}$$

Finally, for any $\varepsilon > 0$,

$$-2\omega_0\psi'(d(\cdot))\langle Q^{(n)}Dd, D^2uDu\rangle \le 2\omega_0 M_2\,|D^2u|\,|Du| \le \omega_0^2 M_2^2\varepsilon^{-1}|Du|^2 + \varepsilon|D^2u|^2. \qquad (13.2.9)$$

From the estimates (13.2.4)-(13.2.9) and recalling that m satisfies (13.1.7), we get

$$\begin{aligned}
D_t v_n - \mathcal{A}^{(n)}v_n \le & \Big\{a\left(1+\frac{\omega_0\varepsilon_0}{2}\right) \\
&\times\left[1+2T\left(k_1+\gamma\delta+\frac{\omega_0 M_2}{\varepsilon_0}+\frac{\omega_0 M_1}{2}+\frac{\omega_0^2 M_2^2}{\varepsilon}\right)\right] \\
&+\kappa^{(n)}\left[-2+\left(1+\frac{\omega_0\varepsilon_0}{2}\right)\frac{aTN^2}{2\varepsilon}\,\beta_0^2\right]\Big\}|Du_n|^2 \\
&-at\left(2\tilde\varepsilon N^2 r^2+2\gamma\delta-1\right)m\,c^{(n)}|Du_n|^2 \\
&+\left(1+\frac{\omega_0\varepsilon_0}{2}\right)\frac{aT\gamma}{2\delta}u_n^2+\left[1-\left(1+\frac{\omega_0\varepsilon_0}{2}\right)\frac{aT\gamma}{2\delta}\right]c^{(n)}\,u_n^2 \\
&+at\left(-2+2\varepsilon+\frac{1}{2\tilde\varepsilon}+2\frac{\varepsilon}{\kappa_0}\right)m\,\kappa^{(n)}\,|D^2u_n|^2, \qquad (13.2.10)
\end{aligned}$$

in $(0,T]\times\Omega$ for any arbitrarily fixed $T>0$. Since $r\in[0,\sqrt{2}/N)$ we can choose $\tilde\varepsilon\in(1/4,1)$ such that $-1+2\tilde\varepsilon N^2r^2>0$. Then, we choose a sufficiently small value of $\varepsilon>0$ in order to make the last term in the right-hand side of (13.2.10) vanish. At this point, we fix $\delta>0$ such that also the third term vanishes and, then, $a>0$ such that

$$-1+aTN^2\Big(1+\frac{\omega_0\varepsilon_0}{2}\Big)\frac{\beta_0^2}{4\varepsilon}<0$$

and

$$-1+aT\Big(1+\frac{\omega_0\varepsilon_0}{2}\Big)\frac{\gamma}{2\delta}<0.$$

With these choices of the parameters we obtain

$$\begin{aligned}
&D_t v_n - \mathcal{A}^{(n)}v_n \\
&\le\left[a\Big(1+\frac{\omega_0\varepsilon_0}{2}\Big)+aT(2+\omega_0\varepsilon_0)\left(k_1+\gamma\delta+\frac{\omega_0}{\varepsilon_0}M_2+\frac{\omega_0}{2}M_1+\frac{\omega_0^2M_2^2}{\varepsilon}\right)\right. \\
&\quad\left.+\kappa_0\Big(1+\frac{\omega_0\varepsilon_0}{2}\Big)\left(-2+aT\frac{N^2\beta_0^2}{2\varepsilon}\right)\right]|Du_n|^2+\Big(1+\frac{\omega_0\varepsilon_0}{2}\Big)\frac{aT\gamma}{2\delta}u_n^2.
\end{aligned}$$

Choosing a smaller value of a, if necessary, we finally obtain $D_t v_n - \mathcal{A}^{(n)}v_n \le Mv_n$ in $(0,T]\times\Omega$, where $M = aT(2+\omega_0\varepsilon_0)\gamma/(4\delta)$. Note that all the constants involved are independent of n. Hence, the conclusion follows as in Proposition 13.1.3. ■

We are now in position to prove the main theorem of the section.

Theorem 13.2.5 *For any $f \in C_b(\overline{\Omega})$ the Cauchy problem* (13.0.2) *admits a unique bounded classical solution u, which also belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty) \times\overline{\Omega})$. Moreover, the estimate* (13.1.6) *holds true with $k = 0$. Finally, if $f \in C^1_\nu(\overline{\Omega})$, then the estimate* (13.1.6) *holds also with $k = 1$.*

Proof. We limit ourselves to proving the assertion when $f \in C_b(\overline{\Omega})$, since the other case follows straightforward from this one.

For any fixed $f \in C_b(\overline{\Omega})$ and any $n > \overline{R}$, let u_n be the unique bounded classical solution to the problem (13.2.2). As it has been already pointed out, $u_n \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty) \times \overline{\Omega})$. Moreover, Theorem 12.1.5 yields $\|u_n\|_\infty \leq \|f\|_\infty$ and u_n satisfies the gradient estimate

$$|Du_n(t,x)| \leq C_T t^{-\frac{1-k}{2}} \|f\|_{C_b(\overline{\Omega})}, \qquad t \in (0,T], \quad x \in \overline{\Omega}, \quad (13.2.11)$$

for any $T > 0$ and some positive constant C_T, independent of n, as a consequence of Proposition 13.2.4.

In order to prove that there exists a subsequence of $\{u_n\}$, which converges to the bounded classical solution to the problem (13.0.2) in $C^{1,2}([\varepsilon,T] \times \Omega')$ for any $\varepsilon \in (0,T)$ and any $\Omega' \subset\subset \overline{\Omega}$, we argue as in Theorem 13.1.11. Hence, we may skip in this context some details. First, we show that a subsequence $\{u_{n_k}\}$ converges to a solution u to the differential equation in (13.0.2), which satisfies the boundary condition $\partial u/\partial\nu = 0$ on $\partial\Omega$. For this purpose, we apply the Schauder estimates in Theorem C.1.5 to the function u_n, observing that the constants which appear can be taken independent of n, since for any bounded set $\widetilde{\Omega} \subset \Omega$, one has $\mathcal{A}^{(n)} = \mathcal{A}$ in $\widetilde{\Omega}$, if n is sufficiently large. Then, to prove the continuity of u at $t = 0$, we use a localization argument and the gradient estimate (13.2.11). For the same reason as above, also in this case all the constants involved can be taken independent of n. ■

Remark 13.2.6 We remark that all the results in Remark 13.1.12, Theorem 13.1.13 and Proposition 13.1.14 can be extended to the case when Ω is an exterior domain. The proof of Remark 13.1.12 follows from the arguments in the proof of Theorem 13.2.5, whereas the proof of Theorem 13.1.13 can be obtained just as in the case when the q_{ij}'s are bounded $(i,j = 1,\ldots,N)$. In the proof of Proposition 13.1.14 only a slight modification is needed. Since now the coefficients q_{ij} need not to be bounded, one has to change the definition of the approximating operators $A^{(n)}$ replacing the q_{ij}'s with a sequence of bounded coefficients $q_{ij}^{(n)}$ such that the operator $A^{(n)}$ is still uniformly elliptic in Ω and $q_{ij}^{(n)} = q_{ij}$ in $\Omega \cap B(n)$. Then, the proof can be obtained just repeating the same arguments as in the quoted proposition.

13.3 Pointwise gradient estimates and their consequences

In this section we are devoted to proving some pointwise gradient estimates for the function $T(t)f$. We start by proving the following estimate

$$|(DT(t)f)(x)|^p \le C_p\, e^{M_p t}\, (T(t)(|Df|^p))(x), \qquad t>0, \quad x\in\Omega, \quad (13.3.1)$$

in the case when the diffusion coefficients are bounded. Here, M_p is a real constant that can be explicitly determined (see (13.3.3)). As it is immediately seen, $T(\cdot)\mathbb{1}\equiv\mathbb{1}$ is a necessary condition in order to have (13.3.1), hence we need to assume $c\equiv 0$.

Theorem 13.3.1 *Let Hypotheses* 13.1.1(i), 13.1.1(ii), 13.1.1(iv)–13.1.1(vi) *be satisfied. Further, assume that $c\equiv 0$. Then, for any $p\in(1,+\infty)$, there exist two constants $C_p>0$ and $M_p\in\mathbb{R}$, such that* (13.3.1) *holds for any $f\in C^1_\nu(\overline{\Omega})$.*

Proof. The techniques used are similar to those in the proof of Proposition 12.3.1. The only differences are in the case when $p\in(1,2]$. Hence, we sketch the proof only in this case, pointing out the main differences. We assume that $f\in C^{2+\alpha}_c(\overline{\Omega})$ and $\partial f/\partial\nu=0$ on $\partial\Omega$. The general case then follows by density by virtue of Lemma 13.1.10 and Theorem 13.1.13. We replace the function w defined in the proof of Proposition 12.3.1 with the new function w defined by

$$w(t,x)=\big(m(x)|Du(t,x)|^2+\delta\big)^{\frac{p}{2}}, \qquad t>0, \quad x\in\Omega,$$

where $u=T(\cdot)f$. We sketch the proof in the case when $p\in(1,2]$, pointing out the main differences.

According to Remark 13.1.12, w is differentiable once with respect to the time variable and twice with respect to the space variables in $(0,+\infty)\times\Omega$ and it is bounded and continuous in $[0,+\infty)\times\overline{\Omega}$. Moreover,

$$D_t w-\mathcal{A}w=mf_1+f_2+f_3,$$

where f_1 is given by (12.3.3) (with $m|Du|^2+\delta$ instead of $|Du|^2+\delta$),

$$f_2=p\big(m|Du|^2+\delta\big)^{\frac{p}{2}-1}\bigg(-\frac{1}{2}|Du|^2\mathcal{A}m-2\sum_{i,j,k=1}^N q_{ij}D_jmD_kuD_{ik}u\bigg),$$

$$f_3=p(2-p)(m|Du|^2+\delta)^{\frac{p}{2}-2}\sum_{i,j=1}^N q_{ij}\left(\frac{1}{2}|Du|^2D_im+m(D^2uDu)_i\right)$$
$$\times\left(\frac{1}{2}|Du|^2D_jm+m(D^2uDu)_j\right).$$

Using (13.1.14), (13.1.15) (with $\|DQ\|_\infty^{-1}\varepsilon$ instead of δ) and (13.1.16) we get

$$f_1 \le p\left(m|Du|^2+\delta\right)^{\frac{p}{2}-1} \times\left\{\left(\frac{1}{4\varepsilon}\|DQ\|_\infty^2+k_1\right)|Du|^2+\varepsilon|D^2u|^2-\mathrm{Tr}(QD^2uD^2u)\right\}.$$

Since $\mathcal{A}m=\omega_0\psi'(d(\cdot))\mathcal{A}d+\omega_0\psi''(d(\cdot))\langle Qd,d\rangle$, from (13.1.18), (13.1.19) and (13.1.20), with ε instead of δ, we get

$$f_2 \le p\omega_0\left(m|Du|^2+\delta\right)^{\frac{p}{2}-1} \times\left\{\|Q\|_\infty\left(\frac{1}{\varepsilon_0}+\frac{1}{2}\|D^2d\|_\infty+\frac{1}{2\varepsilon}+\frac{k_2}{2}\right)|Du|^2+2\varepsilon\|Q\|_\infty|D^2u|^2\right\},$$

for any $\varepsilon>0$. Finally, using the Cauchy-Schwarz inequality twice, first for the inner product induced by the matrix Q and, then, for the Euclidean inner product, and recalling that

$$(\alpha+\beta)^2\le(1+\varepsilon)\alpha^2+\frac{1+\varepsilon}{\varepsilon}\beta^2,$$

for any $\alpha,\beta,\varepsilon>0$, we get

$$\begin{aligned}
&\sum_{i,j=1}^N q_{ij}\left(\frac{1}{2}|Du|^2D_im+m(D^2uDu)_i\right)\left(\frac{1}{2}|Du|^2D_jm+m(D^2uDu)_j\right)\\
&\le\left[\frac{1}{2}\left(\langle QDm,Dm\rangle\right)^{\frac{1}{2}}|Du|^2+m\left(\mathrm{Tr}(QD^2uD^2u)\right)^{\frac{1}{2}}|Du|\right]^2\\
&\le m|Du|^2\left[\frac{1}{2}\left(\langle QDm,Dm\rangle\right)^{\frac{1}{2}}|Du|+\left(m\mathrm{Tr}(QD^2uD^2u)\right)^{\frac{1}{2}}\right]^2\\
&\le(m|Du|^2+\delta)\\
&\qquad\times\left(m(1+\varepsilon)\mathrm{Tr}(QD^2uD^2u)+m\frac{1+\varepsilon}{4\varepsilon}|Du|^2\langle QDm,Dm\rangle\right).
\end{aligned}\tag{13.3.2}$$

Therefore,

$$f_3\le p(2-p)(m|Du|^2+\delta)^{\frac{p}{2}-1} \times\left(m(1+\varepsilon)\mathrm{Tr}(QD^2uD^2u)+m\omega_0^2\frac{1+\varepsilon}{4\varepsilon}\|Q\|_\infty|Du|^2\right).$$

Now, choosing $\varepsilon > 0$ small enough to have $1-(1+\varepsilon)(2-p) > 0$ from the previous estimate and (13.1.14) we get

$$
\begin{aligned}
&D_t w - \mathcal{A}w \\
&\le p\left(m|Du|^2+\delta\right)^{\frac{p}{2}-1} \\
&\quad\times\Bigg\{\Bigg[\frac{\|DQ\|_\infty^2}{4\varepsilon}+k_1+\frac{\omega_0}{\varepsilon_0}\|Q\|_\infty+\frac{\omega_0}{2}\|D^2 d\|_\infty\|Q\|_\infty+\frac{\omega_0}{2\varepsilon}\|Q\|_\infty \\
&\qquad+\frac{\omega_0}{2}k_2-\frac{\omega_0^2\varepsilon_0}{4+2\omega_0\varepsilon_0}k_2^-+\frac{(1+\varepsilon)(2-p)}{4\varepsilon}\omega_0^2\|Q\|_\infty\Bigg]m|Du|^2 \\
&\qquad+\Big[\varepsilon+\frac{\varepsilon}{2}\omega_0\varepsilon_0+2\omega_0\varepsilon\|Q\|_\infty-(1-(1+\varepsilon)(2-p))\kappa_0\Big]|D^2u|^2\Bigg\},
\end{aligned}
$$

where $k_2^- = \min\{0, k_2\}$. Let us now choose $\varepsilon > 0$ such that

$$\varepsilon+\frac{1}{2}\omega_0\varepsilon_0\varepsilon-\kappa_0+(1+\varepsilon)(2-p)\kappa_0+2\varepsilon\omega_0\|Q\|_\infty=0$$

and set

$$
\begin{aligned}
M_p = p\Bigg\{&\frac{\|DQ\|_\infty^2}{4\varepsilon}+k_1+\frac{\omega_0}{\varepsilon_0}\|Q\|_\infty+\frac{\omega_0}{2}\|D^2 d\|_\infty\|Q\|_\infty+\frac{\omega_0}{2\varepsilon}\|Q\|_\infty+\frac{\omega_0}{2}k_2 \\
&-\frac{\omega_0^2\varepsilon_0}{4+2\omega_0\varepsilon_0}k_2^-+\frac{(1+\varepsilon)(2-p)}{4\varepsilon}\omega_0^2\|Q\|_\infty\Bigg\}. \qquad (13.3.3)
\end{aligned}
$$

Therefore, we get $D_t w - \mathcal{A}w \le M_p w - (M_p \wedge 0)\delta^{\frac{p}{2}}$. Arguing as in the proof of Theorem 7.1.5, taking the maximum principle in Theorem 12.1.5 into account, we deduce that

$$w(t,x) \le e^{M_p t}\left(T(t)\left((f^2+m|Df|^2)^{\frac{p}{2}}\right)\right)(x),$$

for any $t > 0$ and any $x \in \overline{\Omega}$, which gives us (13.3.1) with $C_p = (1+\omega_0\varepsilon_0/2)^{p/2}$. ■

Now, we prove the second type of pointwise estimates.

Theorem 13.3.2 *Let Hypotheses* 13.1.1(*i*), 13.1.1(*ii*), 13.1.1(*iv*)–13.1.1(*vi*) *be satisfied. Further, assume that* $c \equiv 0$. *Then, for any* $p \in (1,+\infty)$ *and any* $f \in C_b(\overline{\Omega})$

$$|(DT(t)f)(x)|^p \le K_{p,t}\, t^{-\frac{p}{2}}\,(T(t)(|f|^p))(x), \qquad t\in(0,+\infty), \quad x\in\overline{\Omega}, \qquad (13.3.4)$$

where

$$K_{p,t} = \begin{cases} \dfrac{(2+\omega_0\varepsilon_0)^{\frac{p}{2}}}{[2p(p-1)\kappa_0]^{\frac{p}{2}}} \dfrac{M_p t}{1-e^{-M_p t}}, & \text{if } p \in (1,2],\\ K_{2,t}^{\frac{p}{2}}, & \text{if } p > 2, \end{cases}$$

with M_p being given by (13.3.3)*. In the case when $M_p = 0$, the term $M_p t(1 - e^{-M_p t})^{-1}$ should be replaced with* 1*.*

Proof. We limit ourselves to sketching the proof (13.3.4) in the case when $f \in C_c^{2+\alpha}(\overline{\Omega})$ is such that $\partial f/\partial\nu = 0$ on $\partial\Omega$, and $p \in (1,2]$, since it is similar to that of Proposition 12.3.3. The general case then will follow from an approximation argument (see Lemma 13.1.10) and the Jensen inequality.

For any $\delta > 0$, we introduce the function $\Phi : (0,t) \to C_b(\overline{\Omega})$ defined by

$$\Phi(s) = T(s)\Big((|T(t-s)f|^2 + \delta)^{\frac{p}{2}}\Big),$$

and we set, for any $s \in (0,t)$, $g(s) = (|T(t-s)f|^2 + \delta)^{p/2}$. The first step consists in showing that

$$\Phi'(s) \ge p(p-1)\kappa_0 T(s)\left((g(s))^{1-\frac{2}{p}}|DT(t-s)f|^2\right),$$

for any $s \in (0,t)$. This can be done as in the proof of Proposition 12.3.3, after proving that $g(s) \in D(\widehat{A})$ for any $s \in (0,t)$, where $D(\widehat{A})$ is defined in Proposition 13.1.14. This is not immediate since now the operator $\mathcal{A}$ has unbounded coefficients. By Remark 13.1.12, $g(s) \in C_b^1(\overline{\Omega}) \cap C^2(\Omega)$ and it satisfies $\partial g(s)/\partial\nu = 0$ on $\partial\Omega$. To conclude that $g(s) \in D(A)$, we just need to show that $\mathcal{A}g(s) \in C_b(\overline{\Omega})$. For this purpose, we observe that

$$\begin{aligned}\mathcal{A}g(s) = {}& p(g(s))^{1-\frac{2}{p}}\left\{T(t-s)f\,\mathcal{A}T(t-s)f + \langle QDT(t-s)f, DT(t-s)f\rangle\right\}\\ &+p(p-2)(g(s))^{1-\frac{4}{p}}|T(t-s)f|^2\langle QDT(t-s)f, DT(t-s)f\rangle.\end{aligned}$$

Since $f \in D(A)$ then, by Proposition 13.1.14, $\mathcal{A}T(t-s)f = T(t-s)\mathcal{A}f$ and, therefore, $\mathcal{A}g(s)$ is bounded and continuous in $\overline{\Omega}$. This implies that $\mathcal{A}\Phi(s) = T(s)\mathcal{A}g(s)$ for any $s \in (0,t)$.

The second step consists in showing that

$$\begin{aligned}&\left(T(t)\left((|f|^2+\delta)^{\frac{p}{2}}\right)\right)(x)\\ &\ge p(p-1)\kappa_0\int_0^t \left(T(s)\left((g(s))^{1-\frac{2}{p}}|DT(t-s)f|^2\right)\right)(x)ds,\end{aligned}$$

for any $x \in \overline{\Omega}$. So, arguing as in the proof of Proposition 12.3.3, we are led to showing that $\left(T(t-\varepsilon)\left((|T(\varepsilon)f|^2+\delta)^{\frac{p}{2}}\right)\right)(x)$ tends to $\left(T(t)\left((f^2+\delta)^{\frac{p}{2}}\right)\right)(x)$

as ε tends to 0, for any $x \in \overline{\Omega}$. This was immediate in the proof of the quoted proposition, since $\{T_n(t)\}$ was a strongly continuous semigroup, whereas, now, it is guaranteed by Theorem 13.1.13.

From now on, the same arguments in the proof of Proposition 12.3.3 yield to (13.3.4). Hence, we skip the details. ∎

Now, we show that the pointwise estimate (13.3.4) can be extended to the case when Ω is an exterior domain. For this purpose, we apply the same technique as in Chapter 7, first proving that, for any $p \in (1, +\infty)$, there exists a constant $C_p > 0$ such that

$$|(DT(t)f)(x)|^p \leq C_p\big(T(t)(f^2 + |Df|^2)^{\frac{p}{2}}\big)(x), \qquad t > 0,\ x \in \overline{\Omega}, \quad (13.3.5)$$

for any $f \in C_b^1(\overline{\Omega})$ such that $\partial f/\partial \nu = 0$ on $\partial\Omega$. As above we still assume that $c \equiv 0$.

Theorem 13.3.3 *Under Hypotheses* 13.2.1 *and* 13.2.2 (*where we take* $c \equiv 0$) *there exists a constant* $C_p > 0$ *such that* (13.3.5) *holds true.*

Proof. Since the proof is similar to that of Theorem 13.3.1, we limit ourselves to sketching it, pointing out the main differences. In the case when $p \in (1, 2]$ and $f \in C_c^{2+\alpha}(\overline{\Omega})$, we introduce, for any $\delta > 0$, the function w defined by

$$w(t, x) = \big(a|u(t, x)|^2 + m(x)|Du(t, x)|^2 + \delta\big)^{\frac{p}{2}}, \qquad t > 0, \quad x \in \Omega. \quad (13.3.6)$$

Here $a > 0$ is a real parameter to be fixed later on. By virtue of Remark 13.2.6, the function w is differentiable once with respect to the time variable and twice with respect to the space variables in $(0, +\infty) \times \Omega$ and it is continuous in $[0, +\infty) \times \overline{\Omega}$, due to Remark 13.2.6. Moreover, it satisfies the equation $D_t w - \mathcal{A}w = mf_1 + f_2 + f_3$, where f_1 is given by (12.3.11) (with w as in (13.3.6) and a being replaced by a/m),

$$f_2 = -p\, w^{1-\frac{2}{p}} \left(\frac{1}{2}|Du|^2 \mathcal{A}m + 2\langle QDm, D^2uDu\rangle\right)$$

and

$$f_3 = \frac{p}{2}\left(1 - \frac{p}{2}\right) \langle Q(D(au^2 + m|Du|^2)), D(au^2 + m|Du|^2)\rangle.$$

The function f_1 can be immediately estimated as in (13.2.5) and (13.2.6), where we take $c^{(n)} = 0$ and replace $\kappa^{(n)}$ with κ. Similarly, since

$$\mathcal{A}m = \omega_0 \psi'(d(\cdot))\mathcal{A}d + \omega''(d(\cdot))\langle Qd, d\rangle,$$

f_2 can be estimated by (13.2.7) and (13.2.9). To estimate f_3 we observe that, arguing as in the proof of (13.3.2), we get

$$\begin{aligned}
&\langle Q(D(au^2+m|Du|^2), D(au^2+m|Du|^2)\rangle \\
&\leq \Big[(\langle QDm, Dm\rangle)^{\frac12}|Du| + 2m\left(\mathrm{Tr}(QD^2uD^2u)\right)^{\frac12}|Du| \\
&\qquad +2a|u|\left(\langle QDu, Du\rangle\right)^{\frac12}\Big]^2 \\
&\leq (a|u|^2+m|Du|^2) \\
&\qquad \times\Big\{ m\Big[(\langle QDm, Dm\rangle)^{\frac12}|Du| + 2\left(\mathrm{Tr}(QD^2uD^2u)\right)^{\frac12}\Big]^2 \\
&\qquad\qquad +4a\langle QDu, Du\rangle\Big\} \\
&\leq (a|u|^2+m|Du|^2)\Big(4m(1+\varepsilon)\mathrm{Tr}(QD^2uD^2u) \\
&\qquad\qquad +m\frac{1+\varepsilon}{\varepsilon}|Du|^2\langle QDm, Dm\rangle + a\langle QDu, Du\rangle\Big).
\end{aligned}$$

Hence, we obtain for $1-(1+\varepsilon)(2-p)>0$,

$$\begin{aligned}
&D_tw - \mathcal{A}w \\
&\leq p\left(u^2+m|Du|^2+\delta\right)^{\frac{p}{2}-1} \\
&\quad\times\Big\{\Big[\Big(\frac{N^2\beta_0^2}{4\varepsilon}+\frac{2a(1-p)}{2+\omega_0\varepsilon_0}\Big)\kappa + k_1 + \frac{\omega_0}{2}M_1 - \frac{\omega_0^2\varepsilon_0}{4+2\omega_0\varepsilon_0}M_1^- \\
&\qquad + \Big(\frac{1}{\varepsilon_0}+\frac{1}{2\varepsilon}+\frac{(1+\varepsilon)(2-p)}{4\varepsilon}\omega_0\Big)\omega_0M_2\Big]m|Du|^2 \\
&\qquad +\Big[\Big(1+\frac12\omega_0\varepsilon_0\Big)\varepsilon + 2\varepsilon\omega_0M_2\kappa_0^{-1} - 1 + (1+\varepsilon)(2-p)\Big]\kappa|D^2u|^2\Big\}.
\end{aligned}$$

Now we can choose $\varepsilon=\varepsilon_1>0$ such that

$$\Big(1+\frac12\omega_0\varepsilon_0\Big)\varepsilon_1 - 1 + (1+\varepsilon_1)(2-p) + 2\varepsilon_1\omega_0M_2\kappa_0^{-1} = 0. \tag{13.3.7}$$

Then, we fix a sufficiently large such that

$$L_a := \frac{N^2\beta_0^2}{4\varepsilon_1} + \frac{2a(1-p)}{2+\omega_0\varepsilon_0} < 0$$

and

$$L_a\kappa_0 + k_1 + \frac{\omega_0}{2}M_1 - \frac{\omega_0^2\varepsilon_0}{4+2\omega_0\varepsilon_0}M_1^- + \left(\frac{1}{\varepsilon_0} + \frac{1}{2\varepsilon_1} + \frac{(1+\varepsilon_1)(2-p)}{4\varepsilon_1}\omega_0\right)\omega_0 M_2 \le 0.$$

With this choice of ε_1 and a, we easily see that

$$D_t w - \mathcal{A}w \le 0.$$

Therefore, using the same technique as in the proof of Theorem 13.3.1, we finally get (13.3.5) in the case when $p \in (1,2]$, with

$$C_p = \max\left\{a^{p/2}, \left(1 + \frac{\omega_0\varepsilon_0}{2}\right)^{\frac{p}{2}}\right\}.$$

The other cases follow just as in the proof of the quoted theorem. ∎

The estimates (13.3.5) can be improved in the case when the dissipative condition is satisfied with the constant k_1 replaced with a function (still denoted by k_1) which, roughly speaking, balance the growth of the coefficients at infinity. In such a situation, we can recover the results in Theorem 13.3.1 and, in some particular cases, we can allow the exponential term appearing in the estimate of $DT(t)f$ to be of negative type. We reformulate Hypotheses 13.2.1 and 13.2.2 as follows:

Hypotheses 13.3.4 (i) the coefficients $q_{ij} = q_{ji}$ $(i,j = 1,\ldots,N)$ satisfy

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \ge \kappa(x)|\xi|^2, \qquad \xi \in \mathbb{R}^N, \quad x \in \overline{\Omega},$$

for some function κ such that $0 < \kappa_0 = \inf_{x\in\overline{\Omega}}\kappa(x)$. Moreover, there exist $\sigma \in (0,1]$ and $\beta_0 > 0$ such that

$$|Dq_{ij}(x)| \le \beta_0(\kappa(x))^\sigma, \qquad x \in \overline{\Omega};$$

(ii) Hypotheses 13.1.1(i), 13.1.1(v), 13.1.1(vi) and 13.2.2 are satisfied;

(iii) there exist a function $k_1 : \mathbb{R}^N \to \mathbb{R}$ and $p_0 \in (1,2)$ such that

$$\sum_{i,j=1}^N D_i b_j(x)\xi_i\xi_j \le k_1(x)|\xi|^2, \qquad \xi \in \mathbb{R}^N, \quad x \in \overline{\Omega}.$$

Moreover,

$$L_1 := \sup_{x\in\mathbb{R}^N}\left\{k_1(x) + \frac{N^2\beta_0^2}{4\varepsilon_1}(\kappa(x))^\sigma\right\} < +\infty,$$

where

$$\varepsilon_1 = \frac{2(p_0 - 1)\kappa_0}{(6 + \omega_0\varepsilon_0 - 2p_0)\kappa_0 + 4\omega_0 M_2}.$$

Remark 13.3.5 In the case when $\lim_{|x|\to+\infty} k_1(x)(\kappa(x))^{-\sigma} = -\infty$, Hypothesis 13.3.4(iii) is satisfied by any $p_0 \in (1,2]$.

Theorem 13.3.6 *Under Hypotheses* 13.3.4 *the estimate* (13.3.1) *holds true for any $p \in [p_0, +\infty)$, with M_p being replaced with ω_p, where*

$$\omega_p = L_1 + \frac{\omega_0}{2}M_1 - \frac{\omega^2\varepsilon_0}{4 + 2\omega_0\varepsilon_0}M_1^- + \left(\frac{1}{2\varepsilon_1} + \frac{1}{\varepsilon_0} + \frac{(1+\varepsilon_1)(2-p)}{4\varepsilon_1}\omega_0\right)\omega_0 M_2, \tag{13.3.8}$$

if $p \in (1,2]$ and $\omega_p = \omega_2^{p/2}$ if $p \in (2,+\infty)$.

Proof. The proof can be obtained arguing as in the proof of Theorem 13.3.1, estimating the terms containing Q and its gradient as in the proof of Theorem 13.3.3. ■

We are now in a position to extend the results in Theorem 13.3.2 to the case when Ω is an exterior domain.

Theorem 13.3.7 *Under the same assumptions as in Theorem* 13.3.3, *the estimate* (13.3.4) *holds true with $K_{p,t}t^{-p/2}$ replaced with $\widetilde{K}_{p,t}$, where*

$$\widetilde{K}_{p,t} = \left(\frac{1}{[p(p-1)t\kappa_0]^{p/2}} + 1\right)\left(1 + \frac{\omega_0\varepsilon_0}{2}\right)^{\frac{p}{2}},$$

if $p \in (1,2]$, and $\widetilde{K}_{p,t} = \widetilde{K}_{2,t}^{p/2}$ if $p \in (2,+\infty)$.
Similarly, under the assumptions of Theorem 13.3.6, *$\{T(t)\}$ satisfies* (13.3.4) *for any $p \ge p_0$, with M_p being replaced with ω_p given by* (13.3.8).

Proof. The proof of the second part of the theorem can be obtained arguing as in the proof of Theorem 13.3.2, whereas the proof of the first part can be obtained applying the technique in the proof of Theorem 12.3.8 to the semigroup $\{T_n(t)\}$ associated with the operator $\mathcal{A}^{(n)}$ defined in (13.2.1). This yields (13.3.4) with $T_n(t)$ instead of $T(t)$. Recalling that a suitable subsequence of $T_n(t)f$ converges to $T(t)f$ in $C^{1,2}(F)$ for any compact set $F \subset (0,+\infty)\times\overline{\Omega}$ (see the proof of Theorem 13.2.5), we deduce the assertion. ■

To conclude this section, we briefly see some consequence of all the previous pointwise gradient estimates. First of all, as the following proposition shows, they allow us to improve the uniform gradient estimate (13.1.6) with $k = 0$, in the case when $c \equiv 0$. We skip the proof since it is immediate.

Proposition 13.3.8 *Under the assumptions of Theorem* 13.3.2*, for any* $f \in C_b(\overline{\Omega})$ *it holds that*

$$\|DT(t)f\|_\infty \le \left(\frac{M_2(2+\omega_0\varepsilon_0)}{4\kappa_0(1-e^{-M_2 t})}\right)^{\frac{1}{2}} \|f\|_\infty, \qquad t>0, \tag{13.3.9}$$

if $M_2 \neq 0$ *and*

$$\|DT(t)f\|_\infty \le \left(\frac{2+\omega_0\varepsilon_0}{4\kappa_0 t}\right)^{\frac{1}{2}} \|f\|_\infty, \qquad t>0, \tag{13.3.10}$$

if $M_2 = 0$.

Similarly, under the assumptions of Theorem 13.3.3*, for any* $f \in C_b(\overline{\Omega})$*, it holds that*

$$\|DT(t)f\|_\infty \le \left(\frac{2+\omega_0\varepsilon_0}{4t\kappa_0}(1+2\kappa_0 t)\right)^{\frac{1}{2}} \|f\|_\infty, \qquad t>0.$$

Finally, under the assumptions of Theorem 13.3.6*,* $\{T(t)\}$ *satisfies* (13.3.9) (*with* M_2 *replaced with* ω_2 *given by* (13.3.8)) *and* (13.3.10).

Proposition 13.3.9 *Let the assumptions of Theorem* 13.3.2 *be satisfied and suppose that* $M_p \le 0$ *for some* $p \in (1,+\infty)$*. If* $f \in D(\widehat{A})$ (*see* (13.1.39)) *is such that* $\mathcal{A}f = 0$*, then* f *is constant. The same result holds in the case when* Ω *is an exterior domain, under the assumptions of Theorem* 13.3.6*, provided that* $\omega_p \le 0$ *for some* $p \in [p_0,+\infty)$.

Proof. The proof is similar to that of Theorem 7.2.5. Hence it is omitted. ■

13.4 The invariant measure of the semigroup

In this section we deal with the invariant measure μ of the semigroup whenever it exists. To simplify the notation, we simply write L^p_μ and $W^{1,p}_\mu$ ($p \in [1,+\infty)$) instead of $L^p(\Omega,\mu)$ and $W^{1,p}(\Omega,\mu)$. Moreover, we denote by $\|\cdot\|_p$ the usual norm of L^p_μ.

In view of the extension of the gradient estimates in Section 13.3 to this setting, we assume that c identically vanishes in $\overline{\Omega}$.

First of all we observe that, repeating just the same proofs as in Section 12.4, we can show that, when the invariant measure exists, it is unique and is absolutely continuous with respect to the restriction of the Lebesgue measure to the σ-algebra of Borel sets of Ω. Moreover, its density ϱ is everywhere positive in Ω and it belongs to $W^{1,p}_{\rm loc}(\Omega)$ for any $p \in [1,+\infty)$. In particular, ϱ is a continuous function in Ω. The Khas'minskii theorem, in the version of Theorem 12.4.3, is still the main existence theorem of an invariant measure.

When $\{T(t)\}$ admits an invariant measure it can be extended to a strongly continuous semigroup of contractions in L^p_μ for any $p \in [1,+\infty)$, and the set $D(\widehat{A})$ (see (13.1.39)) is a core of the infinitesimal generator L_p of the semigroup.

The following proposition is the main result of this section: under the assumptions of Section 13.3 it allows us to prove that the semigroup maps L^p_μ into $W^{1,p}_\mu$. Moreover, it provides us some useful estimates of the behaviour of the gradient of $T(t)f$ near $t = 0$, when $f \in L^p_\mu$.

Proposition 13.4.1 *Assume that the semigroup* $\{T(t)\}$ *admits an invariant measure and let Hypotheses* 13.1.1(*i*), 13.1.1(*ii*), 13.1.1(*iv*)–13.1.1(*vi*) *be satisfied. Further, suppose that* $c \equiv 0$. *Then,* $T(t)$ *maps* L^p_μ *into* $W^{l,p}_\mu$, *for any* $t > 0$ *and any* $p \in (1,+\infty)$. *Moreover,*

$$\|DT(t)f\|_p \le \begin{cases} \dfrac{(2+\omega_0\varepsilon_0)^{\frac{1}{2}}}{[2p(p-1)\kappa_0]^{\frac{1}{2}}}\left(\dfrac{M_pt}{1-e^{-M_pt}}\right)^{\frac{1}{p}} t^{-\frac{1}{2}}\|f\|_p, & p \in (1,2),\\[2ex] \left(\dfrac{2+\omega_0\varepsilon_0}{4\kappa_0}\dfrac{M_2t}{1-e^{-M_2t}}\right)^{\frac{1}{2}} t^{-\frac{1}{2}}\|f\|_p, & p \ge 2, \end{cases} \tag{13.4.1}$$

for any $t > 0$ *and any* $p \in (1,+\infty)$, *where* M_p *is given by* (13.3.3). *In the case when* $M_p = 0$, *the term* $M_pt(1-e^{-M_pt})^{-1}$ *should be replaced with* 1.

The same results hold in the particular case when Ω *is an exterior domain, and Hypotheses* 13.3.4 *are satisfied. In such a case, the estimate* (13.4.1) *is satisfied for any* $p \ge p_0$, *where* p_0 *is given by Hypothesis* 13.3.4(*iii*), *with the constant* M_p *being replaced with the constant* ω_p *defined by* (13.3.8).

Finally, if Hypotheses 13.2.1 *and* 13.2.2 (*where we take* $c \equiv 0$) *are satisfied, then* $T(t)$ *maps* L^p_μ *into* $W^{1,p}_\mu$ *for any* $t > 0$ *and any* $p \in (1,+\infty)$. *Moreover,*

$$\|DT(t)f\|_p \le \begin{cases} \left(\dfrac{1}{[p(p-1)t\kappa_0]^{p/2}}+1\right)^{\frac{1}{p}}\left(1+\dfrac{\omega_0\varepsilon_0}{2}\right)^{\frac{1}{2}}\|f\|_p, & p \in [p_0,2),\\[2ex] \left(\dfrac{1}{2t\kappa_0}+1\right)^{\frac{1}{2}}\left(1+\dfrac{\omega_0\varepsilon_0}{2}\right)^{\frac{1}{2}} t^{-\frac{1}{2}}\|f\|_p, & p \ge 2. \end{cases} \tag{13.4.2}$$

A partial characterization of the domain of the infinitesimal generator of the semigroup $\{T(t)\}$ in L^p_μ now follows easily from the estimates in Proposition 13.4.1. We skip the proof since it can be obtained arguing as in the proof of Proposition 8.3.3.

Proposition 13.4.2 *Suppose that Hypotheses* 13.1.1(*i*), 13.1.1(*ii*), 13.1.1(*iv*)–13.1.1(*vi*), *or Hypotheses* 13.3.4 *are satisfied. Then,* $D(L_p)$ *is continuously* *embedded in* $W^{1,p}_\mu$ *for any* $p \in (1,+\infty)$ (*under Hypothesis* 13.1.1) *and for any* $p \ge p_0$ (*under Hypothesis* 13.3.4). *Moreover, for any* $\omega > 0$, *there exists a positive constant* M_ω *such that*

$$\|Df\|_p \le M_{\omega,p}\|f\|_p^{\frac{1}{2}}\|(L_p-\omega)f\|_p^{\frac{1}{2}}, \qquad f \in D(L_p). \tag{13.4.3}$$

In the case when the estimates in Proposition 13.4.1 *are satisfied with an exponential term of negative type, we can take* $\omega = 0$ *in* (13.4.3).

The same results hold under Hypotheses 13.2.1 *and* 13.2.2 (*where we take* $c \equiv 0$), *for any* $p \in (1,+\infty)$.

13.5 Final remarks

To conclude the chapter, we observe that the results of these chapter have been very recently generalized in [17, 71] in the case when Ω is an exterior domain Ω. To be more precise, under assumptions comparable with those in Hypothesis 6.1.1, uniform estimates up to the third-order have been proved in [71], for the semigroup associated with the operator

$$\mathcal{A}\varphi = \sum_{i,j=1}^N q_{ij}D_{ij}u + \sum_{j=1}^N b_jD_ju,$$

with boundary conditions $\mathcal{B}u = 0$ on $\partial\Omega$, where $\mathcal{B}$ is a rather general class of differential boundary operators, including the Dirichlet and Neumann boundary operators.

Such estimates have been the keystone to prove optimal Schauder estimates for both the elliptic equation

$$\begin{cases} \lambda u - \mathcal{A}u = f, & \text{in } \Omega, \\ \mathcal{B}u = 0, & \text{on } \partial\Omega, \end{cases}$$

(for λ positive) and the parabolic problem

$$\begin{cases} D_tu - \mathcal{A}u = f, & t>0,\ x \in \Omega, \\ \mathcal{B}u = 0, & t>0,\ x \in \partial\Omega, \\ u(0,\cdot) = u_0, & x \in \overline{\Omega}, \end{cases}$$

adapting the arguments in Chapter 6.

Finally, in [17], the pointwise estimates of Section 13.3 have been obtained removing the Hypothesis 13.1.1 and 13.2.2.

Part III

A class of Markov semigroups in $\mathbb{R}^N$ associated with degenerate elliptic operators

Chapter 14

The Cauchy problem in $C_b(\mathbb{R}^N)$

14.0 Introduction

In this chapter we generalize the results in Chapters 2 and 6 to a class of linear degenerate elliptic operators in $C_b(\mathbb{R}^N)$ of the type

$$(\mathcal{A}u)(x) = \sum_{i,j=1}^{r} q_{ij}(x) D_{ij} u(x) + \sum_{i,j=1}^{N} b_{ij} x_j D_i u(x), \qquad x \in \mathbb{R}^N, \quad (14.0.1)$$

under the following assumptions on the coefficients and on r.

Hypotheses 14.0.1 (i) $N/2 \le r < N$ and

$$\sum_{i,j=1}^{r} q_{ij}(x)\xi_i\xi_j \ge \kappa(x)|\xi|^2, \qquad \xi \in \mathbb{R}^r, \ x \in \mathbb{R}^N,$$

for some function $\kappa : \mathbb{R}^N \to (0, +\infty)$ such that $\kappa_0 := \inf_{x\in\mathbb{R}^N} \kappa(x) > 0$;

(ii) there exists a positive constant C such that

$$|D^\alpha q_{ij}(x)| \le C|x|^{(1-|\alpha|)^+} \sqrt{\kappa(x)}, \quad (14.0.2)$$

for any $x \in \mathbb{R}^N$, any $i, j = 1, \ldots, r$ and any $|\alpha| \le 3$;

(iii) the matrix B can be split as

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

with $B_1 \in L(\mathbb{R}^r)$, $B_2, B_3^* \in L(\mathbb{R}^{N-r}, \mathbb{R}^r)$, $B_4 \in L(\mathbb{R}^{N-r})$. Moreover, $\mathrm{rank}(B_3) = N - r$.

The prototype of this class of elliptic operators is the degenerate Ornstein-Uhlenbeck operator

$$\mathcal{A}u(x) = \frac{1}{2}\sum_{i,j=1}^{N} q_{ij} D_{ij} u(x) + \sum_{i,j=1}^{N} b_{ij} x_j D_i u(x), \qquad x \in \mathbb{R}^N, \quad (14.0.3)$$

where the matrix $Q = (q_{ij})$ is singular and positive definite, while the matrix

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \qquad t > 0 \tag{14.0.4}$$

is strictly definite positive for any $t > 0$ (see Chapter 9). In such a case some results, similar to those that we present here, have been proved by A. Lunardi in [107], without assuming that $r \geq N/2$ (see also Remark 9.2.4). Note that the condition (14.0.4) is equivalent to the hypoellipticity of the operator $\mathcal{A}$. Elliptic operators similar to those considered in this chapter have been also investigated, generally with different techniques, by other authors. Among them, we quote the papers [63, 84, 95, 96, 97, 98, 110, 111, 122].

Here, we show that, if $\mathcal{A}$ is given by (14.0.1) with the coefficients satisfying Hypotheses 14.0.1, then the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{14.0.5}$$

admits, for any $f \in C_b(\mathbb{R}^N)$, a unique classical solution u (i.e., a unique function u continuous in $[0,+\infty) \times \mathbb{R}^N$ and continuously differentiable in $(0,+\infty) \times \mathbb{R}^N$, once with respect to time and twice with respect to the space variables, which solves (14.0.5)), which is bounded in $[0,+\infty) \times \mathbb{R}^N$. Therefore, as in the nondegenerate case, we can then associate a semigroup $\{T(t)\}$ of bounded linear operators with the operator $\mathcal{A}$ by setting $T(\cdot)f = u$. The results that we present are taken from [102, 103].

We stress that the coefficients q_{ij} may be unbounded in $\mathbb{R}^N$, but Hypotheses 14.0.1 give some bounds on their growth at infinity (see (14.1.3)). Notice that, due to our assumptions on Q and B, the hypoellipticity condition (14.0.4) is satisfied for any $t > 0$ and any $x \in \mathbb{R}^N$.

We construct the semigroup $\{T(t)\}$ as the "limit" (in a sense to be made precise later on) as ε tends to 0^+ of a family of semigroups $\{T_\varepsilon(t)\}$ associated with the uniformly elliptic operators

$$\mathcal{A}_\varepsilon = \mathcal{A} + \varepsilon \sum_{j=r+1}^{N} D_{jj}, \qquad \varepsilon > 0, \tag{14.0.6}$$

and to which all the results of Chapter 2 can be applied.

To prove the convergence of the function $T_\varepsilon(\cdot)f$ to $T(\cdot)f$, we first determine suitable estimates for the sup-norm of the space derivatives of the function $T_\varepsilon(t)f$, for any $t > 0$, with the constants therein appearing, being independent of ε (see (14.2.1)-(14.2.4)). Such estimates replace the interior estimate (C.1.15) and make the compactness argument used in the proof of Theorem 2.2.1 work also in this situation.

Once the convergence of $T_\varepsilon(\cdot)f$ to a solution to the Cauchy problem (14.0.5) is proved, sharp estimates on the behaviour of the sup-norm of the space derivatives of $T_\varepsilon(t)f$ as t approaches 0^+ can be obtained, just letting ε go to 0^+ in (14.2.1)-(14.2.4).

We remark also that some properties that we proved for Markov semigroups associated with uniformly elliptic operators in $C_b(\mathbb{R}^N)$ can be generalized also to this situation (see Section 14.3). In particular, the semigroup $\{T(t)\}$ is strong Feller and irreducible.

The uniform estimates are used in Section 14.4 to prove some regularity properties for the distributional solutions to both the elliptic equation

$$\lambda u(x) - \mathcal{A}u(x) = f(x), \qquad x \in \mathbb{R}^N, \qquad f \in C_b(\mathbb{R}^N), \quad \lambda > 0,$$

and to the nonhomogeneous Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + g(t,x), & t > 0,\, x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N. \end{cases}$$

when f and g are sufficiently smooth function.

To conclude this section, let us introduce a few notations.

Notations. Throughout this chapter, for any $x \in \mathbb{R}^N$, we denote by $Q(x) \in L(\mathbb{R}^r)$ the (strictly) positive definite matrix defined by $(Q(x))_{ij} = q_{ij}(x)$ for any $1 \le i,j \le r$. Similarly, for any $\varepsilon \ge 0$ and any $x \in \mathbb{R}^N$, we denote by $Q^{(\varepsilon)}(x) = (q_{ij}^{(\varepsilon)}(x)) \in L(\mathbb{R}^N)$ the matrix defined as follows: $q_{ij}^{(\varepsilon)}(x) = q_{ij}(x)$ if $i,j \le r$, $q_{ij}^{(\varepsilon)}(x) = \delta_{ij}\varepsilon$ if $i \vee j > r$.

14.1 Some preliminary results on the operator $\mathcal{A}_\varepsilon$

In this section we recall and prove some remarkable properties of the semigroup $\{T_\varepsilon(t)\}$ in $C_b(\mathbb{R}^N)$, associated with the operator $\mathcal{A}_\varepsilon$ ($\varepsilon > 0$) (see (14.0.6)). Such results will be used in following sections.

To begin with, we recall that, according to the results in Section 2.1, for any $\varepsilon > 0$, $u = T_\varepsilon(\cdot)f$ belongs to $C_b([0,+\infty) \times \mathbb{R}^N) \cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty) \times \mathbb{R}^N)$ and it solves the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}_\varepsilon u(t,x), & t > 0,\, x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{14.1.1}$$

for any $f \in C_b(\mathbb{R}^N)$. Moreover,

$$\|T_\varepsilon(t)f\|_{C_b(\mathbb{R}^N)} \le \|f\|_\infty, \qquad t > 0, \quad f \in C_b(\mathbb{R}^N). \tag{14.1.2}$$

Actually, $T_\varepsilon(\cdot)f$ is the unique classical solution to the Cauchy problem (14.1.1), which is bounded in $[0,T]\times\mathbb{R}^N$ for any $T>0$. Indeed, the assumptions on the growth of the diffusion coefficients q_{ij} $(i,j=1,\dots,N)$ in Hypotheses 14.0.1 imply that there exists a positive constant $\hat{C}$ such that

$$\kappa(x)\le\hat{C}|x|^2,\qquad |q_{ij}(x)|\le\hat{C}|x|^2,\quad i,j=1,\dots,N,\tag{14.1.3}$$

for any $x\in\mathbb{R}^N$. Therefore, the function $\varphi:\mathbb{R}^N\to\mathbb{R}$, defined by $\varphi(x)=1+|x|^2$ for any $x\in\mathbb{R}^N$, is a Lyapunov function for the operator $\mathcal{A}_\varepsilon$ $(\varepsilon>0)$, (i.e., it satisfies Hypothesis 4.0.2). The uniqueness of the bounded classical solution to the problem (14.1.1) now follows immediately from Theorem 4.1.3.

Now, we prove some boundedness and continuity properties up to $t=0$ of the functions $(t,x)\mapsto t^{(j-k)/2}(T_\varepsilon(t)f)(x)$ when $f\in C^k_b(\mathbb{R}^N)$ $(j,k\in\mathbb{N}$, $1\le j\le k\le 3)$, similar to those in Theorem 6.1.7.

Proposition 14.1.1 *Suppose that Hypotheses* 14.0.1 *are satisfied. Then, for any* $T>0$, *any* $\varepsilon>0$ *and any* $k=0,1,2,3$, *there exists a positive constant* $\widetilde{C}=\widetilde{C}_{T,\varepsilon}$ *such that*

$$\sum_{j=0}^{3}t^{\frac{(j-k)^+}{2}}\|D^jT_\varepsilon(t)f\|_\infty\le\widetilde{C}\|f\|_{C^k(\mathbb{R}^N)},\qquad t\in(0,T].\tag{14.1.4}$$

Proof. To begin with, we note that the estimate (14.1.4) does not follow immediately from Theorem 6.1.7. The techniques therein used need to be adapted to our situation. Indeed, the quoted theorem to be applied requires that the modulus of derivatives of the diffusion coefficients of the operator $\mathcal{A}_\varepsilon$ at any $x\in\mathbb{R}^N$ could be estimated by $C\overline{\kappa}(x)$, where $\overline{\kappa}(x)$ is the minimum eigenvalue of the matrix $Q^{(\varepsilon)}(x)$. Therefore, it requires that the derivatives of the coefficients are bounded in $\mathbb{R}$, which may be not the case in our situation.

We limit ourselves to proving (14.1.4) in the case when $(k,l)=(0,3)$, the other cases being similar and even easier. For any $n\in\mathbb{N}$, let $v_{n,\varepsilon}:(0,+\infty)\times\mathbb{R}^N\to\mathbb{R}$ be the function defined by

$$\begin{aligned}v_{n,\varepsilon}(t,x)=&\,|u_{n,\varepsilon}(t,x)|^2+at\vartheta_n^2|Du_{n,\varepsilon}(t,x)|^2+a^2t^2\vartheta_n^4|D^2u_{n,\varepsilon}(t,x)|^2\\&+a^3t^3\vartheta_n^6|D^3u_{n,\varepsilon}(t,x)|^2,\end{aligned}$$

for any $t>0$ and any $x\in\overline{B}(n)$, where $u_{n,\varepsilon}$ denotes the classical solution to the problem

$$\begin{cases}D_tu_{n,\varepsilon}(t,x)=\mathcal{A}_\varepsilon u_{n,\varepsilon}(t,x), & t>0,\ x\in B(n),\\ u_{n,\varepsilon}(t,x)=0, & t>0,\ x\in\partial B(n),\\ u_{n,\varepsilon}(0,x)=\vartheta_n(x)f(x), & x\in B(n),\end{cases}$$

and $\vartheta_n\in C^\infty_c(\mathbb{R}^N)$ is such that $\chi_{B(n/2)}\le\vartheta_n\le\chi_{B(n)}$. From now on, to simplify the notation, when there is no damage of confusion, we do not write explicitly the dependence on n and ε of the functions that we consider. As it is immediately seen, the function v solves the Cauchy problem

$$\begin{cases} D_t v(t,x) = \mathcal{A}_\varepsilon v(t,x) + g(t,x), & t \in [0,T],\, x \in B(n), \\ v(t,x) = 0, & t \in [0,T],\, x \in \partial B(n), \\ v(0,x) = (f(x))^2, & x \in B(n), \end{cases}$$

where $g = \sum_{j=1}^5 g_j$ is given by

$$\begin{aligned} g_1 = & -2\sum_{i,j=1}^N q_{ij}^{(\varepsilon)} D_i u D_j u - 2at\vartheta^2 \sum_{i,j,h=1}^N q_{ij}^{(\varepsilon)} D_{ih} u D_{jh} u \\ & -2a^2t^2\vartheta^4 \sum_{i,j,h,k=1}^N q_{ij}^{(\varepsilon)} D_{ihk} u D_{jhk} u - 2a^3t^3\vartheta^6 \sum_{i,j,h,k,l=1}^N q_{ij}^{(\varepsilon)} D_{ihkl} u D_{jhkl} u, \end{aligned}$$

$$\begin{aligned} g_2 = & -2at\vartheta(\mathcal{A}_\varepsilon\vartheta)|Du|^2 - 4a^2t^2\vartheta^3(\mathcal{A}_\varepsilon\vartheta)|D^2u|^2 - 6a^3t^3\vartheta^5(\mathcal{A}_\varepsilon\vartheta)|D^3u|^2 \\ & -8at\vartheta \sum_{i,j,h=1}^N q_{ij}^{(\varepsilon)} D_j\vartheta D_h u D_{ih} u - 16a^2t^2\vartheta^3 \sum_{i,j,h,k=1}^N q_{ij}^{(\varepsilon)} D_j\vartheta D_{hk} u D_{ihk} u \\ & -24a^3t^3\vartheta^5 \sum_{i,j,h,k,l=1}^N q_{ij}^{(\varepsilon)} D_j\vartheta D_{hkl} u D_{ihkl} u + 2at\vartheta^2 \sum_{i,j=1}^N b_{ij} D_i u D_j u \\ & +4a^2t^2\vartheta^4 \sum_{i,j,h=1}^N b_{ij} D_{ih} u D_{jh} u + 6a^3t^3\vartheta^6 \sum_{i,j,h,k=1}^N b_{ij} D_{ihk} u D_{jhk} u, \end{aligned}$$

$$\begin{aligned} g_3 = & \; 2at\vartheta^2 \sum_{i,j=1}^r \sum_{h=1}^N D_h q_{ij} D_h u D_{ij} u + 4a^2t^2\vartheta^4 \sum_{i,j=1}^r \sum_{h,k=1}^N D_h q_{ij} D_{hk} u D_{ijk} u \\ & +6a^3t^3\vartheta^6 \sum_{i,j=1}^r \sum_{h,k,l=1}^N D_h q_{ij} D_{hkl} u D_{ijkl} u \\ & +6a^3t^3\vartheta^6 \sum_{i,j=1}^r \sum_{h,k,l=1}^N D_{hk} q_{ij} D_{ijl} u D_{hkl} u \\ & +2a^2t^2\vartheta^4 \sum_{i,j=1}^r \sum_{h,k=1}^N D_{hk} q_{ij} D_{ij} u D_{hk} u \\ & +2a^3t^3\vartheta^6 \sum_{i,j=1}^r \sum_{h,k,l=1}^N D_{hkl} q_{ij} D_{ij} u D_{hkl} u, \end{aligned}$$

$$g_4 = -2at|Du|^2 \sum_{i,j=1}^N q_{ij}^{(\varepsilon)} D_i\vartheta D_j\vartheta - 12a^2t^2\vartheta^2|D^2u|^2 \sum_{i,j=1}^N q_{ij}^{(\varepsilon)} D_i\vartheta D_j\vartheta$$
$$-30a^3t^3\vartheta^4|D^3u|^2 \sum_{i,j=1}^N q_{ij}^{(\varepsilon)} D_i\vartheta D_j\vartheta,$$

$$g_5 = a\vartheta^2|Du|^2 + 2a^2t\vartheta^4|D^2u|^2 + 3a^3t^2\vartheta^6|D^3u|^2.$$

In the rest of the proof, to simplify the notation, we denote by $D^k_{\#,1}u$ (resp. $D^k_{\#,2}u$) ($k = 1, 2, 3, 4$) the vectors whose entries are the k-th-order derivatives of u corresponding, respectively, to multi-indices α with at least a component less than $r+1$, and multi-indices α with all the components greater than r.

Taking Hypothesis 14.0.1(i) into account, we easily deduce that

$$\begin{aligned} g_1 \le& -2(\varepsilon \wedge \kappa_0)|Du|^2 - 2at\vartheta^2\kappa|D^2_{\#,1}u|^2 \\ &-2at\vartheta^2\varepsilon|D^2_{\#,2}u|^2 - 2a^2t^2\vartheta^4\kappa|D^3_{\#,1}u|^2 \\ &-2a^2t^2\vartheta^4\varepsilon|D^3_{\#,2}u|^2 - 2a^3t^3\vartheta^6\kappa|D^4_{\#,1}u|^2 - 2a^3t^3\vartheta^6\varepsilon|D^4_{\#,2}u|^2. \end{aligned} \tag{14.1.5}$$

To estimate the function g_2 we observe that, arguing as in the proof of (6.1.23)-(6.1.25) and taking (14.1.3) into account, we get

$$|(\mathcal{A}\vartheta)(x)| \le C_1, \qquad \left|(Q^{(\varepsilon)}(x)D\vartheta(x))_i\right| \le C_1 \begin{cases} \sqrt{\kappa(x)}, & \text{if } i \le r, \\ 1, & \text{if } i > r, \end{cases} \tag{14.1.6}$$

for any $x \in \mathbb{R}^N$ and some positive constant C_1, independent of ε and x. Now, using (14.1.6), we obtain

$$\begin{aligned} &\left|8at\vartheta \sum_{i,j,h=1}^N q_{ij}^{(\varepsilon)} D_j\vartheta D_h u D_{ih}u\right| \\ \le& 8at\vartheta \left|\sum_{i=1}^r \sum_{j,h=1}^N q_{ij}^{(\varepsilon)} D_j\vartheta D_h u D_{ih}u\right| + 8at\vartheta \left|\sum_{i=r+1}^N \sum_{j,h=1}^N q_{ij}^{(\varepsilon)} D_j\vartheta D_h u D_{ih}u\right| \\ \le& 8at\vartheta C_1\sqrt{\kappa} \sum_{i=1}^r \sum_{h=1}^N |D_h u||D_{ih}u| + 8at\vartheta C_1 \sum_{i=r+1}^N \sum_{h=1}^N |D_h u||D_{ih}u| \\ \le& 8atC_1\left(\frac{r}{4\delta}|Du|^2 + \delta\vartheta^2\kappa|D^2_{\#,1}u|^2\right) \\ &+8atC_1\left(\frac{N-r}{4\delta}|Du|^2 + \frac{\delta}{\kappa_0}\vartheta^2\kappa|D^2_{\#,1}u|^2 + \delta\vartheta^2|D^2_{\#,2}u|^2\right) \\ \le& 8atC_1\frac{N}{4\delta}|Du|^2 + 8atC_1\delta\left(1 + \frac{1}{\kappa_0}\right)\vartheta^2\kappa|D^2_{\#,1}u|^2 + 8atC_1\delta\vartheta^2|D^2_{\#,2}u|^2. \end{aligned}$$

Similarly, we can estimate all the other terms in the definition of g_2 and g_3, so that, finally, we get

$$
\begin{aligned}
g_2 \le\ & 2at\left\{C_1 + C_1\frac{N}{\delta} + \|B\|\right\}|Du|^2 \\
&+4at\left\{2C_1\delta\left(1+\frac{1}{\kappa_0}\right) + at\left(\frac{C_1}{\kappa_0} + \frac{NC_1}{\delta\kappa_0} + \frac{\|B\|}{\kappa_0}\right)\right\}\vartheta^2\kappa|D^2_{\#,1}u|^2 \\
&+4at\left\{2C_1\delta + at\left(C_1 + C_1\frac{N}{\delta} + \|B\|\right)\right\}\vartheta^2|D^2_{\#,2}u|^2 \\
&+2a^2t^2\left\{8C_1\delta\left(1+\frac{1}{\kappa_0}\right) + 3at\left(\frac{C_1N}{\delta\kappa_0} + \frac{C_1}{\kappa_0} + \frac{\|B\|}{\kappa_0}\right)\right\}\vartheta^4\kappa|D^3_{\#,1}u|^2 \\
&+2a^2t^2\left\{8C_1\delta + 3at\left(\frac{C_1N}{\delta} + C_1 + \|B\|\right)\right\}\vartheta^4|D^3_{\#,2}u|^2 \\
&+24a^3t^3C_1\delta\left(1+\frac{1}{\kappa_0}\right)\vartheta^6\kappa|D^4_{\#,1}u|^2 + 24a^3t^3C_1\delta\vartheta^6|D^4_{\#,2}u|^2 \qquad (14.1.7)
\end{aligned}
$$

and

$$
\begin{aligned}
g_3 \le\ & atC\frac{N^2}{2\delta}|Du|^2 + atCN\left\{2\delta + at\left(\frac{N}{\delta} + N + \frac{N}{\kappa_0}\right) + a^2t^2\frac{N}{2\delta}\right\}\vartheta^2\kappa|D^2_{\#,1}u|^2 \\
&+a^2t^2CN^2\left(\frac{1}{\delta} + 1 + atC\frac{N}{2\delta}\right)\vartheta^2|D^2_{\#,2}u|^2 \\
&+a^2t^2CN\left\{4\delta + atN\left(\frac{3}{2\delta\kappa_0} + 3 + \frac{3}{\kappa_0} + 2\delta\right)\right\}\vartheta^4\kappa|D^3_{\#,1}u|^2 \\
&+a^3t^3CN^2\left(\frac{3}{2\delta} + 3\right)\vartheta^4|D^3_{\#,2}|^2 + 6a^3t^3C\delta N\vartheta^6\kappa|D^4_{\#,1}u|^2, \qquad (14.1.8)
\end{aligned}
$$

for any $\delta > 0$, where C is given by (14.0.2). For further details, see the proof of the estimate (6.1.26).

Now, let us fix $T > 0$. Observing that $g_4 \le 0$ in $(0,+\infty)\times\mathbb{R}^N$, from (14.1.5), (14.1.7) and (14.1.8) we deduce that

$$
\begin{aligned}
g \le\ & \left\{-2(\varepsilon\wedge\kappa_0) + aT\left(2C_1 + 2C_1\frac{N}{\delta} + 2\|B\| + \frac{CN^2}{2\delta}\right)\right\}|Du|^2 \\
&+at\left\{-2 + 8C_1\delta\left(1+\frac{1}{\kappa_0}\right) + 2CN\delta + 4aT\left(\frac{C_1}{\kappa_0} + \frac{NC_1}{\delta\kappa_0} + \frac{\|B\|}{\kappa_0}\right)\right. \\
&\qquad\left. +aTCN\left(\frac{N}{\delta} + N + \frac{N}{\kappa_0} + aT\frac{N}{2\delta}\right)\right\}|D^2_{\#,1}u|^2
\end{aligned}
$$

$$
\begin{aligned}
&+at\bigg\{-2\varepsilon+8C_1\delta+4aT\left(C_1+\frac{C_1N}{\delta}+\|B\|\right)\\
&\qquad+aTCN^2\left(\frac{1}{\delta}+1+aT\frac{CN}{2\delta}\right)\bigg\}\vartheta^2|D^2_{\#,2}u|^2\\
&+a^2t^2\bigg\{-2+16C_1\delta\left(1+\frac{1}{\kappa_0}\right)+4CN\delta+6aT\left(\frac{C_1N}{\delta\kappa_0}+\frac{C_1}{\kappa_0}+\frac{\|B\|}{\kappa_0}\right)\\
&\qquad+aTCN^2\left(\frac{3}{2\delta\kappa_0}+3+\frac{3}{\kappa_0}+2\delta\right)\bigg\}\vartheta^4\kappa|D^3_{\#,1}u|^2\\
&+a^2t^2\bigg\{-2\varepsilon+16C_1\delta+6aT\left(\frac{C_1N}{\delta}+C_1+\|B\|\right)\\
&\qquad+aTCN^2\left(\frac{3}{2\delta}+3\right)\bigg\}\vartheta^4|D^3_{\#,2}u|^2\\
&+2a^3t^3\bigg\{-1+3aT\delta\left[3CN+4C_1\left(1+\frac{1}{\kappa_0}\right)\right]\bigg\}\vartheta^6|D^4_{\#,1}u|^2\\
&+2a^3t^3\left\{-\varepsilon+12TC_1\delta\right\}\vartheta^6|D^4_{\#,2}u|^2,
\end{aligned}
$$

for any $t\in(0,T]$. A straightforward computation shows that we can choose (a,δ) such that $g\le 0$ in $(0,T]\times B(n)$. Theorem 4.1.3 now implies that

$$
|v_{n,\varepsilon}(t,x)|\le \widetilde{C}\|f\|_\infty,\qquad t\in[0,T],\quad x\in\overline{B}(n),
$$

for some positive constant $\widetilde{C}$, independent of u and n. Since, by Proposition 2.2.1, $u_{n,\varepsilon}$ converges to u_ε on compact sets of $(0,+\infty)\times\mathbb{R}^N$, as n tends to $+\infty$, letting n go to $+\infty$ we get (14.1.4). ∎

The following theorem guarantees the continuity of the functions $(t,x)\mapsto t^{j/2}(D^jT_\varepsilon(t)f)(x)$ at $t=0$, for any $j=1,2,3$ and any $f\in C_b(\mathbb{R}^N)$. Its proof is essentially based upon Proposition 2.2.9 and the interior estimates in Theorem C.1.4.

Proposition 14.1.2 *Under Hypotheses* 14.0.1, *for any* $f\in C^k_b(\mathbb{R}^N)$ $(k=0,\ldots,3)$ *and any* $\varepsilon>0$, *the function* $(t,x)\mapsto t^{(j-k)^+/2}(D^jT_\varepsilon(t)f)(x)$ *is continuous in* $[0,+\infty)\times\mathbb{R}^N$ *for any* $j=0,\ldots,3$. *In particular,*

$$
\begin{aligned}
&(i)\ \lim_{t\to 0^+}t^{\frac{j-k}{2}}(D^jT_\varepsilon(t)f)(x)=0,\qquad if\ j>k,\\
&(ii)\ \lim_{t\to 0^+}(D^jT_\varepsilon(t)f)(x)=D^jf(x),\quad if\ j\le k,
\end{aligned}
\tag{14.1.9}
$$

for any $x\in\mathbb{R}^N$ *and* $j,k=0,1,2,3$.

Proof. The continuity of the functions $(t,x)\mapsto t^{(j-k)^+/2}(D^jT_\varepsilon(t)f)(x)$ in $(0,+\infty)\times\mathbb{R}^N$ is a classical result recalled in Theorem C.1.4. Moreover, the formula (14.1.9)(ii) can be proved by arguing as in the proof of Proposition

7.1.1. So, we just need to show that the formula (14.1.9)(i) holds. We limit ourselves to dealing with the case when $k = 0$, since the other cases are similar and even simpler. We use a localization argument. For any $x \in \mathbb{R}^N$ we fix $r, R > 0$ with $R - r > 1$. Applying (C.1.16) and (C.1.18) with $\Omega' = x + B(r)$ and $\Omega = x + B(R)$ yields

$$\begin{aligned}
&\sup_{(t,y)\in(0,1)\times(x+B(r))} t^{\frac{1}{2}}|Dv(t,y)| + \sup_{(t,y)\in(0,1)\times(x+B(r))} t|D^2v(t,y)| \\
&\quad + \sup_{(t,y)\in(0,1)\times(x+B(r))} t^{\frac{3}{2}}|D^3v(t,y)| \\
&\leq C \sup_{(t,y)\in(0,1)\times(x+B(R))} |v(t,y)|, \qquad (14.1.10)
\end{aligned}$$

for any solution to the parabolic equation $D_t v - \mathcal{A}_\varepsilon v = 0$ and some positive constant $C = C(r, R, \varepsilon)$. Now, we fix a sequence $\{f_n\} \in C_b^3(\mathbb{R}^N)$ of smooth functions, bounded in the sup-norm and converging locally uniformly in $\mathbb{R}^N$ to f as n tends to $+\infty$. Applying (14.1.10) with $v = T_\varepsilon(\cdot)(f - f_n)$ and taking Proposition 2.2.9 into account, we deduce that the function $(t,x) \mapsto \psi_{j,n}(t,x) := t^{j/2}(D^jT_\varepsilon(t)f_n)(x)$ $(j = 1,2,3)$ converges uniformly in $(0,T) \times (x + B(\overline{R}))$ to the function $(t,x) \mapsto \psi_j(t,x) := t^{j/2}(D^jT_\varepsilon(t)f)(x)$. By (14.1.9)(ii), each function $\psi_{n,j}$ is continuous in $[0,1] \times (x + \overline{B}(r))$ and it vanishes at $t = 0$. Hence, each ψ_j $(j = 1,2,3)$ is continuous in $[0,1] \times (x + \overline{B}(r))$ and it vanishes at $t = 0$, as well. The estimate (14.1.9)(i) follows. ∎

14.2 Existence, uniqueness results and uniform estimates

In this section we show that, for any $f \in C_b(\mathbb{R}^N)$, the problem (14.0.5) admits a unique classical solution u bounded in $[0,+\infty) \times \mathbb{R}^N$, where we recall that by classical solution we mean, as usual, a function $u : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ continuous in $[0,+\infty) \times \mathbb{R}^N$, continuously differentiable, once with respect to time and twice with respect to the space variables in $(0,+\infty) \times \mathbb{R}^N$, which satisfies (14.0.5). Moreover, we show that the family of linear operators $\{T(t)\} \in L(C_b(\mathbb{R}^N))$ defined by $T(t)f = u(t,\cdot)$, for any $t > 0$ and any $f \in C_b(\mathbb{R}^N)$, gives rise to a semigroup of linear operators satisfying

$$\|D_iT(t)f\|_\infty \leq Ce^{\omega t}t^{-\frac{1}{2}-H(i-r)}\|f\|_\infty, \qquad t > 0, \tag{14.2.1}$$

$$\|D_{ij}T(t)f\|_\infty \leq Ce^{\omega t}t^{-(1+H(i-r)+H(j-r))}\|f\|_\infty, \qquad t > 0, \tag{14.2.2}$$

$$\|D_{ijh}T(t)f\|_\infty \leq Ce^{\omega t}t^{-\frac{3}{2}-H(i-r)-H(j-r)-H(h-r)}\|f\|_\infty, \qquad t > 0, \tag{14.2.3}$$

for any $i, j, h = 1, \ldots, N$ and some positive constants C and ω, independent of f and t. Here, $H(s) = 0$ if $s \leq 0$ and $H(s) = 1$ if $s > 0$. Furthermore, we prove that the more f is regular, the more the estimates (14.2.1)-(14.2.3) can be improved. More precisely, we show that

$$\|T(t)f\|_{C_b^k(\mathbb{R}^N)} \le Ce^{\omega t}\|f\|_{C_b^k(\mathbb{R}^N)}, \qquad t>0, \quad k=1,2,3, \tag{14.2.4}$$

$$\|D_{ij}T(t)f\|_\infty \le Ce^{\omega t}t^{-\frac{1}{2}-H(i-r)}\|f\|_{C_b^1(\mathbb{R}^N)}, \qquad t>0, \quad i\le j, \tag{14.2.5}$$

$$\|D_{ijh}T(t)f\|_\infty \le Ce^{\omega t}t^{c_{ijk}^k}\|f\|_{C_b^k(\mathbb{R}^N)}, \qquad t>0, \quad 1\le j\le h, \quad k=0,1,2, \tag{14.2.6}$$

where

$$c_{ijh}^k = \frac{3-k}{2} + H(i-r) + ((2-k)-(1-k)^+)H(j-r) + (1-k)^+H(h-r),$$

and C, ω are as above.

As it has already been claimed in the introduction, we will construct the semigroup $\{T(t)\}$ as the limit as ε tends to 0^+ (in a sense to be made clear) of the semigroup $\{T_\varepsilon(t)\}$. As a first step in this direction, in subsection 14.2.2 we show that each semigroup $\{T_\varepsilon(t)\}$ satisfies (14.2.1)-(14.2.6) with constants being independent of ε. This will allow us to apply a compactness argument to prove the convergence of the function $T_\varepsilon(\cdot)f$ to what we will denote by $T(\cdot)f$, for any $f\in C_b(\mathbb{R}^N)$. Moreover, letting ε go to 0^+ in the previous estimates written for the approximating semigroups, we will see that $\{T(t)\}$ satisfies (14.2.1)-(14.2.6) as well.

14.2.1 Some preliminary lemmata

In this section we collect some very easy results of linear algebra that we need in this chapter. For the reader's convenience we give the proofs.

Lemma 14.2.1 *Suppose that $A\in L(\mathbb{R}^n,\mathbb{R}^m)$ is a given matrix. Then, there exists a matrix $C\in L(\mathbb{R}^m,\mathbb{R}^n)$ such that*

$$AC + C^*A^*$$

is strictly positive (resp. negative) definite if and only if $n\ge m$ and $\operatorname{rank}(A)=m$.

Let $B\in L(\mathbb{R}^n,\mathbb{R}^m)$. Then, there exists a matrix $C\in L(\mathbb{R}^m,\mathbb{R}^n)$ such that

$$CB + B^*C^*$$

is uniformly positive (resp. negative) definite if and only if $m\ge n$ and $\operatorname{rank}(B)=n$.

Proof. As far as the first statement is concerned, we limit ourselves to proving that we can choose C such that the matrix $AC+C^*A^*$ (resp. $CB+B^*C^*$) is strictly positive definite. Replacing C by $-C$ we get the assertion also in the other case.

Suppose that the matrix $AC + C^*A^*$ is strictly positive definite. Then, for any $\xi \in \mathbb{R}^m \setminus \{0\}$, we have

$$0 < \langle (AC + C^*A^*)\xi, \xi\rangle = 2\langle AC\xi, \xi\rangle.$$

Hence, the matrix AC is not singular, i.e., its rank equals m. Suppose that $n < m$. Then the ranks of A and C equal at most n and, consequently, the rank of AC is at most n. This implies that AC is singular. Hence, $n \geq m$. Moreover, $\mathrm{rank}(A) = m$, otherwise $\mathrm{rank}(AC)$ would be strictly less than m and AC would be singular.

Vice versa, let us assume that $n \geq m$ and $\mathrm{rank}(A) = n$. Moreover, let $D \in L(\mathbb{R}^n)$ be an invertible matrix such that

$$AD = \begin{pmatrix} A_1 \ A_2 \end{pmatrix},$$

where $A_1 \in L(\mathbb{R}^m)$ is invertible and $A_2 \in L(\mathbb{R}^{n-m}, \mathbb{R}^m)$. Let $\widetilde{C} \in L(\mathbb{R}^m, \mathbb{R}^n)$ be the matrix defined by

$$\widetilde{C} = \begin{pmatrix} A_1^{-1}K \\ 0 \end{pmatrix},$$

$K \in L(\mathbb{R}^m)$ being any strictly positive definite matrix. We set $C = D\widetilde{C}$ and observe that

$$AC = AD\widetilde{C} = A_1A_1^{-1}K = K.$$

Hence, $AC + C^*A^* = 2K$ so that the matrix $AC + C^*A^*$ is strictly positive definite.

Let us prove the last part of the lemma. The same arguments as above show that if $CB + B^*C^*$ is strictly positive definite, then $m \geq n$. Vice versa, suppose that $m \geq n$, $\mathrm{rank}(B) = n$, and let $C = B^*$. Then,

$$\langle (CB + B^*C^*)\xi, \xi\rangle = 2\langle B^*B\xi, \xi\rangle = 2\|B\xi\|^2,$$

for any $\xi \in \mathbb{R}^n$. Since $\mathrm{rank}(B) = n$, then the mapping $T : \mathbb{R}^n \to \mathbb{R}^m$, defined by $T(\xi) = B\xi$ for any $\xi \in \mathbb{R}^N$, is injective. It follows that the matrix B^*B is strictly positive definite and this finishes the proof. ■

Lemma 14.2.2 *Fix $k, n \in \mathbb{N}$ and let $A(t) \in L(\mathbb{R}^N)$ be the matrix defined by*

$$\begin{pmatrix} A_{11}t^k & A_{12}t^{k+1} & \cdots & \cdots & A_{1n}t^{k+n-1} \\ A_{12}^*t^{k+1} & A_{22}t^{k+2} & \cdots & \cdots & A_{2n}t^{k+n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A_{1n}^*t^{k+n-1} & A_{2n}^*t^{k+n} & \cdots & \cdots & A_{nn}t^{k+2n-2} \end{pmatrix}, \qquad t > 0,$$

where $A_{ij} \in L(\mathbb{R}^{N_j}, \mathbb{R}^{N_i})$ ($N_1 + \ldots + N_n = N$) and $A_{ii} = A_{ii}^*$ for any $i = 1, \ldots, n$. Then, $A(t)$ is positive (resp. negative) definite, for any $t > 0$, if and only if it is positive (resp. negative) definite at $t = 1$. In such a case if, for any $\xi \in \mathbb{R}^N$, we split $\xi = (\xi_1, \ldots, \xi_n)$ with $\xi_i \in \mathbb{R}^{N_i}$, we have

$$\langle A(t)\xi, \xi\rangle \geq \lambda_{\min}(A(1)) \sum_{j=0}^{n-1} t^{k+2j}|\xi_{j+1}|^2, \qquad t > 0, \tag{14.2.7}$$

if $A(1)$ is positive definite, and

$$\langle A(t)\xi, \xi\rangle \leq \lambda_{\max}(A(1)) \sum_{j=0}^{n-1} t^{k+2j}|\xi_{j+1}|^2, \qquad t > 0, \tag{14.2.8}$$

if $A(1)$ is negative definite.

Proof. Of course, if $A(t)$ is strictly positive (resp. negative) definite for any $t > 0$, then, in particular, it is strictly positive (resp. negative) definite at $t = 1$.

Vice versa, assume that $A(1)$ is strictly positive definite. For any $t > 0$ and any $\xi \in \mathbb{R}^N$, split as in the statement of the lemma, let

$$\overline{\xi}^T = (t^{k/2}\xi_1, t^{k/2+1}\xi_2, \ldots, t^{k/2+n-1}\xi_n)^T.$$

As it is immediately seen,

$$\langle A(t)\xi, \xi\rangle = \langle A(1)\overline{\xi}, \overline{\xi}\rangle.$$

Therefore, $A(t)$ is strictly positive (resp. negative) definite and the inequalities (14.2.7), (14.2.8) follow. ∎

Lemma 14.2.3 *Suppose that $Q, A \in L(\mathbb{R}^N)$ are two positive definite matrices. Further, assume that the submatrix $Q_0 = (q_{ij})_{i,j=1}^r$ is strictly positive definite and $q_{ij} = 0$ if $i \vee j > r$. Then,*

$$\mathrm{Tr}\,(QA) \geq \lambda_{\min}(Q_0)\mathrm{Tr}\,(A_1),$$

where A_1 is the submatrix obtained from A by erasing the last $N - r$ rows and lines.

Proof. Since Q_0 is strictly positive definite, then there exist an orthogonal matrix $B = (b_{ij})_{i,j=1}^N \in L(\mathbb{R}^r)$ and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$ such that $Q_0 = B^*\Lambda B$. This implies that the matrix $\widetilde{B} \in L(\mathbb{R}^N)$, defined by $\widetilde{b}_{ij} = b_{ij}$ if $1 \leq i, j \leq r$, $\widetilde{b}_{ij} = \delta_{ij}$ if $i \vee j > r$, is orthogonal and $\widetilde{B}Q\widetilde{B}^* = \widetilde{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$. Hence,

$$\mathrm{Tr}\,(QA) = \mathrm{Tr}\,(\widetilde{B}^{-1}\widetilde{\Lambda}\widetilde{B}A) = \mathrm{Tr}\,(\widetilde{\Lambda}\widetilde{B}A\widetilde{B}^{-1}) = \mathrm{Tr}\,(\widetilde{\Lambda}\widetilde{B}A\widetilde{B}^*).$$

We now observe that since A is positive definite, then $\widetilde{B}A\widetilde{B}^* =: (c_{ij})_{i,j=1}^N$ is. This, in particular, implies that $c_{ii} \geq 0$ for any $i = 1, \ldots, N$. It follows that

$$\mathrm{Tr}\,(\Lambda\widetilde{B}A\widetilde{B}^*) = \sum_{j=1}^{r} \lambda_j\, c_{jj} \geq \lambda \mathrm{Tr}\,(BA_1B^*) = \lambda_{\min}(Q_0)\mathrm{Tr}\,(A_1),$$

and the assertion follows. ■

14.2.2 Uniform estimates for the space derivatives of the approximating semigroups $\{T_\varepsilon(t)\}$

To begin with, we introduce a few notations. We set $D_{*,1}w = (D_1w, \ldots, D_rw)$ and $D_{*,2}w = (D_{r+1}w, \ldots, D_Nw)$. Then, we introduce, instead of the corresponding tensors, the vectors $D_*^k w$ $(k = 2, 3, 4)$ consisting of all the derivatives $D_{i_1,\ldots,i_k}w$ ordered as follows: $D_{i_1,\ldots,i_k}w$ precedes $D_{j_1,\ldots,j_k}w$ if $i_l \leq j_l$ for any $l = 1, \ldots, k$ and there exists $l_0 \in \{1, \ldots, k\}$ such that $i_{l_0} < j_{l_0}$, or $\{j_1, \ldots, j_k\}$ contains more indexes $j_l \geq r+1$ than the set $\{i_1, \ldots, i_k\}$. Finally, we set $D_*^k w^T = ((D_{*,1}^k w)^T, \ldots, (D_{*,k+1}^k w)^T)$, where the vector $D_{*,j}^k w$ contains all the derivatives $D_{i_1,\ldots,i_k}^k w$ with $i_{k+1-j} \leq r < i_{k+2-j}$ (when such inequalities are meaningful). For instance, if $N = 4$, $k = 3$ and $r = 2$, then

$$\begin{aligned}
(D_{*,1}^3 w)^T &= (D_{111}w, D_{112}w, D_{122}w, D_{222}w),\\
(D_{*,2}^3 w)^T &= (D_{113}w, D_{114}w, D_{123}w, D_{124}w, D_{223}w, D_{224}w),\\
(D_{*,3}^3 w)^T &= (D_{133}w, D_{134}w, D_{144}w, D_{233}w, D_{234}w, D_{244}w),\\
(D_{*,4}^3 w)^T &= (D_{333}w, D_{334}w, D_{344}w, D_{444}w).
\end{aligned}$$

Theorem 14.2.4 *Let $\varepsilon > 0$ and assume that Hypotheses* 14.0.1 *are satisfied. Then, for any $k = 1, 2, 3$ and any $\omega > 0$, there exists a positive constant $C = C(\omega)$, independent of ε, such that*

$$\|D_{*,j}^k T_\varepsilon(t)f\|_\infty \leq Ce^{\omega t}t^{-\frac{k+2j-2}{2}}\|f\|_\infty, \qquad t > 0, \quad j = 1, \ldots, k+1, \tag{14.2.9}$$

for any $f \in C_b(\mathbb{R}^N)$ and any $\varepsilon > 0$.

Proof. In order to simplify the notation, throughout the proof, we simply write u instead of $T_\varepsilon(\cdot)f$. Moreover, for any function v depending on t and x we write $v(t)$ when we want to point out only the dependence on t. Finally, we set

$$n_m^1 = \frac{1}{2}m(m+1), \qquad n_m^2 = \frac{1}{6}m(m+1)(m+2), \quad m \in \mathbb{N}. \tag{14.2.10}$$

Let us introduce the function $\xi : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$\xi(t,x) = \frac{1}{2}\alpha^3(u(t,x))^2 + \langle F(t)Du(t,x), Du(t,x)\rangle$$
$$+\langle G(t)D_*^2u(t,x), D_*^2u(t,x)\rangle + \langle H(t)D_*^3u(t,x), D_*^3u(t,x)\rangle,$$

for any $t>0$ and any $x \in \mathbb{R}^N$, where

$$F(t) = \begin{pmatrix} \alpha t I_r & 4t^2F_1 \\ 4t^2F_1^* & \iota t^3 I_{N-r} \end{pmatrix}, \tag{14.2.11}$$

$$G(t) = \begin{pmatrix} t^2 I_{n_r^1} & 0 & 0 \\ 0 & \alpha^{-\frac{7}{16}} t^4 I_{r(N-r)} & \alpha^{-\frac{4}{5}} t^5 G_1 \\ 0 & \alpha^{-\frac{4}{5}} t^5 G_1^* & \alpha^{-\frac{7}{8}} t^6 I_{n_{N-r}^1} \end{pmatrix}, \tag{14.2.12}$$

$$H(t) = \begin{pmatrix} \alpha^{-\frac{7}{16}} t^3 I_{n_r^2} & 0 & 0 & 0 \\ 0 & \alpha^{-\frac{7}{8}} t^5 I_{(N-r)n_r^1} & 0 & 0 \\ 0 & 0 & \alpha^{-1} t^7 I_{rn_{N-r}^1} & \alpha^{-\frac{13}{12}} t^8 H_1 \\ 0 & 0 & \alpha^{-\frac{13}{12}} t^8 H_1^* & \alpha^{-\frac{9}{8}} t^9 I_{n_{N-r}^2} \end{pmatrix}, \tag{14.2.13}$$

for any $t>0$. Here,

(i) F_1 is any matrix such that $B_3F_1 + F_1^*B_3^*$ is strictly negative definite (such a matrix exists by virtue of Lemma 14.2.1);

(ii) $-\iota = \lambda_{\max}(B_3F_1 + F_1^*B_3^*)$;

(iii) $G_1 \in L(\mathbb{R}^{n_{N-r}^1}, \mathbb{R}^{r(N-r)})$ and $H_1 \in L(\mathbb{R}^{n_{N-r-2}^2}, \mathbb{R}^{rn_{N-r-1}^1})$ are suitable matrices (with entries being independent of α) to be determined later on, as well as the positive constant α.

We require that the matrices $F(t)$, $G(t)$ and $H(t)$ are strictly positive definite for any $t>0$. According to Lemma 14.2.2 it suffices to assume that $F(1)$, $G(1)$ and $H(1)$ are strictly positive definite and, as a straightforward computation shows, this is the case if we assume that

$$\begin{cases} \iota\alpha - 4\|F_1\|^2 > 0, \\ \alpha^{-\frac{7}{16}} - \alpha^{-\frac{4}{5}}\|G_1\|^2 > 0, \\ \alpha^{-\frac{17}{8}} - \alpha^{-\frac{13}{12}}\|H_1\|^2 > 0. \end{cases} \tag{14.2.14}$$

From Propositions 14.1.1 and 14.1.2 it follows that the function ξ is a classical solution of the Cauchy problem

$$\begin{cases} D_t\xi(t,\cdot) = \mathcal{A}_\varepsilon\xi(t,\cdot) + g_\varepsilon(t,\cdot), \ t>0, \\ \xi(0,\cdot) = \frac{1}{2}\alpha^3 f^2, \end{cases}$$

where $g^{(\varepsilon)} = g_1^{(\varepsilon)} + g_2 + g_3$ and

$$\begin{aligned} g_1^{(\varepsilon)}(t,x) = & -\alpha^3\langle Q^{(\varepsilon)}(x)Du(t,x), Du(t,x)\rangle \\ & -2\mathrm{Tr}(Q^{(\varepsilon)}(x)D^2u(t,x)F(t)D^2u(t,x)) \\ & -2\sum_{i,j=1}^N q_{ij}^{(\varepsilon)}(x)\langle G(t)D_*^2D_iu(t,x), D_*^2D_ju(t,x)\rangle \\ & -2\sum_{i,j=1}^N q_{ij}^{(\varepsilon)}(x)\langle H(t)D_*^3D_iu(t,x), D_*^3D_ju(t,x)\rangle; \end{aligned} \tag{14.2.15}$$

$$\begin{aligned} g_2(t,x) = & \langle F'(t)Du(t,x), Du(t,x)\rangle + \langle G'(t)D_*^2u(t,x), D_*^2u(t,x)\rangle \\ & +\langle H'(t)D_*^3u(t,x), D_*^3u(t,x)\rangle \\ & +\langle (BF(t)+F(t)B^*)Du(t,x), Du(t,x)\rangle \\ & +2\langle G(t)[D_*^2, \langle Bx, D\rangle]u(t,x), D_*^2u(t,x)\rangle \\ & +2\langle H(t)[D_*^3, \langle Bx, D\rangle]u(t,x), D_*^3u(t,x)\rangle; \end{aligned} \tag{14.2.16}$$

$$\begin{aligned} g_3(t,x) = & 2\sum_{i,j=1}^r D_{ij}u(t,x)\langle F(t)Dq_{ij}(x), Du(t,x)\rangle \\ & +2\sum_{i,j=1}^r \langle G(t)[D_*^2, q_{ij}(x)D_{ij}]u(t,x), D_*^2u(t,x)\rangle \\ & +2\sum_{i,j=1}^r \langle H(t)[D_*^3, q_{ij}(x)D_{ij}]u(t,x), D_*^3u(t,x)\rangle, \end{aligned} \tag{14.2.17}$$

for any $t > 0$ and any $x \in \mathbb{R}^N$.

We claim that we can fix α and the matrices G_1 and H_1 such that both $g_\varepsilon \le 0$ in $(0, T_0] \times \mathbb{R}^N$ for some $T_0 > 0$, independent of ε, and the conditions (14.2.14) are satisfied. Theorem 4.1.3 then will imply that

$$\xi(t,x) \le \frac{1}{2}\alpha^3(f(t,x))^2, \qquad t \in (0,T_0], \quad x \in \mathbb{R}^N. \tag{14.2.18}$$

Since we are assuming that the matrices $F(t)$, $G(t)$ and $H(t)$ are strictly positive definite for any $t > 0$, then, by (14.2.18), all the terms in the definition of ξ will turn out to be bounded from above by $\frac{1}{2}\alpha^3 f^2$ in $(0, T_0] \times \mathbb{R}^N$ and, consequently, Lemma 14.2.2 will lead us to the estimate (14.2.9) in $(0, T_0]$ with $\omega = 0$ and C replaced with a new constant C_1. The semigroup rule, then, will allow us to extend the previous estimate to all the positive t, as in the proof of Theorem 6.1.7. So, let us prove the claim. For this purpose, we begin by observing that, since the matrices $F(t)$, $(\langle G(t)D_*^2D_iu(t), D_*^2D_ju(t)\rangle)_{ij}$ and

$(\langle H(t)D_*^3D_iu(t), D_*^3D_ju(t)\rangle)_{ij}$ are strictly positive definite, then $g_1^{(\varepsilon)} \le g_1$ in $(0,+\infty)\times\mathbb{R}^N$, where the function g_1 is obtained from $g_1^{(\varepsilon)}$ replacing, at any $x\in\mathbb{R}^N$, the matrix $Q^{(\varepsilon)}(x)$ with $Q^{(0)}(x)$. Now, Lemma 14.2.3 implies that

$$\begin{aligned}g_1(t) \le& -\alpha^3\kappa|D_{*,1}u(t)|^2 - 2\kappa\sum_{i=1}^r (D^2u(t)F(t)D^2u(t))_{ii}\\ &-2\kappa\sum_{i=1}^r\langle G(t)D_*^2D_iu(t), D_*^2D_iu(t)\rangle\\ &-2\kappa\sum_{i=1}^r\langle H(t)D_*^3D_iu(t), D_*^3D_iu(t)\rangle.\end{aligned}$$

Hence, using properly the inequality

$$2\alpha^\gamma t^\beta ab \le \alpha^{\gamma_1}t^{\beta_1}a^2 + \alpha^{\gamma_2}t^{\beta_2}b^2 \qquad (2\gamma=\gamma_1+\gamma_2,\quad 2\beta=\beta_1+\beta_2), \tag{14.2.19}$$

holding for any $\alpha, a, b, t > 0$, we obtain

$$\begin{aligned}g_1(t) \le& -\alpha^3\kappa|D_{*,1}u(t)|^2 - 2\alpha t\kappa\langle K_1D_{*,1}^2u(t), D_{*,1}^2u(t)\rangle - 2\iota\kappa t^3|D_{*,2}^2u(t)|^2\\ &+\|K_2\|\kappa(\alpha^{\frac12}t|D_{*,1}^2u(t)|^2 + \alpha^{-\frac12}t^3|D_{*,2}^2u(t)|^2)\\ &-2t^2\kappa\langle K_3D_{*,1}^3u(t), D_{*,1}^3u(t)\rangle - 2\alpha^{-\frac{7}{16}}t^4\kappa\langle K_4D_{*,2}^3u(t), D_{*,2}^3u(t)\rangle\\ &-2\alpha^{-\frac78}t^6\kappa|D_{*,3}^3u(t)|^2 + \alpha^{-\frac35}t^4\|K_5\|\,\kappa\,|D_{*,2}^3u(t)|^2\\ &+\alpha^{-1}t^6\|K_5\|\,\kappa\,|D_{*,3}^3u(t)|^2 - 2\alpha^{-\frac{7}{16}}t^3\kappa\langle K_6D_{*,1}^4u(t), D_{*,1}^4u(t)\rangle\\ &-2\alpha^{-\frac78}t^5\kappa\langle K_7D_{*,2}^4u(t), D_{*,2}^4u(t)\rangle\\ &-2\alpha^{-1}t^7\kappa\langle K_8D_{*,3}^4u(t), D_{*,3}^4u(t)\rangle - 2\alpha^{-\frac98}t^9\kappa|D_{*,4}^4u(t)|^2\\ &+\|K_9\|\kappa(\alpha^{-\frac{49}{48}}t^7|D_{*,3}^4u(t)|^2 + \alpha^{-\frac{55}{48}}t^9|D_{*,4}^4u(t)|^2),\end{aligned}$$

for any $t>0$, where $K_1, K_3, K_4, K_6, K_7, K_8$ are suitable diagonal matrices whose minimum eigenvalue is 1, whereas the entries of the matrices K_2, K_5 and K_9 depend linearly only on the entries of F_1, G_1 and H_1, respectively. In particular, all the previous matrices are independent of α. Therefore,

$$\begin{aligned}g_1(t) \le& -\alpha^3\kappa|D_{*,1}u(t)|^2 + (-2\alpha + \alpha^{\frac12}\|K_2\|)t\kappa|D_{*,1}^2u(t)|^2\\ &+(-2\iota + \alpha^{-\frac12}\|K_2\|)\kappa t^3|D_{*,2}^2u(t)|^2\\ &-2t^2\kappa|D_{*,1}^3u(t)|^2 + (-2\alpha^{-\frac{7}{16}} + \alpha^{-\frac35}\|K_5\|)\kappa t^4|D_{*,2}^3u(t)|^2\\ &+(-2\alpha^{-\frac78} + \alpha^{-1}\|K_5\|)\kappa t^6|D_{*,3}^3u(t)|^2 - 2\alpha^{-\frac{7}{16}}t^3\kappa|D_{*,1}^4u(t)|^2\\ &-2\alpha^{-\frac78}t^5\kappa|D_{*,2}^4u(t)|^2 + (-2\alpha^{-1} + \alpha^{-\frac{49}{48}}\|K_9\|)\kappa t^7|D_{*,3}^4u(t)|^2\\ &+(-2\alpha^{-\frac98} + \alpha^{-\frac{55}{48}}\|K_9\|)\kappa t^9|D_{*,4}^4u(t)|^2,\end{aligned} \tag{14.2.20}$$

for any $t > 0$.

Throughout the rest of the proof, to simplify the notation, we will denote by $o_t(t^k)$ ($k \ge 0$) any function h, depending on t, possibly on α, $\|G_1\|$ and $\|H_1\|$, but being independent of x, such that $\lim_{t\to 0} t^{-k}h(t) = 0$. Moreover, in the estimates for g_2 and g_3 we will write explicitly only the terms which are not negligible, as t tends to 0^+, with respect to the terms in the right-hand side of (14.2.20), and we use the notation now introduced to denote all the other ones. For instance, if a term is negligible with respect to $-2\iota\kappa t^3|D^2_{*,2}u(t)|^2$ we simply denote it by $o_t(t^3)\kappa|D^2_{*,2}u(t)|^2$, or by $o_t(t^3)|D^2_{*,2}u(t)|^2$, if it is independent of κ.

In order to estimate the function $g_{2,\varepsilon}$ we observe that

$$\begin{cases} [D^2_*, \langle Bx, D\rangle]u(t,x) = \mathcal{L}D^2_*u(t,x), \\ [D^3_*, \langle Bx, D\rangle]u(t,x) = \mathcal{M}D^3_*u(t,x), \end{cases} \qquad t>0,\ \ x\in\mathbb{R}^N, \tag{14.2.21}$$

where

$$\mathcal{L} = \begin{pmatrix} L_1 & L_2 & 0 \\ L_3 & L_4 & L_5 \\ 0 & L_6 & L_7 \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} M_1 & M_2 & 0 & 0 \\ M_3 & M_4 & M_5 & 0 \\ 0 & M_6 & M_7 & M_8 \\ 0 & 0 & M_9 & M_{10} \end{pmatrix}, \tag{14.2.22}$$

L_j ($j = 1, \dots, 7$) and M_j ($j = 1, \dots, 10$) being suitable matrices whose entries linearly depend on the entries of B, but are independent of α, G_1, H_1. Using properly the inequality (14.2.19), we get

$$\begin{aligned}
&2\langle G(t)[D^2_*, \langle Bx, D\rangle]u(t), D^2_*u(t)\rangle \\
&\le 2t^2\|L_1\||D^2_{*,1}u(t)|^2 + \|L_2\|(\alpha^{\frac{1}{2}}t|D^2_{*,1}u(t)|^2 + \alpha^{-\frac{1}{2}}t^3|D^2_{*,2}u(t)|^2) \\
&\quad +\alpha^{-\frac{4}{5}}\|G_1^*\|\|L_3\|\left(t^2|D^2_{*,1}u(t)|^2 + t^8|D^2_{*,3}u(t)|^2\right) \\
&\quad +\alpha^{-\frac{7}{16}}\|L_3\|\left(t^2|D^2_{*,1}u(t)|^2 + t^6|D^2_{*,2}u(t)|^2\right) + 2\alpha^{-\frac{7}{16}}t^4\|L_4\||D^2_{*,2}u(t)|^2 \\
&\quad +2\alpha^{-\frac{4}{5}}t^5\|G_1\|\|L_6\||D^2_{*,2}u(t)|^2 \\
&\quad +\|L_5\|(\alpha^{-\frac{1}{24}}t^3|D^2_{*,2}u(t)|^2 + \alpha^{-\frac{5}{6}}t^5|D^2_{*,3}u(t)|^2) \\
&\quad +\alpha^{-\frac{4}{5}}\|G_1\|\left(\|L_4\| + \|L_7\|\right)\left(t^4|D^2_{*,2}u(t)|^2 + t^6|D^2_{*,3}u(t)|^2\right) \\
&\quad +\alpha^{-\frac{7}{8}}t^6\|L_6\|\left(|D^2_{*,2}u(t)|^2 + |D^2_{*,3}u(t)|^2\right) \\
&\quad +\alpha^{-\frac{4}{5}}\lambda_{\max}(G_1^*L_5 + L_5^*G_1)t^5|D^2_{*,3}u(t)|^2 + 2\alpha^{-\frac{7}{8}}t^6\|L_7\||D^2_{*,3}u(t)|^2,
\end{aligned}$$

that we can rewrite, with the notation introduced above, as

$$\begin{aligned}
&2\langle G(t)[D_*^2,\langle Bx,D\rangle]u(t),D_*^2u(t)\rangle\\
=\ &\{\alpha^{\frac12}\|L_2\|+o_t(1)\}t|D_{*,1}^2u(t)|^2\\
&+\{\alpha^{-\frac12}\|L_2\|+\alpha^{-\frac{1}{24}}\|L_5\|+o_t(1)\}t^3|D_{*,2}^2u(t)|^2\\
&+\{\alpha^{-\frac45}\lambda_{\max}(G_1^*L_5+L_5^*G_1)+\alpha^{-\frac56}\|L_5\|+o_t(1)\}t^5|D_{*,3}^2u(t)|^2.
\end{aligned}\tag{14.2.23}$$

Similarly,

$$\begin{aligned}
&2\langle H(t)[D_*^3,\langle Bx,D\rangle]u(t),D_*^3u(t)\rangle\\
\le\ &\{\alpha^{-\frac18}\|M_2\|+o_t(1)\}|D_{*,1}^3u(t)|^2\\
&+\{\alpha^{-\frac34}\|M_2\|+\alpha^{-\frac12}\|M_5\|+o_t(1)\}t^2|D_{*,2}^3u(t)|^2\\
&+\{\alpha^{-\frac{43}{48}}\|M_8\|+\alpha^{-\frac54}\|M_5\|+o_t(1)\}t^6|D_{*,3}^3u(t)|^2\\
&+\{\alpha^{-\frac{53}{48}}\|M_8\|+\alpha^{-\frac{13}{12}}\lambda_{\max}(H_1^*M_8+M_8^*H_1)+o_t(1)\}t^8|D_{*,4}^3u(t)|^2.
\end{aligned}\tag{14.2.24}$$

Now, observing that

$$\langle(4(B_3F_1+F_1^*B_3^*)D_{*,2}u(t),D_{*,2}u(t)\rangle\le -4\iota|D_{*,2}u(t)|^2,\qquad t>0,$$

from (14.2.23) and (14.2.24) we get

$$\begin{aligned}
g_2(t)\le\ &a_1(t)|D_{*,1}u(t)|^2+a_2(t)|D_{*,2}u(t)|^2+a_3(t)|D_{*,1}^2u(t)|^2\\
&+a_4(t)|D_{*,2}^2u(t)|^2+a_5(t)|D_{*,3}^2u(t)|^2+a_6(t)|D_{*,1}^3u(t)|^2\\
&+a_7(t)|D_{*,2}^3u(t)|^2+a_8(t)|D_{*,3}^3u(t)|^2+a_9(t)|D_{*,4}^3u(t)|^2,
\end{aligned}\tag{14.2.25}$$

where

$$\begin{aligned}
a_1(t)&=\alpha+8\alpha^{\frac12}\|F_1\|+\alpha^{\frac52}\|B_3\|+o_t(1),\\
a_2(t)&=\{-\iota+\alpha^{-\frac12}(8\|F_1\|+\|B_3\|)+o_t(1)\}t^2,\\
a_3(t)&=\{4+\alpha^{\frac12}\|L_2\|+o_t(1)\}t,\\
a_4(t)&=\{4\alpha^{-\frac{7}{16}}+\alpha^{-\frac12}\|L_2\|\|L_5\|+\alpha^{-\frac{1}{24}}+o_t(1)\}t^3,\\
a_5(t)&=\{6\alpha^{-\frac78}+\alpha^{-\frac45}\lambda_{\max}(G_1^*L_5+L_5^*G_1)+\alpha^{-\frac56}\|L_5\|+o_t(1)\}t^5,\\
a_6(t)&=\{15\alpha^{-\frac{7}{16}}+\alpha^{-\frac18}\|M_2\|+o_t(1)\}t^2,\\
a_7(t)&=\{10\alpha^{-\frac78}+\alpha^{-\frac34}\|M_2\|+\alpha^{-\frac12}\|M_5\|+o_t(1)\}t^4,\\
a_8(t)&=\{\alpha^{-\frac{43}{48}}\|M_8\|+7\alpha^{-1}+\alpha^{-\frac54}\|M_5\|+o_t(1)\}t^6,\\
a_9(t)&=\{9\alpha^{-\frac98}+\alpha^{-\frac{53}{48}}\|M_8\|+\alpha^{-\frac{13}{12}}\lambda_{\max}(H_1^*M_8+M_8^*H_1)+o_t(1)\}t^8.
\end{aligned}$$

Finally, we consider the function $g_{3,\varepsilon}$ and we observe that, for any $i,j = 1,\dots,N$,

$$\begin{cases} [D^2_*, q_{ij}(x)D_{ij}]u(t,x) = D_{ij}u(t,x)D^2_*q_{ij}(x) + \mathcal{N}D^3_*u(t,x), \\ [D^3_*, q_{ij}(x)D_{ij}]u(t,x) = D_{ij}u(t,x)D^3_*q_{ij}(x) + \mathcal{R}(x)D^3_*u(t,x) \\ \qquad\qquad\qquad\qquad\qquad +\mathcal{S}(x)D^4_*u(t,x), \end{cases} \tag{14.2.26}$$

where, for any $x \in \mathbb{R}^N$, the matrices $\mathcal{N}(x) \in L(\mathbb{R}^{n^2_N}, \mathbb{R}^{n^1_N})$, $\mathcal{P}(x) \in L(\mathbb{R}^{n^2_N})$ and $\mathcal{R}(x) \in L(\mathbb{R}^{n^3_N}, \mathbb{R}^{n^2_N})$ ($n^3_N := N(N+1)(N+2)(N+3)/24$), split according to the splitting of the vectors D^k_*u ($k = 2,3,4$), are given by

$$\mathcal{N}(x) = \begin{pmatrix} N_1(x) & 0 & 0\,0 \\ N_2(x) & N_3(x) & 0\,0 \\ 0 & N_4(x) & 0\,0 \end{pmatrix}, \tag{14.2.27}$$

$$\mathcal{P}(x) = \begin{pmatrix} P_1(x) & 0 & 0\,0 \\ P_2(x) & P_3(x) & 0\,0 \\ P_4(x) & P_5(x) & 0\,0 \\ 0 & P_6(x) & 0\,0 \end{pmatrix}, \tag{14.2.28}$$

$$\mathcal{R}(x) = \begin{pmatrix} R_1(x) & 0 & 0 & 0\,0 \\ R_2(x) & R_3(x) & 0 & 0\,0 \\ 0 & R_4(x) & R_5(x) & 0\,0 \\ 0 & 0 & R_6(x) & 0\,0 \end{pmatrix}, \tag{14.2.29}$$

the entries of the matrices $N_j(x)$ ($j = 1,\dots,4$), $P_j(x), R_j(x)$ ($j = 1,\dots,6$) being linear combinations of the entries of the derivatives of the diffusion coefficients $q_{ij}(x)$. In particular, $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{R}$ are independent of α, G_1, H_1 and there exists a positive constant C_2, independent of x, such that

$$\|N_i(x)\| + \|P_j(x)\| + \|R_j(x)\| \le C_2\sqrt{\kappa(x)}, \qquad x \in \mathbb{R}^N, \tag{14.2.30}$$

for any $i = 1,\dots,4$ and any $j = 1,\dots,6$. Hence, using properly the inequality (14.2.19) (where now a and b are given, respectively, by $\sqrt{\kappa}|D^k_{*,i}u|$ and $|D^m_{*,j}u|$ for suitable i,j,k,m), and taking Hypothesis 14.0.1(ii) into account, it is easy to check that

$$
\begin{aligned}
g_3(t) \le\ & o_t(1)|D_{*,1}u(t)|^2 + o_t(t^2)|D_{*,2}u(t)|^2 + o_t(t)\kappa|D^2_{*,1}u(t)|^2\\
&+o_t(t^3)\kappa|D^2_{*,2}u(t)|^2 + o_t(t^5)|D^2_{*,3}u(t)|^2 + o_t(t^2)\kappa|D^3_{*,1}u(t)|^2\\
&+o_t(t^4)\kappa|D^3_{*,2}u(t)|^2 + o_t(t^6)\kappa|D^3_{*,3}u(t)|^2 + o_t(t^8)|D^3_{*,4}u(t)|^2\\
&+o_t(t^3)\kappa|D^4_{*,1}u(t)|^2 + o_t(t^5)\kappa|D^4_{*,2}u(t)|^2 + o_t(t^7)\kappa|D^4_{*,3}u(t)|^2,
\end{aligned}
\tag{14.2.31}
$$

for any $t > 0$. Summing up, from (14.2.20), (14.2.25) and (14.2.31) we deduce that

$$
\begin{aligned}
g^{(\varepsilon)}(t) \le\ & \{-\kappa\alpha^3 + o_\alpha(\alpha^3) + o_t(1)\}|D_{*,1}u(t)|^2\\
&+\{-\iota + o_\alpha(1) + o_t(1)\}t^2|D_{*,2}u(t)|^2\\
&+\{-2\alpha + o_\alpha(\alpha) + o_t(1)\}\kappa t|D^2_{*,1}u(t)|^2\\
&+\{-2\iota + o_\alpha(1) + o_t(1)\}\kappa t^3|D^2_{*,2}u(t)|^2\\
&+\{\lambda_{\max}(G_1^*L_5 + L_5^*G_1)\alpha^{-\frac{4}{5}} + o_\alpha(\alpha^{-\frac{4}{5}}) + o_t(1)\}t^5|D^2_{*,3}u(t)|^2\\
&+\{-2\kappa + o_\alpha(1) + o_t(1)\kappa\}t^2|D^3_{*,1}u(t)|^2\\
&+\{-2\alpha^{-\frac{7}{16}} + o_\alpha(\alpha^{-\frac{7}{16}}) + o_t(1)\}\kappa t^4|D^3_{*,2}u(t)|^2\\
&+\{-2\alpha^{-\frac{7}{8}} + o_\alpha(\alpha^{-\frac{7}{8}}) + o_t(1)\}\kappa t^6|D^3_{*,3}u(t)|^2\\
&+\{\alpha^{-\frac{13}{12}}\lambda_{\max}(H_1^*M_8 + M_8^*H_1) + o_\alpha(\alpha^{-\frac{13}{12}}) + o_t(1)\}t^8|D^3_{*,4}u(t)|^2\\
&+\{-2\alpha^{-\frac{7}{16}}\kappa + o_t(1)\kappa\}t^3|D^4_{*,1}u(t)|^2\\
&+\{-2\alpha^{-\frac{7}{8}} + o_t(1)\}\kappa t^5|D^4_{*,2}u(t)|^2\\
&+\{-2\alpha^{-1} + o_\alpha(\alpha^{-1}) + o_t(1)\}\kappa t^7|D^4_{*,3}u(t)|^2\\
&+\{-2\alpha^{-\frac{9}{8}} + o_\alpha(\alpha^{-\frac{9}{8}})\}\kappa t^9|D^4_{*,4}u(t)|^2,
\end{aligned}
\tag{14.2.32}
$$

where by $o_\alpha(\alpha^k)$ ($k \in \mathbb{R}$) we have denoted any function $h : (0,+\infty) \to \mathbb{R}$, depending on α, and possibly on G_1 and H_1, but being independent of t, such that $\lim_{\alpha\to+\infty} \alpha^{-k}h(\alpha) = 0$.

To prove that $g^{(\varepsilon)}(t,x) \le 0$ for any t in a right neighborhood of 0 (independent of ε) and any $x \in \mathbb{R}^N$, we show that we can fix $\alpha, T_0 > 0$ and the matrices G_1 and H_1 so that all the terms in the right-hand side of (14.2.32) are negative in $(0,T_0] \times \mathbb{R}^N$. As a first step we prove that we can fix $\alpha > 0$ and the matrices G_1 and H_1 so that

$$
\sup_{x\in\mathbb{R}^N} \hat{a}_j(x) < 0, \qquad j = 1,\dots,9,
\tag{14.2.33}
$$

where $\hat{a}_j(x)$ are obtained from the terms in curly brackets in the right-hand side of (14.2.32), disregarding the terms of type $o_t(1)$. Once (14.2.33) is proved, it will be an easy task to check that we can fix $T_0 > 0$ such that the right-hand side of (14.2.32) is negative in $(0, T_0] \times \mathbb{R}^N$.

An easy asymptotic analysis shows that all the coefficients $\hat{a}_j$ $(j = 1, \ldots, 9)$ satisfy (14.2.33) for $\alpha > 0$ sufficiently large, provided that G_1 and H_1 can be chosen so that the matrices $G_1^* L_5 + L_5^* G_1$ and $H_1^* M_8 + M_8^* H_1$ are strictly negative definite. By virtue of Lemma 14.2.1 and Hypothesis 14.0.1(i), this is the case if the ranks of the matrices $L_5 \in L(\mathbb{R}^{n^1_{N-r}}, \mathbb{R}^{r(N-r)})$ and $M_8 \in L(\mathbb{R}^{n^2_{N-r}}, \mathbb{R}^{rn^1_{N-r}})$ are, respectively, n^1_{N-r} and n^2_{N-r}. Straightforward computations show that, up to rearranging the rows, we can split L_5 and M_8 into blocks (according to the splitting of the vectors $D^3_* u$, $k = 3, 4$) as follows:

$$L_5 = S_0, \qquad M_8 = \begin{pmatrix} S_0 & 0 & \cdots & \cdots & 0 \\ \star & S_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \star & \cdots & \cdots & \star & S_{N-r-1} \end{pmatrix},$$

where $S_j \in L(\mathbb{R}^{n^1_{N-r-j-1}}, \mathbb{R}^{r(N-r-j)})$ is given by

$$S_j = \begin{pmatrix} B_3^j & 0 & \cdots & \cdots & 0 \\ \star & B_3^{j+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \star & \cdots & \cdots & \star & B_3^{N-r-1} \end{pmatrix}, \qquad j = 0, \ldots, N-r-1,$$

and the matrix B_3^j is obtained from B_3^* by removing the first j columns. Here and above by "$\star$" we denote suitable matrices whose entries depend linearly only on the entries of B_3 (recall that the entries of the matrices L_5 and M_8 are independent of α). Since, by Hypothesis 14.0.1(iii), $\mathrm{rank}(B_3) = N - r$, then $\mathrm{rank}(L_5) = n^1_{N-r}$ and $\mathrm{rank}(M_8) = n^2_{N-r}$. Hence, from Lemma 14.2.1, we deduce that, if we set $G_1 = -L_5$, and $H_1 = -M_8$, then the matrices $G_1^* L_5 + L_5^* G_1$ and $H_1^* M_8 + M_8^* H_1$ are strictly negative definite. Therefore, if $\alpha > 0$ is large enough, the estimate (14.2.33) holds. Up to choosing a larger α, we can also assume that the conditions (14.2.14) are satisfied. Then, fixing T_0 sufficiently small, but independent of ε, we obtain that $g^{(\varepsilon)} \le 0$ in $(0, T_0] \times \mathbb{R}^N$ and $F(t)$, $G(t)$ and $H(t)$ strictly positive definite for any $t > 0$. By the above remarks, this concludes the proof. ∎

As in the nondegenerate case, the more the initial datum f is regular, the more we can improve the estimates of the derivatives of $T_\varepsilon(t)f$ near $t=0$. We state this fact in the following theorem.

Theorem 14.2.5 *Under the same assumptions as in Theorem 14.2.4, for any $k=1,2,3$, $h\in\mathbb{N}$ with $h\le k$, there exist two positive constants ω and C, independent of ε, such that*

$$\|D^k_{*,j}T_\varepsilon(t)f\|_\infty \le Ce^{\omega t}t^{-(j-h-1)^+-\frac{k-h}{2}}\|f\|_{C^h_b(\mathbb{R}^N)}, \qquad t>0, \quad j\le k+1, \tag{14.2.34}$$

for any $f\in C^h_b(\mathbb{R}^N)$ and any $\varepsilon>0$. In particular, if $h=k$, (14.2.34) *holds true for any $\omega>0$ and some positive constant $C=C(\omega)$.*

Proof. Since the proof is close to that of Theorem 14.2.4, we just sketch it, pointing out the main differences. We confine ourselves to proving (14.2.34) when $h=k=3$, the other cases being similar and even simpler. As in the previous proof we write u for u_ε, and $u(t)$ when we want to stress the dependence on t of u but we are not interested in the dependence on the space variables.

Let us introduce the function $\xi_{3,\varepsilon}:[0,+\infty)\times\mathbb{R}^N\to\mathbb{R}$ defined by

$$\xi_3=\frac{1}{2}\alpha^3u^2+\langle F\,Du,Du\rangle+\langle G\,D^2_*u,D^2_*u\rangle+\langle H\,D^3_*u,D^3_*u\rangle, \tag{14.2.35}$$

where $F=F(1)$ (with F_1 being replaced with $(\|B_4\|+1)F_1$), $G=G(1)$ (with $G_1=-L_5$) and $H=H(1)$ (with $H_1=-M_8$) (see (14.2.11)-(14.2.13) and (14.2.22)) and α is a positive constant to be determined later on. Using Proposition 14.1.2 we can show that ξ_3 is continuous up to $t=0$. Moreover, straightforward computations show that it satisfies

$$\begin{cases} D_t\xi_3(t,\cdot)=\mathcal{A}_\varepsilon\xi_3(t,\cdot)+\hat g_\varepsilon(t,\cdot), & t>0,\\ \xi_3(0,\cdot)=\frac{1}{2}\alpha^3f^2+\langle F\,Df,Df\rangle+\langle G\,D^2_*f,D^2_*f\rangle+\langle HD^3_*f,D^3_*f\rangle, \end{cases}$$

where $\hat g^{(\varepsilon)}=\hat g_1^{(\varepsilon)}+\hat g_2+\hat g_3$, the functions $g_1^{(\varepsilon)}$ and $\hat g_j$ $(j=1,2)$ being obtained from the corresponding functions $g_1^{(\varepsilon)}$ and g_j $(j=1,2)$ in the proof of Theorem 14.2.4, by replacing everywhere $F(t)$, $G(t)$ and $H(t)$ with F, G and H, and disregarding the terms containing the matrices F', G' and H'.

The proof now follows the same ideas as in the proof of Theorem 14.2.4. The function $\hat g_{1,\varepsilon}$ can be estimated, as the right-hand side of (14.2.20), by

$$\begin{aligned} g_1(t)\le{}& -\alpha^3\kappa|D_{*,1}u(t)|^2+(-2\alpha+\alpha^{\frac{1}{2}}\|K_2\|)\kappa|D^2_{*,1}u(t)|^2\\ &+(-2\iota+\alpha^{-\frac{1}{2}}\|K_2\|)\kappa|D^2_{*,2}u(t)|^2\\ &-2\kappa|D^3_{*,1}u(t)|^2+(-2\alpha^{-\frac{7}{16}}+\alpha^{-\frac{3}{5}}\|K_5\|)\kappa|D^3_{*,2}u(t)|^2\\ &+(-2\alpha^{-\frac{7}{8}}+\alpha^{-1}\|K_5\|)\kappa|D^3_{*,3}u(t)|^2-2\alpha^{-\frac{7}{16}}\kappa|D^4_{*,1}u(t)|^2 \end{aligned}$$

$$-2\alpha^{-\frac{7}{8}}\kappa|D^4_{*,2}u(t)|^2+(-2\alpha^{-1}+\alpha^{-\frac{49}{48}}\|K_9\|)\kappa|D^4_{*,3}u(t)|^2$$
$$+(-2\alpha^{-\frac{9}{8}}+\alpha^{-\frac{55}{48}}\|K_9\|)\kappa t^9|D^4_{*,4}u(t)|^2. \qquad (14.2.36)$$

Here, the matrices $K_1,\dots,K_8$ are defined as the corresponding ones in the formula (14.2.20), with the only difference that now $G_1=-L_5$, $H_1=-M_8$. As far as the functions $\hat{g}_2$ and $\hat{g}_3$ are concerned, using properly the inequality (14.2.19) (with $\beta=0$) and arguing as in the proof of (14.2.25) and (14.2.31), we get

$$\begin{aligned}\hat{g}_2(t)+\hat{g}_3(t)\le{}&-\iota|D_{*,2}u(t)|^2-2\alpha^{-\frac{4}{5}}\lambda_{\min}(L_5^*L_5)|D^2_{*,3}u(t)|^2\\&-2\lambda_{\min}(M_8^*M_8)\alpha^{-\frac{13}{12}}|D^3_{*,4}u(t)|^2+o_\alpha(\alpha^3)|D_{*,1}u(t)|^2\\&+o_\alpha(1)|D_{*,2}u(t)|^2+o_\alpha(\alpha)|D^2_{*,1}u(t)|^2+o_\alpha(1)|D^2_{*,2}u(t)|^2\\&+o_\alpha(1)|D^2_{*,3}u(t)|^2+o_\alpha(1)|D^3_{*,1}u(t)|^2+o_\alpha(\alpha^{-\frac{7}{16}})|D^3_{*,2}u(t)|^2\\&+o_\alpha(\alpha^{-\frac{7}{8}})|D^3_{*,3}u(t)|^2+o_\alpha(\alpha^{-\frac{13}{12}})|D^3_{*,4}u(t)|^2\\&+o_\alpha(\alpha^{-\frac{7}{16}})\kappa|D^4_{*,1}u(t)|^2+o_\alpha(\alpha^{-\frac{7}{8}})\kappa|D^4_{*,2}u(t)|^2\\&+o_\alpha(\alpha^{-1})\kappa|D^4_{*,3}u(t)|^2,\end{aligned} \qquad (14.2.37)$$

for any $t>0$, where, as in the proof of Theorem 14.2.4, $o_\alpha(\alpha^k)$ ($k\in\mathbb{R}$) denote real functions such that $\lim_{\alpha\to+\infty}\alpha^{-k}o_\alpha(\alpha^k)=0$. Therefore, from (14.2.36) and (14.2.37), we obtain

$$\begin{aligned}\hat{g}^{(\varepsilon)}(t)\le{}&-\{\alpha^3\kappa+o_\alpha(\alpha^3)\}|D_{*,1}u(t)|^2-\{\iota+o_\alpha(1)\}|D_{*,2}u(t)|^2\\&-\{2\alpha+o_\alpha(\alpha)\}\kappa|D^2_{*,1}u(t)|^2-\{2\iota+o_\alpha(1)\}\kappa|D^2_{*,2}u(t)|^2\\&-\{2\alpha^{-\frac{4}{5}}\lambda_{\min}(L_5^*L_5)+o_\alpha(\alpha^{-\frac{4}{5}})\}|D^2_{*,3}u(t)|^2\\&-\{2\kappa+o_\alpha(1)\}|D^3_{*,1}u(t)|^2-\{2\alpha^{-\frac{7}{16}}+o_\alpha(\alpha^{-\frac{7}{16}})\}\kappa|D^3_{*,2}u(t)|^2\\&-\{2\alpha^{-\frac{7}{8}}+o_\alpha(\alpha^{-\frac{7}{8}})\}\kappa|D^3_{*,3}u(t)|^2\\&-\{2\alpha^{-\frac{13}{12}}\lambda_{\min}(M_8^*M_8)+o_\alpha(\alpha^{-\frac{13}{12}})\}|D^3_{*,4}u(t)|^2\\&-2\alpha^{-\frac{7}{16}}\kappa|D^4_{*,1}u(t)|^2-2\alpha^{-\frac{7}{8}}\kappa t^5|D^4_{*,2}u(t)|^2\\&-\{2\alpha^{-1}+o_\alpha(\alpha^{-1})\}\kappa|D^4_{*,3}u(t)|^2-\{2\alpha^{-\frac{9}{8}}+o_\alpha(\alpha^{-\frac{9}{8}})\}\kappa|D^4_{*,4}u(t)|^2.\end{aligned} \qquad (14.2.38)$$

Now, from (14.2.38) and the condition (14.2.14) (where we replace F_1 with $(\|B_4\|+1)F_1$), it follows that we can fix $\alpha>0$ such that F,G,H are strictly positive definite and $\hat{g}^{(\varepsilon)}\le 0$ in $(0,+\infty)\times\mathbb{R}^N$. This implies that

$$\|T_\varepsilon(t)f\|_{C^3_b(\mathbb{R}^N)}\le C_0\|f\|_{C^3_b(\mathbb{R}^N)}, \qquad t\in(0,T_0],\quad f\in C^3_b(\mathbb{R}^N), \qquad (14.2.39)$$

for some positive T_0, and some positive constant $C = C(T_0)$. The semigroup rule then allows us to extend (14.2.39) to all the positive t obtaining (14.2.34).

To prove (14.2.34) with $h, k = 1, 2$, it suffices to apply the previous arguments to the functions ξ_k defined by

$$\xi_k = \frac{1}{2}\alpha^3 u^2 + \langle FDu, Du\rangle + (k-1)\langle GD_*^2 u, D_*^2 u\rangle, \qquad k = 1, 2,$$

where F and G are as in (14.2.35).

Finally, to prove (14.2.34) with $(h, k) = (1, 2)$ and with $h = 1, 2$, $k = 3$, it suffices to apply the previous arguments, respectively, to the functions $\xi_{1,2}$, $\xi_{1,3}$ and $\xi_{2,3}$ defined by

$$\begin{aligned}\xi_{1,k}(t,x) = \xi_1(t,x) &+ \langle G(t)D_*^2 u(t,x), D_*^2 u(t,x)\rangle \\ &+(k-1)\langle H_1(t)D_*^3 u(t,x), D_*^3 u(t,x)\rangle, \qquad k = 2, 3,\end{aligned}$$

for any $t > 0$, $x \in \mathbb{R}^N$, and

$$\xi_{2,3}(t,x) = \xi_2(t,x) + \langle H_2(t)D_*^3 u(t,x), D_*^3 u(t,x)\rangle, \qquad t > 0, \quad x \in \mathbb{R}^N,$$

where

$$G(t) = \begin{pmatrix} tI_{n_r^1} & 0 & 0 \\ 0 & \alpha^{-\frac{7}{16}} tI_{(N-r)n_r^1} & -\alpha^{-\frac{4}{5}} t^2 L_5 \\ 0 & -\alpha^{-\frac{4}{5}} t^2 L_5^* & \alpha^{-\frac{7}{8}} t^3 I_{n_{N-r}^1} \end{pmatrix},$$

$$H_1(t) = \begin{pmatrix} \alpha^{-\frac{7}{16}} t^2 I_{n_r^2} & 0 & 0 & 0 \\ 0 & \alpha^{-\frac{7}{8}} t^2 I_{(N-r)n_r^1} & 0 & 0 \\ 0 & 0 & \alpha^{-1} t^4 I_{rn_{N-r}^1} & -\alpha^{-\frac{13}{12}} t^5 M_8 \\ 0 & 0 & -\alpha^{-\frac{13}{12}} t^5 M_8^* & \alpha^{-\frac{9}{8}} t^6 I_{n_{N-r}^2} \end{pmatrix},$$

$$H_2(t) = \begin{pmatrix} \alpha^{-\frac{7}{16}} tI_{n_r^2} & 0 & 0 & 0 \\ 0 & \alpha^{-\frac{7}{8}} tI_{(N-r)n_r^1} & 0 & 0 \\ 0 & 0 & \alpha^{-1} tI_{rn_{N-r}^1} & -\alpha^{-\frac{13}{12}} t^2 M_8 \\ 0 & 0 & -\alpha^{-\frac{13}{12}} t^2 M_8^* & \alpha^{-\frac{9}{8}} t^3 I_{n_{N-r}^2} \end{pmatrix},$$

for any $t > 0$. Here, as in the proof of Theorem 14.2.4, n_m^1 and n_m^2 $(m \in \mathbb{N})$ are given by (14.2.10). ∎

14.2.3 Construction of the semigroup

In this section we prove that, for any $f \in C_b(\mathbb{R}^N)$, the Cauchy problem (14.0.5), associated with the degenerate elliptic operator $\mathcal{A}$, admits a unique classical solution u_f. This will allow us to define a semigroup of bounded linear operators in $C_b(\mathbb{R}^N)$ by setting $T(t)f = u_f(t,\cdot)$ for any $t > 0$ and any $f \in C_b(\mathbb{R}^N)$. At the same time we will also show that the semigroup $\{T(t)\}$ satisfies the uniform estimates (14.2.1)-(14.2.6).

The following theorem will be fundamental in order to prove our results, since it provides us a useful maximum principle for the classical solution to the Cauchy problem (14.0.5).

Theorem 14.2.6 *Assume Hypotheses* 4.0.1 *and* 4.0.2*. Fix* $T > 0$, $f \in C(\mathbb{R}^N)$, $g : (0,T] \times \mathbb{R}^N \to \mathbb{R}$*. Further suppose that the function* $u \in C([0,T] \times \mathbb{R}^N)$ *is such that* $D_t u \in C((0,T] \times \mathbb{R}^N)$, $u(t,\cdot) \in W^{2,p}_{\rm loc}(\mathbb{R}^N)$, $\mathcal{A}u(t,\cdot) \in C(\mathbb{R}^N)$ *for any* $t \in (0,T]$ *and any* $p \in [1,+\infty)$ *and it solves the Cauchy problem*

$$\begin{cases} D_t u(t,x) - \mathcal{A}u(t,x) = g(t,x), & t \in (0,T],\ x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N. \end{cases} \tag{14.2.40}$$

Then, the following properties are met:

(i) if

$$\sup_{x \in \mathbb{R}^N} f(x) < +\infty, \qquad g \le 0 \ \text{ in } \ (0,T] \times \mathbb{R}^N,$$

and u *satisfies*

$$\limsup_{|x| \to +\infty} \left(\sup_{t \in [0,T]} \frac{u(t,x)}{\varphi(x)} \right) \le 0, \tag{14.2.41}$$

then

$$u(t,x) \le \max\left\{0, \sup_{\mathbb{R}^N} f\right\}, \qquad t \in [0,T], \quad x \in \mathbb{R}^N.$$

(ii) If

$$\inf_{x \in \mathbb{R}^N} f(x) > -\infty, \qquad g \ge 0 \ \text{ in } \ (0,T] \times \mathbb{R}^N,$$

u *satisfies*

$$\liminf_{|x| \to +\infty} \left(\inf_{t \in [0,T]} \frac{u(t,x)}{\varphi(x)} \right) \ge 0,$$

then

$$u(t,x) \ge \min\left\{0, \inf_{\mathbb{R}^N} f\right\}, \qquad t \in [0,T], \quad x \in \mathbb{R}^N.$$

(*iii*) *In particular, for any* $f \in C(\mathbb{R}^N)$ *such that*

$$\lim_{|x|\to+\infty} \frac{f(x)}{\varphi(x)} = 0,$$

there exists at most one solution of the parabolic problem (4.1.7) (*with* $g \equiv 0$) *in* $C([0,+\infty) \times \mathbb{R}^N)$ *such that* $D_t u \in C((0,T] \times \mathbb{R}^N)$, $u(t,\cdot) \in W^{2,p}_{\rm loc}(\mathbb{R}^N)$, $\mathcal{A}u(t,\cdot) \in C(\mathbb{R}^N)$ *for any* $p \in [1,+\infty)$ *and any* $t \in (0,T]$, *and such that* (14.2.41) *holds for any* $T > 0$.

Proof. The proof can be obtained arguing just as in the proof of Theorem 4.1.3. Indeed, such a proof can be repeated with no changes also in the case when the operator $\mathcal{A}$ is elliptic but not uniformly elliptic. The essential tool needed in the proof of Theorem 4.1.3 is the existence of a Lyapunov function $\varphi \in C^2(\mathbb{R}^N)$ such that $\sup_{\mathbb{R}^N}(\mathcal{A}\varphi - \lambda\varphi) \le 0$ for some $\lambda > 0$. In our situation, using (14.1.3) it is easy to check that the function φ defined by $\varphi(x) = 1+|x|^2$ for any $x \in \mathbb{R}^N$ is a Lyapunov function for the operator $\mathcal{A}$ corresponding to $\lambda = 2(\sqrt{N}\hat{C} + \|B\|)$, where $\hat{C}$ is given by (14.1.3). ∎

We can now prove the main result of this section.

Theorem 14.2.7 *Under Hypotheses* 14.0.1, *for any* $f \in C_b(\mathbb{R}^N)$ *there exists a unique classical solution* u *to the problem* (14.0.5). *Moreover, if we set* $u(t,\cdot) = T(t)f$ *for any* $t > 0$, *the family* $\{T(t)\}$ *is an order preserving semigroup of contractions in* $C_b(\mathbb{R}^N)$ *satisfying the estimates* (14.2.1)-(14.2.6). *Finally, for any compact set* $D \subset (0,+\infty) \times \mathbb{R}^N$ *and any* $f \in C_b(\mathbb{R}^N)$, $T_\varepsilon(\cdot)f$ *converges to* $T(\cdot)f$ *in* $C^{1,2}(D)$ *as* ε *goes to* 0^+.

Proof. We split the proof into several steps. First in Steps 1 to 3 we prove that, for any $f \in C_b(\mathbb{R}^N)$, the problem (14.0.5) admits a (unique) classical solution u_f; we define the operator $T(t)$ ($t > 0$) and we show that $T_\varepsilon(\cdot)f$ converges to $T(\cdot)f$ in $C^{1,2}(D)$ as ε tends to 0^+ for any compact set $D \subset (0,+\infty)\times\mathbb{R}^N$. Moreover, we show that $\{T(t)\}$ is an order preserving semigroup of contractions in $C_b(\mathbb{R}^N)$ and it satisfies (14.2.1), (14.2.2), (14.2.4) (with $k = 1,2$) and (14.2.5). Then, in Steps 4 and 5, we show that u_f is thrice-continuously differentiable with respect to the space variables in $(0,+\infty)\times\mathbb{R}^N$ and $T(t)$ satisfies the estimates (14.2.3), (14.2.4) (with $k = 3$) and (14.2.6) (with $k = 0,1,2$), for any $t > 0$.

Step 1. For any $f \in C_b(\mathbb{R}^N)$ and any $\varepsilon \in (0,1]$, we set $u_\varepsilon = T_\varepsilon(\cdot)f$. Using the estimates (14.2.1) and (14.2.2), we deduce that the family of functions $\{u_\varepsilon : \varepsilon \in (0,1)\}$ is contained in $B([T_0,T]; C^3_b(\mathbb{R}^N))$, for any $0 < T_0 < T$, with norm being independent of ε. Moreover, since $D_t u_\varepsilon = \mathcal{A}_\varepsilon u_\varepsilon$, then $\{D_t u_\varepsilon : \varepsilon \in (0,1)\}$ is equibounded in $[T_0,T] \times \overline{B}(R)$ for any $R > 0$. It follows that $u_\varepsilon \in {\rm Lip}([T_0,T]; C(\overline{B}(R))) \cap B([T_0,T] \times C^3(\overline{B}(R)))$. From Propositions A.4.4 and A.4.6, we deduce that $u_\varepsilon \in C^{(1-\alpha)/3}([T_0,T]; C^{2+\alpha}(\overline{B}(R)))$ and $D_t u_\varepsilon \in$

$C^{(1-\alpha)/3}([T_0,T];C(\overline{B}(R)))$ with norms being independent of ε. It follows that the families of functions $\{D_t^\alpha D_x^\beta u_\varepsilon : \varepsilon \in (0,1)\}$ ($2\alpha+|\beta| \le 2$) are equibounded and equicontinuous in $[T_0,T] \times \overline{B}(R)$ for any $0 < T_0 < T$ and any $R > 0$. Hence, there exists an infinitesimal sequence $\{\varepsilon_n\}$ such that, for any compact set $D \subset (0,+\infty) \times \mathbb{R}^N$, u_{ε_n} converges in $C^{1,2}(D)$ to some function u_f, which, of course, solves the differential equation $D_t u_f - \mathcal{A} u_f = 0$.

We now assume that $f \in C_c^2(\mathbb{R}^N)$ and prove that u_f is continuous up to $t = 0$. For this purpose we observe that, since $C_c^2(\mathbb{R}^N)$ is contained in the domain of the weak generator of the operator $\mathcal{A}_\varepsilon$ (see Propositions 2.3.6 and 4.1.10), by Lemma 2.3.3 and Proposition 2.3.5 we can write

$$\begin{aligned}(T_{\varepsilon_n}(t)f)(x) - f(x) &= \int_0^t (D_t T_{\varepsilon_n}(s)f)(x)ds \\ &= \int_0^t (\mathcal{A}_{\varepsilon_n} T_{\varepsilon_n}(s)f)(x)ds \\ &= \int_0^t (T_{\varepsilon_n}(s)\mathcal{A}_{\varepsilon_n} f)(x)ds,\end{aligned}$$

for any $t > 0$ and any $x \in \mathbb{R}^N$. Using (14.1.2) we get

$$\|T_{\varepsilon_n}(t)f - f\|_\infty \le \sup_{x\in\mathbb{R}^N} \int_0^t |(T_{\varepsilon_n}(s)\mathcal{A}_{\varepsilon_n} f)(x)|ds \le \|\mathcal{A}_{\varepsilon_n} f\|_\infty t \le Ct\|f\|_{C_b^2(\mathbb{R}^N)},$$

for any $t > 0$ and any $n \in \mathbb{N}$, where C is a positive constant, independent of n. Letting n go to $+\infty$, we deduce that $u_f(t,\cdot)$ tends to f uniformly as t tends to 0^+. Hence, the function u_f is a classical solution to the problem (14.0.5).

Repeating the same arguments as above and taking Theorem 14.2.6 into account, we can show that any sequence $\{u_{\varepsilon_n'}\}$, with ε_n' vanishing as n go to $+\infty$, admits a subsequence converging to u_f in $C^{1,2}(D)$ for any compact set $D \subset (0,+\infty) \times \mathbb{R}^N$. This implies that $T_\varepsilon(\cdot)f$ converges to u_f in $C^{1,2}(D)$ as ε tends to 0^+ for any D as above.

Step 2. We now assume that $f \in C_0(\mathbb{R}^N)$ and let $u_{\varepsilon_n} = T_{\varepsilon_n}(\cdot)f$ be as in Step 1. Moreover, we denote by $\{f_m\} \in C_c^2(\mathbb{R}^N)$ any sequence converging to f uniformly in $\mathbb{R}^N$. We fix $m \in \mathbb{N}$ and observe that, by Step 1, $T_{\varepsilon_n}(t)f_m$ converges to $T(t)f_m$ as n tends to $+\infty$, locally uniformly in $\mathbb{R}^N$ and for any $t > 0$. According to (14.1.2),

$$\|T_{\varepsilon_n}(t)f - T_{\varepsilon_n}(t)f_m\|_\infty \le \|f - f_m\|_\infty, \qquad t > 0,\ \ n,m \in \mathbb{N}.$$

So, letting n go to $+\infty$, we get

$$\|u_f(t,\cdot) - T(t)f_m\|_\infty \le \|f - f_m\|_\infty, \qquad t > 0, \quad m \in \mathbb{N}. \tag{14.2.42}$$

Hence, from (14.2.42) it follows that

$$\begin{aligned}\|u_f(t,\cdot) - f\|_\infty &\le \|u_f(t,\cdot) - T(t)f_m\|_\infty + \|T(t)f_m - f_m\|_\infty + \|f_m - f\|_\infty \\ &\le 2\|f - f_m\|_\infty + \|T(t)f_m - f_m\|_\infty,\end{aligned}$$

for any $t > 0$ and any $m \in \mathbb{N}$, and, from Step 1, we deduce that $u_f(t,\cdot)$ tends to f uniformly in $\mathbb{R}^N$ as t tends to 0^+. Hence, the function u_f is a classical solution to the problem (14.0.5). Then, with the same arguments as in Step 1 we can easily show that $T_\varepsilon(\cdot)f$ converges to $T(\cdot)f$ in $C^{1,2}(D)$ as ε goes to 0^+, for any compact set $D \subset (0,+\infty)\times\mathbb{R}^N$.

Step 3. We now assume that $f \in C_b(\mathbb{R}^N)$ and adapt to our situation the technique in the proof of Theorem 11.2.1. We preliminarily observe that by Theorem 4.1.3, $T_\varepsilon(t)\mathbb{1} = \mathbb{1}$ for any $t > 0$. Hence, $T(t)\mathbb{1}$ is well defined for any $t > 0$ and $T(\cdot)\mathbb{1} = \mathbb{1}$.

We now fix a compact set $K \subset \mathbb{R}^N$ and a smooth function η, compactly supported in $\mathbb{R}^N$, such that $\eta = \mathbb{1}$ in K and $0 \le \eta \le \mathbb{1}$. By linearity and Step 2, we easily see that for any $t > 0$, the function $T(t)(\mathbb{1}-\eta)$ is well defined. Moreover, since $\{T_\varepsilon(t)\}$ is an order preserving semigroup, $T_\varepsilon(t)(\mathbb{1}-\eta) \ge 0$ for any $t > 0$ and any $\varepsilon \in (0,1)$. Thus,

$$0 \le T(t)(\mathbb{1}-\eta) = \mathbb{1} - T(t)\eta, \qquad t > 0.$$

Since, by Step 2, $T(t)\eta$ tends to η, uniformly in $\mathbb{R}^N$, then $T(t)(\mathbb{1}-\eta)$ tends to 0 as t tends to 0^+, uniformly in K.

Now, let $u_{\varepsilon_n} = T_{\varepsilon_n}(\cdot)f$ and u_f be as in Step 1. Splitting $T_{\varepsilon_n}(t)(\mathbb{1}-\eta)f) = T_{\varepsilon_n}(t)f - T_{\varepsilon_n}(\eta f)$, it is immediate to check that $T_{\varepsilon_n}(\cdot)((\mathbb{1}-\eta)f)$ converges in $C^{1,2}(D)$ to $u_f(t,\cdot) - T(t)(\eta f)$, for any compact set $D \subset (0,+\infty)\times\mathbb{R}^N$. Since

$$|(T_{\varepsilon_n}(t)((\mathbb{1}-\eta)f))(x)| \le \|f\|_\infty \, (T_{\varepsilon_n}(t)(\mathbb{1}-\eta))(x), \qquad t > 0,\ \ x \in \mathbb{R}^N, \quad n \in \mathbb{N},$$

for any $n \in \mathbb{N}$, then

$$|u_f(t,x) - (T(t)(\eta f))(x)| \le \|f\|_\infty \, (T(t)(\mathbb{1}-\eta))(x), \qquad t > 0, \quad x \in \mathbb{R}^N.$$

It follows that $u_f(t,\cdot) - T(t)(\eta f)$ vanishes uniformly in K, as t tends to 0^+. Recalling that $T(t)(\eta f)$ tends to f, uniformly, in K as t tends to 0^+, we easily deduce that $u_f(t,\cdot)$ converges to f, as t tends to 0^+, uniformly in K, as well. By the arbitrariness of K, u_f is continuous up to $t = 0$ and it is a classical solution to the problem (14.0.5). Arguing again as in Step 1, we can then show that $T_\varepsilon(\cdot)f$ converges to $T(\cdot)f$ in $C^{1,2}(D)$ as ε goes to 0^+, for any compact set $D \subset (0,+\infty)\times\mathbb{R}^N$.

Now, the estimates (14.2.1), (14.2.2), (14.2.4) (with $k = 1, 2$) and (14.2.5) easily follow letting ε go to 0^+ in (14.2.9) and (14.2.34), recalling that the constant occurring in these latter estimates are independent of ε.

To conclude the proof of this step, we observe that the family of contractive operators $\{T(t)\}$ ($T(0) = I$) is an order preserving semigroup of linear operators in $C_b(\mathbb{R}^N)$. The semigroup rule follows from Theorem 14.2.6 since, for any $f \in C_b(\mathbb{R}^N)$ and any $s > 0$, both the functions $u(t,\cdot) = T(t)T(s)f$ and $v(t,\cdot) = T(t+s)f$ are both classical solutions to the Cauchy problem (14.0.5) with f replaced with $T(s)f$. The maximum principle also implies that $\{T(t)\}$

is an order preserving semigroup, and, since $T_\varepsilon(t)$ are all contractions, then $T(t)$ is a contraction as well, for any $t \geq 0$.

Step 4. We now prove (14.2.3), (14.2.4) (with $k = 3$) and (14.2.6) (in the case when $i \leq r$), using a localization argument. Without loss of generality, we can assume that $f \in C_b^3(\mathbb{R}^N)$. Indeed, once (14.2.3) and (14.2.6) are established for any $f \in C_b^3(\mathbb{R}^N)$, then they can be extended, respectively, to the cases when $f \in BUC(\mathbb{R}^N)$ and $f \in BUC^k(\mathbb{R}^N)$ ($k = 0, 1, 2$) by a density argument, by approximating $f \in BUC^k(\mathbb{R}^N)$ with a sequence of functions in $C_b^3(\mathbb{R}^N)$, converging uniformly to f in $BUC^k(\mathbb{R}^N)$. Finally, for a general $f \in C_b^k(\mathbb{R}^N)$, it suffices to split, for any $t > 0$, $T(t)f = T(t/2)T(t/2)f$ and apply the above results with f replaced with $T(t/2)f \in BUC^k(\mathbb{R}^N)$.

Observe that, to prove the previous estimates, it suffices to show that, for any $i \in \{1, \ldots, r\}$ and any $j, h \in \{1, \ldots, N\}$, the function $D_{jh}T(t)f$ is continuously differentiable with respect to the i-th space variable. Indeed, suppose, for instance, that $i = 1$, $j \leq r$, $h > r$ and $f \in C_b(\mathbb{R}^N)$. The estimate (14.2.9) (with $k = 3$) implies that, for any $x_2, \ldots, x_N$, the function $x \mapsto (D_{jh}T_\varepsilon(t)f)(x, x_2, \ldots, x_N)$ is Lipschitz continuous in $\mathbb{R}$ and

$$[(D_{jh}T_\varepsilon(t)f)(\cdot, x_2, \ldots, x_N)]_{\mathrm{Lip}(\mathbb{R})} \leq Ce^{\omega t}t^{-\frac{5}{2}}\|f\|_{C_b(\mathbb{R}^N)}, \qquad t > 0,$$

for suitable $C, \omega > 0$, independent of $x_2, \ldots, x_N$. Since $D_{jh}T_\varepsilon(t)f$ converges to $D_{jh}T(t)f$ locally uniformly, then the function $(D_{jh}T(t)f)(\cdot, x_2, \ldots, x_N)$ is Lipschitz continuous in $\mathbb{R}$ as well, and

$$[(D_{jh}T(t)f)(\cdot, x_2, \ldots, x_N)]_{\mathrm{Lip}(\mathbb{R})} \leq Ce^{\omega t}t^{-\frac{5}{2}}\|f\|_{C_b(\mathbb{R}^N)}, \qquad t > 0.$$

Therefore, if $D_{jh}T(t)f$ is continuously differentiable with respect to the direction e_1, then the function $D_{1jh}T(t)f$ satisfies (14.2.6).

So, we fix $i \leq r$ and $j, h \leq N$, and prove that the function $D_{jh}u = D_{jh}T(\cdot)f$ is continuously differentiable in $(0, +\infty) \times \mathbb{R}^N$ with respect to the i-th variable. For this purpose, let $\eta_R : \mathbb{R}^N \to \mathbb{R}$ be a smooth function compactly supported in $B(R)$ ($R > 0$) such that $\eta_R \equiv 1$ in $B(R/2)$. For any $k \in \mathbb{R}$, with $|k| \leq 1$ we introduce the operator τ_k^h defined on $C_b(\mathbb{R}^N)$ by

$$\tau_k^h\psi(x) = \frac{\psi(x + ke_h) - \psi(x)}{k}, \qquad x \in \mathbb{R}^N, \qquad \psi \in C_b(\mathbb{R}^N).$$

Moreover, we set $v_{\varepsilon,k,R}^h = \tau_k^h v_{\varepsilon,R}$ where $v_{\varepsilon,R} = u_\varepsilon \eta_R$ and $u_\varepsilon = T_\varepsilon(\cdot)f$. As it is easily seen, the function $v_{\varepsilon,k,R}^h$ is the classical solution to the Cauchy problem

$$\begin{cases} D_t v_{\varepsilon,k,R}^h(t,x) = \mathcal{A}_\varepsilon v_{\varepsilon,k,R}^h(t,x) + g_{\varepsilon,k,R}^h(t,x), & t > 0,\, x \in \mathbb{R}^N, \\ v_{\varepsilon,h,R}^k(0,x) = \tau_k^h(\eta_R f)(x), & x \in \mathbb{R}^N, \end{cases}$$

where, for any $t>0$,

$$
\begin{aligned}
g^h_{\varepsilon,k,R}(t,\cdot) = & -\tau^h_k(u_\varepsilon(t,\cdot)\mathcal{A}_\varepsilon\eta_R) - 2\sum_{l,m=1}^r q_{lm}D_lu_\varepsilon(t,\cdot+ke_h)\tau^h_k(D_m\eta_R)\\
& -2\sum_{l,m=1}^r q_{lm}(\tau^h_kD_lu_\varepsilon(t,\cdot))D_m\eta_R\\
& -2\sum_{l,m=1}^r (\tau^h_kq_{lm})D_lu_\varepsilon(t,\cdot+ke_h)D_m\eta_R(\cdot+ke_h)\\
& -2\varepsilon\sum_{m=r+1}^N D_mu_\varepsilon(t,\cdot+ke_h)(\tau^h_kD_m\eta_R)\\
& -2\varepsilon\sum_{m=r+1}^N (\tau^h_kD_mu_\varepsilon(t,\cdot))D_m\eta_R\\
& +\sum_{l,m=1}^r(\tau^h_kq_{lm})D_{lm}v_{\varepsilon,R}(t,\cdot)+\sum_{l=1}^N b_{lh}D_lv_{\varepsilon,R}(t,\cdot).
\end{aligned}
\tag{14.2.43}
$$

By the results of Chapter 6, $v^h_{\varepsilon,k,R}$ can be represented by the variation-of-constants formula

$$
v^h_{\varepsilon,k,R}(t,x) = (T_\varepsilon(t)(\tau^h_k(\eta_Rf)))(x) + \int_0^t (T_\varepsilon(t-s)g^h_{\varepsilon,k,R}(s,\cdot))(x)ds, \tag{14.2.44}
$$

for any $t>0$ and any $x\in\mathbb{R}^N$. We are going to show that we can take the limit as ε tends to 0^+ in (14.2.44) and write

$$
\begin{aligned}
v^h_{k,R}(t,x) &:= (\tau^h_k(u\eta_R)(t,\cdot))(x)\\
&= (T(t)(\tau^h_k(\eta_Rf)))(x) + \int_0^t (T(t-s)g^h_{k,R}(s,\cdot))(x)ds,
\end{aligned}
\tag{14.2.45}
$$

for any $t>0$ and any $x\in\mathbb{R}^N$, where $g^h_{k,R}$ is obtained from $g^h_{\varepsilon,k,R}$ by replacing u_ε with u and letting $\varepsilon=0$ in (14.2.43). Of course, thanks to the previous steps, it suffices to deal with the convolution term in (14.2.44). Since u_ε converges to u in $C^{1,2}(D)$ for any compact set $D\subset(0,+\infty)\times\mathbb{R}^N$ and η_R is compactly supported in $\mathbb{R}^N$, then the continuous function $g^h_{\varepsilon,k,R}$ converges uniformly in $\mathbb{R}^N$ to the function $g^h_{k,R}$ as ε tends to 0^+. This implies that, for any $s,t>0$, $T_\varepsilon(t)g^h_{\varepsilon,k,R}(s,\cdot)$ converges to $T(t)g^h_{k,R}(s,\cdot)$ locally uniformly in $\mathbb{R}^N$ as ε tends to 0^+. Indeed, for any compact set $K\subset\mathbb{R}^N$, we have

$$
\begin{aligned}
&\sup_{x\in K}|(T_\varepsilon(t)g^h_{\varepsilon,k,R}(s,\cdot))(x)-(T(t)g^h_{k,R}(s,\cdot))(x)|\\
\le &\sup_{x\in K}|(T_\varepsilon(t)g^h_{\varepsilon,k,R}(s,\cdot))(x)-(T_\varepsilon(t)g^h_{k,R}(s,\cdot))(x)|\\
&+\sup_{x\in K}|(T_\varepsilon(t)g^h_{k,R}(s,\cdot))(x)-(T(t)g^h_{k,R}(s,\cdot))(x)|
\end{aligned}
$$

$$\leq \|g^h_{\varepsilon,k,R} - g^h_{k,R}\|_\infty + \sup_{x\in K} |(T_\varepsilon(t)g^h_{k,R}(s,\cdot))(x) - (T(t)g^h_{k,R}(s,\cdot))(x)|,$$

and, by virtue of Step 1, the last side of the previous chain of inequalities vanishes as ε tends to 0^+. Moreover, since $f \in C^3_b(\mathbb{R}^N)$ and $\{T_\varepsilon(t)\}$ is a semigroup of contractions for any $\varepsilon > 0$, then $T_\varepsilon(\cdot)g^h_{\varepsilon,k,R}$ is bounded in $[0,T]\times\mathbb{R}^N$ for any $T > 0$, uniformly with respect to $\varepsilon \in (0,1)$. Therefore, letting ε go to 0^+ in (14.2.44), by the dominated convergence theorem we get (14.2.45).

Next step consists in showing that we can let k go to 0 in (14.2.45) getting the fundamental representation formula

$$D_h v_R(t,x) = (T(t)(D_h(\eta_R f)))(x) + \int_0^t (T(t-s)g^h_R(s,\cdot))(x)ds, \qquad (14.2.46)$$

for any $t > 0$ and any $x \in \mathbb{R}^N$, where

$$\begin{aligned} g^h_R = &-D_h(u\mathcal{A}\eta_R) - 2\sum_{l,m=1}^r q_{lm}D_l u D_{hm}\eta_R - 2\sum_{l,m=1}^r q_{lm}D_{hl}uD_m\eta_R \\ &-2\sum_{l,m=1}^r (D_h q_{lm})D_l u D_m\eta_R + \sum_{l,m=1}^r (D_h q_{lm})D_{lm}v_R + \sum_{l=1}^N b_{hl}D_l v_R, \end{aligned} \qquad (14.2.47)$$

and $v_R = u\eta_R$. The convergence of the integral term in (14.2.45) follows from the dominated convergence theorem, since $g^h_{k,R}$ converges uniformly in $\mathbb{R}^N$, to the function g^h_R, as k tends to 0. To show this, it suffices to observe that, for any $\psi \in C(\mathbb{R}^N)$ such that $D_h\psi \in C(\mathbb{R}^N)$, the function $\tau^h_k\psi$ converges to $D_h\psi$, locally uniformly in $\mathbb{R}^N$, as k tends to 0. Similarly, $\tau^h_k(\eta_R f)$ and $v^h_{k,R}(t,\cdot)$ ($t > 0$) converge uniformly, respectively, to $D_h(\eta_R f)$ and $D_h v_R(t,\cdot)$ as k tends to 0.

Now, taking advantage of the representation formula (14.2.46), we can show that the function $D_h v_R$ is twice continuously differentiable in $(0,T)\times\mathbb{R}^N$ with respect to the i-th and j-th space variable, for any $T > 0$. To simplify the notation, in the rest of the proof, we denote by C_j positive constants which may depend on R, but are independent of t, ε and k.

Using a simple interpolation argument, from (14.2.34) (with $h = k = 2,3$) we deduce that $T_\varepsilon(\cdot)f \in B((0,T);C^{2+\alpha}_b(\mathbb{R}^N))$ with norm being independent of ε. Therefore, letting ε go to 0^+, we get $T(\cdot)f \in B((0,T);C^{2+\alpha}_b(\mathbb{R}^N))$ as well and, consequently, $g^h_R \in B((0,T);C^\alpha_b(\mathbb{R}^N))$ for any $\alpha \in (0,1)$.

Next, interpolating the estimates (14.2.2) and (14.2.5) we get

$$\|D_{ij}T(t)\psi\|_\infty \leq C_1 t^{-\frac{3}{4}}\|\psi\|_{C^{5/6}_b(\mathbb{R}^N)}, \qquad (14.2.48)$$

for any $\psi \in C^\alpha_b(\mathbb{R}^N)$. The estimate (14.2.48) implies that the function $s \mapsto \|D_{ij}T(t-s)g^h_R(s,\cdot)\|_\infty$ is in $L^1(0,t)$. Hence, (14.2.46) and Step 3 imply that the function $D_h v_R$ is twice continuously differentiable with respect to the i-th and j-th space variables in $(0,T)\times\mathbb{R}^N$. Since $v_R\equiv u$ in $B(R/2)$ and R is arbitrarily fixed, we deduce that $D_h u$ is twice continuously differentiable with respect to the i-th and j-th space variables as well. The estimates (14.2.3) and (14.2.4) (with $k=3$) follow.

Step 5. We now prove the estimate (14.2.6) in the case when $i,j,h>r$. The same arguments as in Step 4 show that it is not restrictive to consider the case when $f\in C^4_b(\mathbb{R}^N)$. We are going to show that the right-hand side of (14.2.46) defines a function which is twice continuously differentiable in $(0,+\infty)\times\mathbb{R}^N$, with respect to space variables x_i and x_j. As a first step, we show that the function g^h_R in (14.2.47) belongs to $B((0,T);C^{3/2}_b(\mathbb{R}^N))$ for any $T>0$. Of course, this is the case if $D_{lm}u\in B((0,T);C^{3/2}(\overline{B}(R)))$ for any $l,m=1,\ldots,r$. To show that $D_{lm}u\in B((0,T);C^{3/2}(\overline{B}(R)))$ we use a bootstrap argument, first showing that it belongs to $B((0,T);C^{1+\theta/3}_b(B(2R)))$ for any $\theta\in(0,1)$. For this purpose, we replace the function η_R defined in Step 4, with the function η_{4R}. Interpolating the estimates (14.2.5) and (14.2.6) (with $k=1$) we get

$$\|D_{lm}T(t)\psi\|_{C^{\theta/3}_b(\mathbb{R}^N)}\le C_2 t^{-\frac{\theta+1}{2}}\|\psi\|_{C^1_b(\mathbb{R}^N)},\qquad t\in(0,T),\quad 1\le l\le r,$$

for any $\psi\in C^1_b(\mathbb{R}^N)$. Hence, the function $t\mapsto\|D_{lm}T(t)\psi\|_{C^{\theta/3}_b(\mathbb{R}^N)}$ is integrable in $(0,T)$. Therefore, since by Step 4, $T(\cdot)\psi$ and $D_{lm}T(\cdot)\psi$ both belong to $B((0,T);C^1_b(\mathbb{R}^N))$ for any $l=1,\ldots,r$, it follows, first, that $g^h_{4R}\in B((0,T);C^1_b(\mathbb{R}^N))$ and, then, using the formula (14.2.46), that $D_{lm}u$ belongs to $B((0,T);C^{1+\theta/3}(\overline{B}(R)))$. As a straightforward consequence, from (14.2.47) (with R replaced with $2R$) we deduce that $g^h_{2R}\in B((0,T);C^{1+\theta/3}_b(\mathbb{R}^N))$ for any $\theta\in(0,1)$. Now, we interpolate first (14.2.6), respectively, with $k=1$ and $k=2$, and then (14.2.2) and (14.2.5), obtaining

$$\|D_{lm}T(t)\psi\|_{C^1_b(\mathbb{R}^N)}\le C_3 t^{\frac{\theta}{2}-2}\|\psi\|_{C^{1+\theta/3}_b(\mathbb{R}^N)},\tag{14.2.49}$$

$$\|D_{lm}T(t)\psi\|_\infty\le C_4 t^{\frac{\theta-3}{6}}\|\psi\|_{C^{1+\theta/3}_b(\mathbb{R}^N)},\tag{14.2.50}$$

for any $t\in(0,T)$ and any $1\le l,m\le r$. Again, interpolating (14.2.49) and (14.2.50) yields

$$\|D_{lm}T(t)\psi\|_{C^{1/2}_b(\mathbb{R}^N)}\le C_5 t^{\frac{4\theta-15}{12}}\|\psi\|_{C^{1+\theta/3}_b(\mathbb{R}^N)},\qquad t\in(0,T),\quad 1\le l\le r.\tag{14.2.51}$$

Taking $\theta=4/5$ in (14.2.51), we easily see that the function

$$s\mapsto\|D_{lm}T(t-s)g^h_{2R}(s)\|_{C^{1/2}_b(\mathbb{R}^N)}$$

is integrable in $(0,T)$. Hence, from the formula (14.2.46) we get $D_{lm}v_{2R}\in B((0,T);C^{3/2}_b(\mathbb{R}^N))$ and, consequently, $D_{lm}u\in B((0,T);C^{3/2}(\overline{B}(R)))$, since $v_{2R}\equiv u$ in $(0,+\infty)\times B(R)$.

Now, we are almost done. Indeed, interpolating (14.2.4) and (14.2.5), we get

$$\|D_{ij}T(t)\psi\|_\infty \le C_6 t^{-\frac{3}{4}}\|\psi\|_{C_b^{3/2}(\mathbb{R}^N)}, \qquad t\in(0,T), \quad i,j>r,$$

that, due to the above results, implies that the map $s\mapsto \|D_{ij}T(t-s)g_R^h(s,\cdot)\|_\infty$ is integrable in $(0,t)$. Therefore, from (14.2.46) we easily deduce that $D_{ijk}u(t,\cdot)$ exists for any $t>0$. ∎

Remark 14.2.8 In fact, in Step 2 of the proof of Theorem 14.2.7 we have shown that, for any $f\in C_0(\mathbb{R}^N)$, $T(t)f$ converges to f uniformly in $\mathbb{R}^N$ as t tends to 0^+.

Remark 14.2.9 Uniform gradient estimates similar to those in Theorem 14.2.4 have been proved in [37] in the case when the operator $\mathcal{A}$ is obtained from the degenerate Ornstein-Uhlenbeck operator in (14.0.3) replacing the drift term Bx, with the term $Bx+F(x)$. The smooth function F is assumed to satisfy

$$\langle DF(x)G_t\xi,\xi\rangle \le C\langle G_t\xi,\xi\rangle, \qquad t>0,\ \ x,\xi\in\mathbb{R}^N,$$

for some constant $C\in\mathbb{R}$, and the matrix G_t is defined as in (14.0.4) with B being replaced with the matrix $-B$. Moreover, F should either belong to $C_b^2(\mathbb{R}^N,\mathbb{R}^N)$ or be Lipschitz continuous and satisfy

$$\langle F(x)-F(y),x-y\rangle \le \eta|x-y|^2, \qquad x,y\in\mathbb{R}^N,$$

for some constant $\eta\in\mathbb{R}$.

14.3 Some remarkable properties of the semigroup

In this section we first prove a continuity property of the semigroup $\{T(t)\}$ that will play a fundamental role in order to prove the Schauder estimates of Section 14.4. Then, we prove that $\{T(t)\}$ is strong Feller and, finally, we characterize the domain of its weak generator.

Proposition 14.3.1 *Let $\{f_n\}\subset C_b(\mathbb{R}^N)$ be a bounded sequence of continuous functions converging to $f\in C_b(\mathbb{R}^N)$ locally uniformly in $\mathbb{R}^N$. Then, $T(\cdot)f_n$ converges to $T(\cdot)f$ locally uniformly in $[0,+\infty)\times\mathbb{R}^N$ and in $C^{1,2}(D)$ for any compact set $D\subset(0,+\infty)\times\mathbb{R}^N$.*

Proof. Without loss of generality, throughout the proof we assume that $f \equiv 0$ and $\sup_{n\in\mathbb{N}} \|f_n\|_\infty \le 1$. As a first step, we show that $T(\cdot)f_n$ tends to 0 in $C^{1,2}(D)$, for any compact set $D \subset (0,+\infty)\times\mathbb{R}^N$. Since the function $T(\cdot)f_n$ solves the differential equation $D_t u - \mathcal{A}u = 0$ in $(0,+\infty)\times\mathbb{R}^N$, by the estimate (14.2.1)-(14.2.3) it follows that the sequence $\{T(\cdot)f_n\}$ is bounded in $C^{1+\alpha/2,2+\alpha}(K)$ for any compact set $K \subset (0,+\infty)\times\mathbb{R}^N$ and any $\alpha \in (0,1)$. Therefore, the Ascoli-Arzelà theorem implies that there exists a subsequence $\{T(\cdot)f_{n_k}\}$ converging in $C^{1,2}(D)$ to some function g, for any compact set $D \subset (0,+\infty)\times\mathbb{R}^N$. As it is easily seen, the function g solves the differential equation $D_t g - \mathcal{A}g = 0$. Let us prove that $g \equiv 0$. For this purpose, according to Theorem 14.2.6, it suffices to show that g can be extended by continuity at $t = 0$ by setting $g(0,\cdot) = 0$. So, let $F \subset \mathbb{R}^N$ be a compact set and take any smooth function η compactly supported in $\mathbb{R}^N$ such that $\eta \equiv \mathbb{1}$ in F and $0 \le \eta \le \mathbb{1}$. Recalling that $\{T(t)\}$ is a semigroup of contractions (see Theorem 14.2.7), we can write

$$\begin{aligned}|(T(t)f_{n_k})(x) - (T(t)(\eta f_{n_k}))(x)| &\le \|f_{n_k}\|_\infty |\mathbb{1} - (T(t)\eta)(x)| \\ &\le |\mathbb{1} - (T(t)\eta)(x)|, \qquad (14.3.1)\end{aligned}$$

for any $t > 0$ and any $x \in \mathbb{R}^N$. Now, since f_{n_k} converges locally uniformly in $\mathbb{R}^N$ as k tends to $+\infty$, the function $T(t)(\eta f_{n_k})$ converges uniformly in $\mathbb{R}^N$ to 0. Hence, letting k go to $+\infty$ in (14.3.1) gives

$$|g(t,x) - (T(t)(\eta f))(x)| \le |\mathbb{1} - (T(t)\eta)(x)|, \qquad t > 0, \quad x \in \mathbb{R}^N.$$

Since both η and ηf are compactly supported in $\mathbb{R}^N$, then, by Remark 14.2.8, $T(t)\eta$ and $T(t)(\eta f)$ tend, respectively, to η and 0 uniformly in $\mathbb{R}^N$, as t tends to 0^+. Therefore, from (14.3.1) we deduce that g tends to 0 as t tends to 0^+, uniformly in F. The arbitrariness of F implies that g can be extended by continuity at $t = 0$ setting $g(0,\cdot) = 0$, and we are done.

Repeating the same arguments as above we can show that any sequence $\{T(\cdot)f_{n_k}\}$ admits a subsequence which converges to 0 in $C^{1,2}(D)$ for any set D as above. This implies that, actually, all the sequence $\{T(\cdot)f_n\}$ converges in $C^{1,2}(D)$ to 0, for any D as above.

Now, to check that $T(\cdot)f_n$ converges to 0 locally uniformly in $[0,+\infty)\times\mathbb{R}^N$ it suffices to argue as in the proof of Proposition 2.2.9. ∎

Corollary 14.3.2 *There exists a family of probability Borel measures*

$$\{p(t,x;dy) : \ t > 0, \ x \in \mathbb{R}^N\},$$

such that

$$(T(t)f)(x) = \int_{\mathbb{R}^N} f(y)p(t,x;dy), \qquad t > 0, \ x \in \mathbb{R}^N, \qquad (14.3.2)$$

for any $f \in C_b(\mathbb{R}^N)$. It follows that, if $\{f_n\} \in C_b(\mathbb{R}^N)$ is a bounded sequence converging pointwise to $f \in C_b(\mathbb{R}^N)$, then $T(\cdot)f_n$ tends to $T(\cdot)f$ pointwise.

Proof. It can be obtained arguing as in the proof of Proposition 12.1.7, taking Proposition 14.3.1 into account. Therefore, we omit the details. ■

Next, arguing as in the proof of Proposition 12.2.1 and taking the gradient estimate in (14.2.9) into account, the following proposition can be proved.

Proposition 14.3.3 *The semigroup $\{T(t)\}$ is strong Feller.*

We now observe that since $\{T(t)\}$ is a semigroup of contractions, then for any $\lambda > 0$ we can define the linear operator $R(\lambda) \in L(C_b(\mathbb{R}^N))$ by setting

$$(R(\lambda)f)(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)dt, \qquad x \in \mathbb{R}^N.$$

As in Chapter 2, it is immediate to check that $\{R(\lambda) : \lambda > 0\}$ is the resolvent family associated with some closed operator $\widehat{A} : D(\widehat{A}) \subset C_b(\mathbb{R}^N) \to C_b(\mathbb{R}^N)$: the weak generator of the semigroup $\{T(t)\}$. From now on, we write $R(\lambda, \widehat{A})$ instead of $R(\lambda)$.

In the next proposition we characterize $D(\widehat{A})$ and we show that $\widehat{A}u = \mathcal{A}u$ for any $u \in D(\widehat{A})$, where $\mathcal{A}u$ is meant in the sense of distributions. The following lemma is the keystone in order to prove the forthcoming Proposition 14.3.5.

Lemma 14.3.4 *For any $f \in C_b^2(\mathbb{R}^N)$ such that $\mathcal{A}f \in C_b(\mathbb{R}^N)$ it holds that*

$$T(t)\mathcal{A}f = \mathcal{A}T(t)f, \qquad t > 0. \tag{14.3.3}$$

Proof. To prove (14.3.3) we first show that

$$T_\varepsilon(t)\mathcal{A}_\varepsilon f = \mathcal{A}_\varepsilon T_\varepsilon(t)f, \qquad t > 0, \tag{14.3.4}$$

for any $\varepsilon > 0$ and any $f \in C_b^2(\mathbb{R}^N)$ such that $\mathcal{A}f \in C_b(\mathbb{R}^N)$. Here, $\mathcal{A}_\varepsilon$ is given by (14.0.6). For this purpose, we recall that, according to Lemma 2.3.3 and Propositions 2.3.6, 4.1.1(i), $T(t)\mathcal{A}_\varepsilon u = \mathcal{A}_\varepsilon T(t)u$ for any $u \in D_{\max}(\mathcal{A}_\varepsilon)$, where

$$D_{\max}(\mathcal{A}_\varepsilon) := \left\{ g \in C_b(\mathbb{R}^N) \cap \bigcap_{1 \le p < +\infty} W^{2,p}_{\rm loc}(\mathbb{R}^N) : \mathcal{A}_\varepsilon g \in C_b(\mathbb{R}^N) \right\},$$

and $A_\varepsilon g = \mathcal{A}_\varepsilon g$ for any $g \in D_{\max}(\mathcal{A}_\varepsilon)$. Since $f \in D_{\max}(\mathcal{A}_\varepsilon)$ for any $\varepsilon > 0$, then (14.3.4) follows.

Now, we are almost done. Indeed, a straightforward computation shows that $\mathcal{A}_\varepsilon f$ converges uniformly in $\mathbb{R}^N$ to $\mathcal{A}f$ as ε tends to 0^+. Moreover, from Theorem 14.2.7 we know that $T_\varepsilon(t)f$ converges to $T(t)f$ in $C^2(K)$, as ε tends to 0^+, for any $t > 0$. Therefore, from (14.1.2) we get

$$\begin{aligned}
&\|T_\varepsilon(t)\mathcal{A}_\varepsilon f - T(t)\mathcal{A}f\|_{C(K)} \\
&\le \|T_\varepsilon(t)(\mathcal{A}_\varepsilon f - \mathcal{A}f)\|_{C(K)} + \|(T_\varepsilon(t) - T(t))\mathcal{A}f\|_{C(K)} \\
&\le \|\mathcal{A}_\varepsilon f - \mathcal{A}f\|_\infty + \|(T_\varepsilon(t) - T(t))\mathcal{A}f\|_{C(K)},
\end{aligned}$$

and the last side of the previous chain of inequalities vanishes as ε tends to 0^+. This implies that $T_\varepsilon(t)\mathcal{A}_\varepsilon f$ tends to $T(t)\mathcal{A}f$, locally uniformly in $\mathbb{R}^N$, for any $t>0$. Similarly, for any $t>0$, $\mathcal{A}_\varepsilon T_\varepsilon(t)f$ tends to $\mathcal{A}T(t)f$ locally uniformly in $\mathbb{R}^N$ as ε tends to 0^+. Hence, taking the limit as ε tends to 0^+ in (14.3.4), we get (14.3.3). ■

We are now in a position to prove the following proposition.

Proposition 14.3.5 *The following characterization holds true:*

$$\begin{aligned} D(\widehat{A}) = \Big\{ f \in C_b(\mathbb{R}^N) : \exists \{f_n\} \subset C_b^2(\mathbb{R}^N), \exists g \in C_b(\mathbb{R}^N) : \\ f_n \to f,\ \mathcal{A}f_n \to g \text{ locally uniformly in } \mathbb{R}^N \\ \text{and } \sup_{n\in\mathbb{N}} (\|f_n\|_\infty + \|\mathcal{A}f_n\|_\infty) < +\infty \Big\}. \end{aligned} \tag{14.3.5}$$

Moreover, $Af = \mathcal{A}f$ for any $f \in D(A)$. Here and above, $\mathcal{A}f$ is meant in the sense of distributions.

Proof. We adapt to our situation the technique of [107, Theorem 6.2]. For this purpose, let us denote by A_0 the realization of the operator $\mathcal{A}$ in $C_b(\mathbb{R}^N)$ with domain $D(A_0) = \{f \in C_b^2(\mathbb{R}^N) : \mathcal{A}f \in C_b(\mathbb{R}^N)\}$. Taking Lemma 14.3.4 into account, it is easy to check that, for any $f \in D(A_0)$,

$$\begin{aligned} (R(\lambda,\widehat{A})\mathcal{A}f)(x) &= \int_0^{+\infty} e^{-\lambda t}(T(t)\mathcal{A}f)(x)dt \\ &= \int_0^{+\infty} e^{-\lambda t}(\mathcal{A}T(t)f)(x)dt \\ &= \int_0^{+\infty} e^{-\lambda t}\left(\frac{\partial}{\partial t}T(t)f\right)(x)dt \\ &= -f(x) + \lambda(R(\lambda,\widehat{A})f)(x), \end{aligned} \tag{14.3.6}$$

for any $x \in \mathbb{R}^N$ and any $\lambda > 0$. Therefore, $R(\lambda,\widehat{A})(\lambda f - \mathcal{A}f) = f$ and, consequently, $f \in D(\widehat{A})$ and $\widehat{A}f = \mathcal{A}f$.

Now let $f \in D_0$ (the function space defined by the right-hand side of (14.3.5)). By the above results we know that

$$f_n = R(\lambda,\widehat{A})(\lambda f_n - \mathcal{A}f_n), \qquad n \in \mathbb{N},\ \ \lambda > 0. \tag{14.3.7}$$

Let us now show that, for any bounded sequence $\{h_n\} \subset C_b(\mathbb{R}^N)$, converging to some function $h \in C_b(\mathbb{R}^N)$ locally uniformly in $\mathbb{R}^N$, the function $R(\lambda,\widehat{A})h_n$ converges to $R(\lambda,\widehat{A})h$, locally uniformly in $\mathbb{R}^N$, as well. For this purpose, we observe that, for any compact set $K \subset \mathbb{R}^N$,

$$\|R(\lambda,\widehat{A})(h_n - h)\|_{C(K)} \le \int_0^{+\infty} e^{-\lambda t}\|T(t)(h_n - h)\|_{C(K)}dt.$$

By Proposition 14.3.1, $\|T(t)(h_n - h)\|_{C(K)}$ converges to 0 as n tends to $+\infty$, for any $t \geq 0$. Moreover,

$$\|T(t)(h_n - h)\|_{C(K)} \leq \|T(t)(h_n - h)\|_\infty \leq \|h_n - h\|_\infty \leq \|h_n\| + \|h\| \leq C,$$

for any $n \in \mathbb{N}$ and some positive constant C, independent of n. A straightforward application of the dominated convergence theorem shows that $R(\lambda, \widehat{A})h_n$ converges to $R(\lambda, \widehat{A})h$ as n tends to $+\infty$, locally uniformly in $\mathbb{R}^N$. Hence, letting n go to $+\infty$ in (14.3.7) gives $f = R(\lambda, \widehat{A})(\lambda f - g)$ and, consequently, $f \in D(\widehat{A})$ and $g = \widehat{A}f$. Moreover, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \mathcal{A}f_n \,\varphi dx = \int_{\mathbb{R}^N} f_n \mathcal{A}^* \varphi dx, \qquad n \in \mathbb{N}, \tag{14.3.8}$$

where by $\mathcal{A}^*$ we denote the adjoint of the operator $\mathcal{A}$, i.e., the operator defined by

$$(\mathcal{A}^*\psi)(x) = \sum_{i,j=1}^{r} D_{ij}(q_{ij}\psi)(x) - \sum_{i,j=1}^{N} b_{ij}x_j D_i\psi(x) - \psi(x)\mathrm{Tr}(B), \qquad x \in \mathbb{R}^N, \tag{14.3.9}$$

for any smooth function ψ. Hence, letting n go to $+\infty$ in (14.3.8), we obtain that $g = \mathcal{A}f$ and, consequently, $\widehat{A}f = \mathcal{A}f$.

Now, we prove that $D(\widehat{A}) \subset D_0$. For this purpose, fix $u \in D(\widehat{A})$ and $\lambda > 0$. Further, let $f \in C_b(\mathbb{R}^N)$ be such that $u = R(\lambda, \widehat{A})f$. By convolution, we can regularize f obtaining a sequence $\{f_n\} \subset C_b^2(\mathbb{R}^N)$ bounded in $C_b(\mathbb{R}^N)$ and converging locally uniformly to f as n tends to $+\infty$. Taking Theorem 14.2.7 into account, we deduce that $R(\lambda, \widehat{A})f_n$ belongs to $C_b^2(\mathbb{R}^N)$ for any $n \in \mathbb{N}$. Moreover,

$$\begin{aligned}(\mathcal{A}R(\lambda, \widehat{A})f_n)(x) &= \int_0^{+\infty} e^{-\lambda t}(\mathcal{A}T(t)f_n)(x)dt \\ &= -f_n(x) + \lambda(R(\lambda, \widehat{A})f_n)(x),\end{aligned} \tag{14.3.10}$$

for any $x \in \mathbb{R}^N$ and any $n \in \mathbb{N}$. Since, by the above results, $R(\lambda, \widehat{A})f_n$ converges to u, locally uniformly in $\mathbb{R}^N$, letting n go to $+\infty$ in (14.3.10) we deduce that $\mathcal{A}R(\lambda, \widehat{A})f_n$ converges to $\lambda u - f$, locally uniformly in $\mathbb{R}^N$. Moreover, as we can easily see, the sequences $\{R(\lambda, \widehat{A})f_n\}$ and $\{\mathcal{A}R(\lambda, \widehat{A})f_n\}$ are bounded in $C_b(\mathbb{R}^N)$. Therefore, $u \in D_0$. ■

14.4 The nonhomogeneous elliptic equation and the nonhomogeneous Cauchy problem

In this section we are devoted to prove existence, uniqueness and regularity properties of the bounded and continuous distributional solution to the elliptic equation

$$\lambda u(x) - \mathcal{A}u(x) = f(x), \qquad x \in \mathbb{R}^N, \tag{14.4.1}$$

and to the parabolic Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + g(t,x), & t > 0,\, x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N. \end{cases} \tag{14.4.2}$$

The main results that we prove are contained in the following two theorems.

Theorem 14.4.1 *Assume that Hypotheses* 14.0.1 *hold and let* $\theta \in (0,1)$, $\lambda > 0$. *Then, for any* $f \in \mathcal{C}_b^{\theta,\theta/3}(\mathbb{R}^N)$, *there exists a function* $u \in \mathcal{C}_b^{2+\theta,(2+\theta)/3}(\mathbb{R}^N)$ *solving the equation* (14.4.1) *in the sense of distributions. Moreover, there exists a positive constant* C, *independent of* u *and* f, *such that*

$$\|u\|_{\mathcal{C}_b^{2+\theta,(2+\theta)/3}(\mathbb{R}^N)} \le C\|f\|_{\mathcal{C}_b^{\theta,\theta/3}(\mathbb{R}^N)}. \tag{14.4.3}$$

Such a function u *is the unique distributional solution to the equation* (14.4.1) *which is bounded and continuous in* $\mathbb{R}^N$ *and it is twice continuously differentiable in* $\mathbb{R}^N$ *with respect to the first* r *variables, with bounded derivatives.*

Theorem 14.4.2 *Assume that Hypotheses* 14.0.1 *are satisfied and let* $\theta \in (0,1)$, $T > 0$. *Further, let* $f \in \mathcal{C}_b^{2+\theta,(2+\theta)/3}(\mathbb{R}^N)$ *and let* $g \in C_b([0,T]\times\mathbb{R}^N)$ *be such that* $g(t,\cdot) \in \mathcal{C}_b^{\theta,\theta/3}(\mathbb{R}^N)$ *for any* $t \in [0,T]$ *and*

$$\sup_{t\in[0,T]} \|g(t,\cdot)\|_{\mathcal{C}_b^{\theta,\theta/3}(\mathbb{R}^N)} < +\infty.$$

Then, there exists a function $u \in C_b([0,T]\times\mathbb{R}^N)$, *solution to the problem* (14.4.2) *in the sense of distributions, such that* $u(t,\cdot) \in \mathcal{C}_b^{2+\theta,(2+\theta)/3}(\mathbb{R}^N)$ *for any* $t \in [0,T]$ *and*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{\mathcal{C}_b^{2+\theta,(2+\theta)/3}(\mathbb{R}^N)} \le C\Big(\|f\|_{\mathcal{C}_b^{2+\theta,(2+\theta)/3}(\mathbb{R}^N)} + \sup_{t\in[0,T]} \|g(t,\cdot)\|_{\mathcal{C}_b^{\theta,\theta/3}(\mathbb{R}^N)}\Big), \tag{14.4.4}$$

for some positive constant C, *independent of* u, f *and* g. *Moreover,* u *is the unique distributional solution to the problem* (14.4.2) *which is bounded and continuous in* $[0,T]\times\mathbb{R}^N$ *and it is twice continuously differentiable with respect to the first* r *space variables in* $[0,T]\times\mathbb{R}^N$, *with bounded derivatives.*

To prove Theorems 14.4.1 and 14.4.2, in the spirit of Chapter 6, we first characterize the behaviour of the semigroup in the anisotropic space $C_b^{3\theta,\theta}(\mathbb{R}^N)$ ($\theta \in [0,1]$). More precisely, we show that for any $0 < \alpha \le \beta \le 1$ and any $\omega > 0$, there exists a positive constant $C = C(\alpha, \beta, \omega)$ such that

$$\|T(t)\|_{L(\mathcal{C}_b^{3\alpha,\alpha}(\mathbb{R}^N),\mathcal{C}_b^{3\beta,\beta}(\mathbb{R}^N))} \le Ce^{\omega t}t^{-\frac{3(\beta-\alpha)}{2}}, \qquad t > 0. \qquad (14.4.5)$$

Two steps are the main ingredients to prove (14.4.5):

Step (i): the interpolation set equality

$$(C_b(\mathbb{R}^N), C_b^{3,1}(\mathbb{R}^N))_{\theta,\infty} = \mathcal{C}_b^{3\theta,\theta}(\mathbb{R}^N), \qquad (14.4.6)$$

with equivalence of the corresponding norms, for any $\theta \in (0,1)$;

Step (ii): the estimate

$$\|T(t)\|_{L(C_b^{3k,k}(\mathbb{R}^N),C_b^{3,1}(\mathbb{R}^N))} \le Ce^{\omega t}t^{-\frac{3(1-k)}{2}}, \qquad t > 0, \quad k = 0, 1. \qquad (14.4.7)$$

Once the Steps (i) and (ii) are established, an interpolation argument allows us to prove (14.4.5). We begin by proving Step (i).

Proposition 14.4.3 *For any $\theta \in (0,1)$, the formula* (14.4.6) *holds true with equivalence of the corresponding norms.*

Proof. Throughout the proof we identify $\mathbb{R}^N$ with $\mathbb{R}^r \times \mathbb{R}^{N-r}$. Moreover, we denote by C_k ($k \in \mathbb{N}$) positive constants which are independent of s, t and $f \in C_b(\mathbb{R}^N)$. Finally, since the proof is rather long, we split it into three steps.

Step 1. Let us denote by A_1 and A_2, respectively, the realization in $C_b(\mathbb{R}^N)$ of the operators $\mathcal{A}_1 = \sum_{i=1}^r D_{ii}$ and $\mathcal{A}_2 = \sum_{i=r+1}^N D_{ii}$ with domains

$$D(A_1) = \bigg\{u \in C_b(\mathbb{R}^N) : u(\cdot, y) \in \bigcap_{1\le p<+\infty} W^{2,p}_{\rm loc}(\mathbb{R}^r), \quad \forall y \in \mathbb{R}^{N-r},$$
$$\mathcal{A}_1 u \in C_b(\mathbb{R}^N)\bigg\}$$

and

$$D(A_2) = \bigg\{u \in C_b(\mathbb{R}^N) : u(x, \cdot) \in \bigcap_{1\le p<+\infty} W^{2,p}_{\rm loc}(\mathbb{R}^{N-r}), \quad \forall x \in \mathbb{R}^r,$$
$$\mathcal{A}_2 u \in C_b(\mathbb{R}^N)\bigg\}.$$

Further, let us set $A_3 = -A_1^2$. In this step of the proof we show that

$$(X, D(A_j))_{\beta,\infty} = \mathcal{C}_b^{2(2-j)\beta,2(j-1)\beta}(\mathbb{R}^N), \quad j = 1, 2, \tag{14.4.8}$$

$$(X, D(A_3))_{\beta,\infty} = \mathcal{C}_b^{4\beta,0}(\mathbb{R}^N), \tag{14.4.9}$$

for any $\beta \in (0, 1)$, with equivalence of the corresponding norms. For this purpose, we will take advantage of the characterization of the interpolation spaces $(X, D(A))_{\beta,\infty}$ given in Proposition B.2.12, when A is the generator of an analytic semigroup in a Banach space X. Indeed, taking Theorem C.3.6(iv) into account, it is easy to check that A_1 and A_2 are sectorial operators. Moreover, since the realization of the Laplacian in $C_b(\mathbb{R}^N)$ satisfies (B.2.1) for any $\varphi \in (\pi/2, \pi)$, then the operator A_1 satisfies (B.2.1) for any $\varphi \in (\pi/2, \pi)$ as well. Therefore, according to Theorem B.2.10, also the operator A_3 is sectorial in $C_b(\mathbb{R}^N)$.

In the sequel, we simply write $D_{A_j}(\beta, \infty)$ and $D_{A_3}(\beta, \infty)$ to denote the interpolation spaces $(X, D(A_j))_{\beta,\infty}$ and $(X, D(A_3))_{\infty}$, $(j = 1, 2)$.

We begin by proving (14.4.8) in the case when $j = 1$. For this purpose, we preliminarily observe that the analytic semigroup $\{T_1(t)\}$, associated with the operator A_1, is given by

$$(T_1(t)f)(x, y) = \frac{1}{(4\pi t)^{r/2}} \int_{\mathbb{R}^r} e^{-\frac{|x-z|^2}{4t}} f(z, y)dz, \qquad t > 0,\ (x, y) \in \mathbb{R}^N,$$

for any $f \in C_b(\mathbb{R}^N)$.

To prove the inclusion $\mathcal{C}_b^{2\beta,0}(\mathbb{R}^N) \subset D_{A_1}(\beta, \infty)$, we fix $f \in \mathcal{C}_b^{2\beta,0}(\mathbb{R}^N)$ and observe that

$$\begin{aligned}&(T_1(t)f)(x, y) - f(x, y)\\ &= \frac{1}{2(4\pi)^{r/2}} \int_{\mathbb{R}^r} e^{-\frac{|z|^2}{4}} \big(f(x - \sqrt{t}z, y) - 2f(x, y) + f(x + \sqrt{t}z, y)\big)dz,\end{aligned}$$

for any $t > 0$ and any $(x, y) \in \mathbb{R}^N$. Therefore,

$$\|T_1(t)f - f\|_\infty \le \frac{t^\beta}{(4\pi)^{r/2}} [f]_{\mathcal{C}_b^{2\beta,0}(\mathbb{R}^N)} \int_{\mathbb{R}^r} e^{-\frac{|z|^2}{4}} |z|^{2\beta} dz, \tag{14.4.10}$$

and Proposition B.2.12 implies that $\mathcal{C}_b^{2\beta,0}(\mathbb{R}^N)$ is continuously embedded in $D_{A_1}(\beta, \infty)$.

Let us now prove the other inclusion in (14.4.8). We begin by observing that

$$(i)\ \|D_x T_1(t)f\|_\infty \le C_1 t^{-\frac{1}{2}} \|f\|_{C_b(\mathbb{R}^N)}, \qquad (ii)\ \|A_1 T_1(t)f\|_\infty \le C_1 t^{-1} \|f\|_\infty, \tag{14.4.11}$$

for any $f \in C_b(\mathbb{R}^N)$, where D_x denotes the gradient with respect to the first r variables. Let us fix $f \in D_{A_1}(\beta, \infty)$ and assume that $\beta \in (0, 1/2)$. Then, for any $t>0$, any $x_1,x_2\in\mathbb{R}^r$ and any $y\in\mathbb{R}^{N-r}$, it holds that

$$\begin{aligned}
&|f(x_2,y)-f(x_1,y)|\\
&\le |(T_1(t)f)(x_2,y)-f(x_2,y)|+|(T_1(t)f)(x_2,y)-(T_1(t)f)(x_1,y)|\\
&\quad+|(T_1(t)f)(x_1,y)-f(x_1,y)|\\
&\le 2[[f]]_{D_{A_1}(\beta,\infty)}t^{\beta}+\|D_xT_1(t)f\|_\infty|x_2-x_1|, \qquad (14.4.12)
\end{aligned}$$

where $[[f]]_{D_{A_1}(\beta,\infty)}$ is given by Proposition B.2.12. So, we need to estimate the sup-norm of $D_xT_1(t)f$. As it is easily seen,

$$D_iT_1(n)f-D_iT_1(t)f=\int_t^n D_iA_1T_1(s)f\,ds, \qquad t\in(0,n), \qquad (14.4.13)$$

for any $i=1,\ldots,r$. Combining (14.4.11)(ii) and Proposition B.2.12, we deduce that

$$\|A_1T_1(t)f\|_\infty\le C_2t^{\beta-1}\|f\|_{D_A(\beta,\infty)}, \qquad t\in(0,+\infty). \qquad (14.4.14)$$

Now, from (14.4.11)(i) and (14.4.14), we get

$$\begin{aligned}
\|D_iA_1T_1(s)f\|_\infty &= \|D_iT_1(s/2)A_1T_1(s/2)f\|_\infty\\
&\le \|D_iT_1(s/2)\|_{L(C_b(\mathbb{R}^N))}\|A_1T_1(s/2)f\|_\infty\\
&\le C_3s^{\beta-\frac{3}{2}}\|f\|_{D_{A_1}(\beta,\infty)}, \qquad (14.4.15)
\end{aligned}$$

for any $s>0$, so that we can let n go to $+\infty$ in (14.4.13) obtaining

$$D_iT_1(t)f=-\int_t^{+\infty}D_iA_1T_1(s)fds, \qquad t>0,$$

and

$$\|D_xT_1(t)f\|_\infty\le C_4t^{\beta-\frac{1}{2}}\|f\|_{D_{A_1}(\beta,\infty)}. \qquad (14.4.16)$$

Now, taking $t=|x_2-x_1|^2$ in (14.4.12) and (14.4.16), we get

$$|f(x_2,y)-f(x_1,y)|\le C_5\|f\|_{D_{A_1}(\beta,\infty)}|x_2-x_1|^{2\beta},$$

for any $|x_2-x_1|\le 1$. Moreover, if $|x_2-x_1|>1$ we have

$$|f(x_2,y)-f(x_1,y)|\le 2\|f\|_\infty\le 2\|f\|_{D_{A_1}(\beta,\infty)}|x_2-x_1|^{2\beta},$$

so that $D_{A_1}(\beta,\infty)\subset C_b^{2\beta,0}(\mathbb{R}^N)$ with a continuous embedding.

Suppose now that $\beta>1/2$. According to (14.4.13) and (14.4.15) we have

$$\|D_iT_1(t_2)f-D_iT(t_1)f\|_\infty\le C_6|t_2-t_1|^{\beta-1/2}\|f\|_{D_{A_1}(\beta,\infty)}, \qquad t_1,t_2>0, \qquad (14.4.17)$$

for any $i = 1, \dots, r$. This implies that $DT_1(t)f$ converges uniformly in $\mathbb{R}^N$ as t tends to 0. Since $T_1(\cdot)f$ is continuous in $[0, +\infty) \times \mathbb{R}^N$ and $T_1(0)f = f$, we obtain that $f \in C_b^1(\mathbb{R}^N)$.

Now, taking the estimates (14.4.11)(i) and (14.4.16) (which, of course, holds true also in the case when $\beta \in (1/2, 1)$) into account, we get

$$\begin{aligned}\|D_x D_i T_1(t)f\|_\infty &= \|D_x T(t/2) D_i T(t)f\|_\infty \\ &\le \|D_x T(t/2)\|_\infty \|D_i T(t)f\|_\infty \\ &\le C_1 C_4 t^{\beta-1} \|f\|_{D_{A_1}(\beta,\infty)}.\end{aligned} \tag{14.4.18}$$

Hence, from (14.4.17) (where we take $t_1 = 0$) and (14.4.18), it follows that

$$\begin{aligned}&|D_i f(x_2, y) - D_i f(x_1, y)| \\ \le\ & |(D_i T_1(t)f)(x_2, y) - D_i f(x_2, y)| + |(D_i T_1(t)f)(x_2, y) - (D_i T_1(t)f)(x_1, y)| \\ &+ |(D_i T_1(t)f)(x_1, y) - D_i f(x_1, y)| \\ \le\ & 2\|D_i T_1(t)f - f\|_\infty + \|D_x D_i T_1(t)f\|_\infty |x - y| \\ \le\ & 2C_6 t^{\beta-1/2} \|u\|_{D_{A_1}(\beta,\infty)} + C_1 C_4 t^{\beta-1} |x_2 - x_1| \|f\|_{D_{A_1}(\beta,\infty)},\end{aligned}$$

for any $t > 0$, any $x_1, x_2 \in \mathbb{R}^r$, any $y \in \mathbb{R}^{N-r}$ and any $1 = 1 \dots, r$. The same arguments as in the case when $\beta < 1/2$ allow us to conclude that $D_{A_1}(\beta, \infty) \subset C_b^{2\beta,0}(\mathbb{R}^N)$ with a continuous embedding.

So far, we have proved (14.4.8) (with $j = 1$) in the case when $\beta \neq 1/2$. Let us now consider the case when $\beta = 1/2$. For this purpose, we observe that

$$\begin{aligned}D_{A_1}(1/2, \infty) &= (D_{A_1}(1/4, \infty), D_{A_1}(3/4, \infty))_{1/2,\infty} \\ &= (C_b^{1/2,0}(\mathbb{R}^N), C_b^{3/2,0}(\mathbb{R}^N))_{1/2,\infty}.\end{aligned}$$

Since $(C_b^{1/2}(\mathbb{R}^r), C_b^{3/2}(\mathbb{R}^r))_{1/2,\infty} = \mathcal{C}_b^1(\mathbb{R}^r)$ (see Theorem A.4.8) and the operator $T_y : C_b(\mathbb{R}^N) \to C_b(\mathbb{R}^r)$, defined by $T_y f = f(\cdot, y)$ for any $f \in C_b(\mathbb{R}^N)$, belongs to $L(C_b^{(2k+1)/2,0}(\mathbb{R}^N), C_b^{(2k+1)/2}(\mathbb{R}^r))$ $(k = 0, 1)$ with norms being independent of y, then Proposition A.4.2 allows us to conclude that $D_{A_1}(1/2, \infty)$ is continuously embedded in $\mathcal{C}_b^{1,0}(\mathbb{R}^N)$. This concludes the proof of (14.4.8), in the case when $j = 1$.

The proof of (14.4.8), with $j = 2$, is completely similar, so we skip it.

Let us prove (14.4.9). From the general interpolation theory, we know that, for any $\beta \in (0, 1)$,

$$D_{A_3}(\beta/2, \infty) = (C_b(\mathbb{R}^N), D(A_1^2))_{\beta/2,\infty} = (D(A_1), D(A_1^2))_{\beta,\infty} = D_{A_1}(1+\beta, \infty), \tag{14.4.19}$$

with equivalence of the corresponding norms. Hence, (14.4.9) follows immediately if we show that

$$D_{A_1}(1 + \beta, \infty) = \mathcal{C}_b^{2+2\beta,0}(\mathbb{R}^N), \tag{14.4.20}$$

with equivalence of the corresponding norms. The embedding $\mathcal{C}_b^{2+2\beta,0}(\mathbb{R}^N) \subset D_{A_1}(1+\beta,\infty)$ is an obvious consequence of (14.4.8) and the characterization of $D(A_1)$. To prove the other embedding, we fix $f \in D_{A_1}(1+\beta,\infty)$ and prove that f admits second-order derivatives, with respect to the first r variables, which belong to $\mathcal{C}_b^{2\beta,0}(\mathbb{R}^N)$. For this purpose, we observe that, by (B.2.6), we can write

$$f = \int_0^{+\infty} e^{-t} T_1(t) g dt, \tag{14.4.21}$$

where $g := f - \mathcal{A}_1 f$ belongs to $D_{A_1}(\beta,\infty) = \mathcal{C}_b^{2\beta,0}(\mathbb{R}^N)$. Taking (14.4.16) into account, it is immediate to check that f is once-continuously differentiable with respect to the first r variables and

$$\|D_x f\|_\infty \le C_7 \|g\|_{D_{A_1}(\beta,\infty)}. \tag{14.4.22}$$

Splitting $D_{ij}T_1(t)g = D_j T_1(t/2) D_i T_1(t/2) g$ for any $t > 0$ and any $i,j = 1,\ldots,r$, and using (14.4.11)(i) and again (14.4.16), we can show that

$$\|D_{ij}T_1(t)g\|_\infty \le C_8 t^{\beta-1} \|g\|_{D_{A_1}(\beta,\infty)}, \qquad i,j = 1,\ldots,r. \tag{14.4.23}$$

Hence, f is twice continuously differentiable in $\mathbb{R}^N$ with respect to the first r variables and

$$\|D_{ij}f\|_\infty \le C_9 \|g\|_{D_{A_1}(\beta,\infty)} \le C_{10} \|f\|_{D_{A_1}(1+\beta,\infty)}, \qquad t > 0, \quad i,j = 1,\ldots,r. \tag{14.4.24}$$

This implies that $f \in C_b^{2,0}(\mathbb{R}^N)$ with norm bounded by $C_{11}\|f\|_{D_{A_1}(1+\beta,\infty)}$. Since we already know that $D_{A_1}(\beta,\infty) = \mathcal{C}_b^{2\beta,0}(\mathbb{R}^N)$, it is sufficient to prove that $D_{ij}f \in D_{A_1}(\beta,\infty)$. Using (14.4.11)(ii) and (14.4.23), we get

$$\begin{aligned}
\|\xi^{1-\beta} A_1 T_1(\xi) D_{ij} f\|_\infty &\le \left\| \int_0^{+\infty} \xi^{1-\beta} e^{-t} A_1 T_1(\xi + t/2) D_{ij} T_1(t/2) g dt \right\| \\
&\le C_{12} \|g\|_{D_{A_1}(\beta,\infty)} \int_0^{+\infty} \frac{1}{(1+t)t^{1-\beta}} dt,
\end{aligned}$$

for any $\xi \in (0,1)$. Therefore, all the second order derivatives of f (with respect to the directions e_i, e_j, with $i,j \le r$) are in $D_{A_1}(\beta,\infty) = \mathcal{C}_b^{2\beta,0}(\mathbb{R}^N)$ (see again Proposition B.2.12) and

$$\|D_{ij}f\|_{\mathcal{C}_b^{2\beta,0}(\mathbb{R}^N)} \le C_{13} \|g\|_{D_{A_1}(\beta,\infty)} \le C_{14} \|f\|_{D_{A_1}(1+\beta,\infty)}.$$

The formula (14.4.20) now follows.

Step 2. Here, taking advantage of the results in Step 1, we show that

$$(C_b(\mathbb{R}^N), \mathcal{C}_b^{3\alpha,\alpha}(\mathbb{R}^N))_{\gamma,\infty} = \mathcal{C}_b^{3\alpha\gamma,\alpha\gamma}(\mathbb{R}^N), \tag{14.4.25}$$

for any $\alpha \in (0, 4/3)$ and any $\gamma \in (0, 1)$. This can be done adapting the proof given in [107, Theorem 2.2]. We begin by proving (14.4.25) in the case when $\alpha \in (0, 2/3)$. By (14.4.8) we can write

$$\mathcal{C}_b^{3\alpha,\alpha}(\mathbb{R}^N) = D_{A_1}(3\alpha/2, \infty) \cap D_{A_2}(\alpha/2, \infty).$$

The characterization (14.4.25) will follow immediately if we show that

$$\begin{aligned}&(C_b(\mathbb{R}^N), D_{A_1}(3\alpha/2, \infty) \cap D_{A_2}(\alpha/2, \infty))_{\gamma,\infty}\\ &= D_{A_1}(3\gamma\alpha/2, \infty) \cap D_{A_2}(\gamma\alpha/2, \infty),\end{aligned} \tag{14.4.26}$$

for any $\gamma \in (0, 1)$, with equivalence of the corresponding norms.

The inclusion "$\subset$" in (14.4.26) is immediate if we observe that, for any triplet of Banach spaces X, Y, Z such that $Z \subset Y \subset X$ with continuous embeddings, it holds that $(X, Z)_{\theta,\infty} \subset (X, Y)_{\theta,\infty}$, for any $\theta \in (0, 1)$, with continuous embedding.

So, let us prove the other inclusion in (14.4.26). For this purpose, we fix $f \in D_{A_1}(3\gamma\alpha/2, \infty) \cap D_{A_2}(\gamma\alpha/2, \infty)$ and define the function $u : [0, +\infty) \to C_b(\mathbb{R}^N)$ by setting

$$u(t) = T_1(t^{2/(3\alpha)})T_2(t^{2/\alpha})f, \qquad t \in (0, 1].$$

Using Proposition B.2.12 yields

$$\begin{aligned}&\|u(t)\|_{D_{A_1}(3\alpha/2,\infty)}\\ &\le C_{15}\Big(\|T_1(t^{2/(3\alpha)})\|_{L(C_b(\mathbb{R}^N))}\|T_2(t^{2/\alpha})\|_{L(C_b(\mathbb{R}^N))}\|f\|_{C_b(\mathbb{R}^N)}\\ &\qquad +\|T_2(t^{2/(3\alpha)})\|_{L(C_b(\mathbb{R}^N))}\sup_{0<\xi\le 1}\|\xi^{1-2/(3\alpha)}A_1T_1(\xi + t^{2/(3\alpha)})f\|_{C_b(\mathbb{R}^N)}\Big)\\ &\le C_{16}\Big(\|f\|_{C_b(\mathbb{R}^N)} + \sup_{0<\xi\le 1}\xi^{1-2/(3\alpha)}(\xi + t^{2/(3\alpha)})^{(3\alpha-2)/2}\|f\|_{D_{A_1}(3\alpha/2,\infty)}\Big)\\ &\le C_{17}t^{(3\alpha-2)/2}\|f\|_{D_{A_1}(3\alpha/2,\infty)},\end{aligned} \tag{14.4.27}$$

for any $t \in (0, 1]$. Similarly,

$$\|u(t)\|_{D_{A_2}(\alpha/2,\infty)} \le C_{18}t^{(\alpha-2)/2}\|f\|_{D_{A_2}(\alpha/2,\infty)} \tag{14.4.28}$$

and

$$\begin{aligned}\|u(t) - f\|_{C_b(\mathbb{R}^N)} &\le \|T_1(t^{2/(3\alpha)})\|_{L(C_b(\mathbb{R}^N))}\|(T_2(t^{2/\alpha}) - 1)f\|_{C_b(\mathbb{R}^N)}\\ &\le C_{19}t^{\gamma}\Big(\|f\|_{D_{A_1}(3/\alpha/2,\infty)} + \|f\|_{D_{A_2}(\alpha/2,\infty)}\Big),\end{aligned} \tag{14.4.29}$$

for any $t \in (0, 1]$. Splitting $f = u(t) + (f - u(t))$ and using (14.4.27)-(14.4.29), we get

$$t^{-\gamma}K(t, f) \le C_{20}\Big(\|f\|_{D_{A_1}(3\alpha/2,\infty)} + \|f\|_{D_{A_2}(\alpha/2,\infty)}\Big), \qquad t \in (0, 1], \tag{14.4.30}$$

which implies that $f \in (C_b(\mathbb{R}^N), D_{A_1}(3\alpha/2,\infty)\cap D_{A_2}(\alpha/2,\infty))_{\gamma,\infty}$. We have so proved both the set equality in (14.4.26) and the equivalence of the norms of the spaces in the left- and right-hand sides of (14.4.26).

Now, suppose that $\alpha \in [2/3, 4/3)$. Then, according to (14.4.9), $\mathcal{C}_b^{3\alpha,0}(\mathbb{R}^N) = D_{A_3}(3\alpha/4,\infty)$ with equivalence of the corresponding norms. Therefore, repeating the above arguments, with the operator A_1 being now replaced with A_3, we can prove (14.4.25).

Step 3. We can now conclude the proof proving (14.4.6). From (14.4.25), with $\alpha \in (1, 4/3)$ and $\gamma = 1/\alpha$, we deduce that

$$C_b^{3,1}(\mathbb{R}^N) \subset \mathcal{C}_b^{3,1}(\mathbb{R}^N) \in K_{1/\alpha}(C_b(\mathbb{R}^N), \mathcal{C}_b^{3\alpha,\alpha}(\mathbb{R}^N)).$$

Since $C_b^3(\mathbb{R}^r)$ and $C_b^1(\mathbb{R}^{N-r})$ belong, respectively, to $J_{1/\alpha}(C_b(\mathbb{R}^r), C_b^{3\alpha}(\mathbb{R}^r))$ and $J_{1/\alpha}(C_b(\mathbb{R}^{N-r}), C_b^{\alpha}(\mathbb{R}^{N-r}))$ (see Proposition A.4.4), then $C_b^{3,1}(\mathbb{R}^N)$ belongs to $J_{1/\alpha}(C_b(\mathbb{R}^N), \mathcal{C}_b^{3\alpha,\alpha}(\mathbb{R}^N))$. The reiteration theorem (see Theorem A.4.7) and (14.4.25) imply that

$$(C_b(\mathbb{R}^N), C_b^{3,1}(\mathbb{R}^N))_{\theta,\infty} = (C_b(\mathbb{R}^N), \mathcal{C}_b^{3\alpha,\alpha}(\mathbb{R}^N))_{\theta/\alpha,\infty} = \mathcal{C}_b^{3\theta,\theta}(\mathbb{R}^N),$$

for any $\theta \in (0,1)$, with equivalence of the corresponding norms. The set equality (14.4.6) now follows. ■

Applying the reiteration Theorem A.4.7, we get the following corollary.

Corollary 14.4.4 *For any $\alpha, \theta \in (0,1)$ and any $\alpha \le \beta < 1$,*

$$(\mathcal{C}_b^{3\alpha,\alpha}(\mathbb{R}^N), C_b^{3,1}(\mathbb{R}^N))_{\theta,\infty} = \mathcal{C}_b^{3(\alpha+(1-\alpha)\theta),\alpha+(1-\alpha)\theta}(\mathbb{R}^N), \tag{14.4.31}$$

with equivalence of the corresponding norms.

Remark 14.4.5 Adapting the techniques in the proof of Proposition 14.4.3 one can show that (14.4.25) holds true for any $\alpha > 1$ and, hence, by reiteration, that for any $\alpha, \beta > 0$ and any $\theta \in (0,1)$,

$$(\mathcal{C}_b^{\alpha,\alpha/3}(\mathbb{R}^N), \mathcal{C}_b^{\beta,\beta/3}(\mathbb{R}^N))_{\theta,\infty} = \mathcal{C}_b^{\alpha+\theta(\beta-\alpha),(\alpha+\theta(\beta-\alpha))/3}(\mathbb{R}^N),$$

with equivalence of the corresponding norms.

As far as Step (ii) is concerned, we observe that, when $k=0$, the estimate (14.4.7) follows easily from (14.2.1)-(14.2.3). Hence, we just need to show (14.4.7) when $k=1$. For this purpose, we adapt the Bernstein method to these anisotropic spaces.

In the following theorem we use the same notation introduced before Theorem 14.2.4.

Theorem 14.4.6 *Let $\varepsilon > 0$ and assume that Hypotheses* 14.0.1 *are satisfied. Then, there exist two positive constants C and ω such that* (14.4.7) *holds true, with $k = 1$.*

Proof. As in the proof of Theorem 14.2.4, we limit ourselves to showing that for any $\omega > 0$, there exists a positive constant C, independent of ε such that

$$\|T_\varepsilon(t)\|_{L(C_b^{3,1}(\mathbb{R}^N),C_b^{3,1}(\mathbb{R}^N))} \le Ce^{\omega t}, \qquad t > 0. \tag{14.4.32}$$

Here, $\{T_\varepsilon(t)\}$ is the semigroup defined in Section 14.2. Once (14.4.32) is proved we will be almost done. Indeed, suppose that (14.4.32) holds true. Since $T_\varepsilon(\cdot)f$ tends to $T(\cdot)f$ in $C^{1,2}(D)$ for any compact set $D \subset (0,+\infty) \times \mathbb{R}^N$ and any $f \in C_b(\mathbb{R}^N)$ (see Theorem 14.2.7), then, for any $t > 0$ the sup-norms of both $DT(t)f$ and $D^2_{*,1}T(t)f$ are bounded by the right-hand side of (14.4.32). As far as the vector $D^3_{*,1}T(t)f$ is concerned, we observe that (14.4.32) implies that

$$[(D^2_{*,1}T_\varepsilon(t)f)(\cdot, x_{r+1}, \ldots, x_N)]_{\mathrm{Lip}(\mathbb{R}^r)} \le Ce^{\omega t}\|f\|_{C_b^{3,1}(\mathbb{R}^N)}, \qquad t > 0,$$

for any $t > 0$ and any $(x_{r+1}, \ldots, x_N) \in \mathbb{R}^{N-r}$, and, letting ε go to 0^+, yields

$$[(D^2_{*,1}T(t)f)(\cdot, x_{r+1}, \ldots, x_N)]_{\mathrm{Lip}(\mathbb{R}^r)} \le Ce^{\omega t}\|f\|_{C_b^{3,1}(\mathbb{R}^N)}, \tag{14.4.33}$$

and we are done. Indeed, since $T(t)f \in C_b^3(\mathbb{R}^N)$ for any $t > 0$ (see Theorem 14.2.7), from (14.4.33) we deduce that

$$\|D^3_{*,1}T(t)f\|_\infty \le Ce^{\omega t}\|f\|_{C_b^{3,1}(\mathbb{R}^N)}, \qquad t > 0,$$

and (14.4.7) follows.

So, let us prove (14.4.32). For notational convenience we simply write u instead of $T_\varepsilon(\cdot)f$. Let us introduce the function $\xi : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$\begin{aligned}\xi(t,x) = {}& \frac{1}{2}\alpha^3(u(t,x))^2 + \langle F\, Du(t,x), Du(t,x)\rangle + \langle G(t)D^2_*u(t,x), D^2_*u(t,x)\rangle \\ &+ \langle H(t)D^3_*u(t,x), D^3_*u(t,x)\rangle,\end{aligned}$$

for any $t > 0$ and any $x \in \mathbb{R}^N$, where the matrices $F \in L(\mathbb{R}^N)$, $G(t) \in L(\mathbb{R}^{n^1_N})$ and $H(t) \in L(\mathbb{R}^{n^2_N})$ $(t \ge 0)$ are defined by

$$F = \begin{pmatrix} \alpha I_r & \beta F_1 \\ \beta F_1^* & -\iota I_{N-r} \end{pmatrix}, \qquad G(t) = \begin{pmatrix} I_{n^1_r} & 0 & 0 \\ 0 & \alpha^{-\frac{7}{16}} t I_{r(N-r)} & -\alpha^{-\frac{4}{5}} t^2 L_5 \\ 0 & -\alpha^{-\frac{4}{5}} t^2 L_5^* & \alpha^{-\frac{7}{8}} t^3 I_{n^1_{N-r}} \end{pmatrix}, \tag{14.4.34}$$

$$H(t)=\begin{pmatrix} \alpha^{-\frac{4}{3}}I_{n_r^2} & -\alpha^{-\frac{4}{3}}tM_2 & \frac{1}{2}t^2\alpha^{-\frac{4}{3}}H_1 & -\frac{1}{6}\alpha^{-\frac{4}{3}}t^3H_2 \\ -\alpha^{\frac{4}{3}}tM_2^* & \alpha^{-\frac{8}{9}}t^2I_{(N-r)n_r^1} & 0 & 0 \\ \frac{1}{2}\alpha^{-\frac{4}{3}}t^2H_1^* & 0 & \alpha^{-1}t^4I_{rn_{N-r}^1} & -\alpha^{-\frac{13}{12}}t^5M_8 \\ -\frac{1}{6}\alpha^{-\frac{4}{3}}t^3H_2^* & 0 & -\alpha^{-\frac{13}{12}}t^5M_8^* & \alpha^{-\frac{9}{8}}t^6I_{n_{N-r}^2} \end{pmatrix}. \tag{14.4.35}$$

Here, n_m^1 and n_m^2 ($m\in\mathbb{N}$) are given by (14.2.10), $\beta=2\|B_4\|+1$, $F_1\in L(\mathbb{R}^{N-r},\mathbb{R}^r)$ is any matrix such that $B_3F_1+F_1^*B_3^*$ is strictly negative definite and $-\iota$ denotes its maximum (negative) eigenvalue. Such a matrix exists since we have assumed that $r\geq N/2$. Moreover, $H_1=M_2M_5$ and $H_2=M_2M_5M_8$, where the matrices M_2, M_5, M_8, as well as L_5, are given by (14.2.22). Finally α is a positive constant to be determined later on.

Taking Propositions 14.1.1 and 14.1.2 into account, it is easy to check that the function ξ is a classical solution of the Cauchy problem

$$\begin{cases} D_t\xi(t,\cdot)=\mathcal{A}_\varepsilon\xi(t,\cdot)+g_\varepsilon(t,\cdot), & t>0,\\ \xi(0,\cdot)=\frac{1}{2}\alpha^3f^2+\langle F\,Df,Df\rangle+\langle E\,D^2_{*,1}f,D^2_{*,1}f\rangle+\alpha^{-\frac{4}{3}}\langle K_1D^3_{*,1}f,D^3_{*,1}f\rangle, \end{cases}$$

where $g^{(\varepsilon)}=g_1^{(\varepsilon)}+g_2+g_3$, and $g_1^{(\varepsilon)}$, g_2 and g_3 are given by (14.2.15)-(14.2.17) with F, $G(t)$ and $H(t)$ as in (14.4.34) and (14.4.35). To get the assertion, as in the proof of Theorem 14.2.4, it suffices to show that there exists $\alpha>0$ such that F, $G(t)$ and $H(t)$ are strictly positive definite for any $t>0$ and $g^{(\varepsilon)}\leq 0$ in $(0,T_0]\times\mathbb{R}^N$ for some T_0 independent of ε. Indeed, then Theorem 14.2.6 and the semigroup rule will lead us to (14.4.32) We refer the reader to the proof of Theorem 14.2.4 for more details.

Due to the particular structure of the matrices F, $G(t)$ and $H(t)$, it is easy to check that the previous matrices are strictly positive definite for any $t>0$, if α is sufficiently large. Hence, we can limit ourselves to showing that $g_\varepsilon\leq 0$ in $(0,T_0]\times\mathbb{R}^N$ for a suitable choice of T_0 and the parameter α. We begin by estimating the function $g_1^{(\varepsilon)}$. As it is immediately seen, $g_1^{(\varepsilon)}\leq g_1$, where g_1 is obtained from $g_1^{(\varepsilon)}$ by replacing everywhere the matrix $Q^{(\varepsilon)}$ with the matrix $Q^{(0)}$. Arguing as in the proof of (14.2.20) we can prove that

$$\begin{aligned} g_1(t)\leq{}& -\alpha^3\kappa|D_{*,1}u(t)|^2-\kappa\{2\alpha-(2\|B_4\|+1)\alpha^{\frac{1}{2}}\|K_2\|\}|D^2_{*,1}u(t)|^2\\ &-\kappa\{2\iota-(2\|B_4\|+1)\alpha^{-\frac{1}{2}}\|K_2\|\}|D^2_{*,2}u(t)|^2-2\kappa|D^3_{*,1}u(t)|^2\\ &-\kappa t(2\alpha^{-\frac{7}{16}}-\alpha^{-\frac{3}{5}}\|K_5\|)|D^3_{*,2}u(t)|^2-\kappa t^3(2\alpha^{-\frac{7}{8}}-\alpha^{-1}\|K_5\|)|D^3_{*,3}u(t)|^2\\ &-\kappa\{2\alpha^{-\frac{4}{3}}-\alpha^{-\frac{3}{2}}(\|K_9\|+\|K_{10}\|+\|K_{11}\|)\}|D^4_{*,1}u(t)|^2\\ &-\kappa t^2(2\alpha^{-\frac{8}{9}}-\alpha^{-\frac{7}{6}}\|K_9\|)|D^4_{*,2}u(t)|^2 \end{aligned}$$

$$-\kappa t^4(2\alpha^{-1}-\alpha^{-\frac{7}{6}}\|K_{10}\|-\alpha^{-\frac{49}{48}}\|K_{12}\|)|D^4_{*,3}u(t)|^2$$
$$-\kappa t^6(2\alpha^{-\frac{9}{8}}-\alpha^{-\frac{7}{6}}\|K_{11}\|-\alpha^{-\frac{55}{48}}\|K_{12}\|)|D^4_{*,4}u(t)|^2. \qquad (14.4.36)$$

The function g_2 can be estimated using properly the inequality (14.2.19) and taking (14.2.21) and (14.2.22) into account. We get

$$\begin{aligned}
g_2(t) \le{}& \{2\alpha\|B_1\|+o_\alpha(\alpha)\}|D_{*,1}u(t)|^2+\{-\iota+o_\alpha(1)\}|D_{*,2}u(t)|^2\\
&+\{2\|L_1\|+o_\alpha(1)+o_t(1)\}|D^2_{*,1}u(t)|^2\\
&+\{\alpha^{-\frac{1}{16}}\|L_5\|+o_\alpha(\alpha^{-\frac{1}{16}})+o_t(1)\}|D^2_{*,2}u(t)|^2\\
&+t^2\{-2\alpha^{-\frac{4}{5}}\lambda_{\min}(L_5^*L_5)+o_\alpha(\alpha^{-\frac{4}{5}})+o_t(1)\}|D^2_{*,3}u(t)|^2\\
&+\{2\alpha^{-\frac{4}{3}}\|M_1\|+o_t(1)\}|D^3_{*,1}u(t)|^2\\
&+t\{\alpha^{-\frac{8}{9}}(\|M_5\|+2)+o_\alpha(\alpha^{-\frac{8}{9}})+o_t(1)\}|D^3_{*,2}u(t)|^2\\
&+t^3\{\alpha^{-\frac{8}{9}}\|M_5\|+o_\alpha(\alpha^{-\frac{8}{9}})+o_t(1)\}|D^3_{*,3}u(t)|^2\\
&+t^5\{-2\alpha^{-\frac{13}{12}}\lambda_{\min}(M_8^*M_8)+o_\alpha(\alpha^{-\frac{13}{12}})+o_t(1)\}|D^3_{*,4}u(t)|^2, \qquad (14.4.37)
\end{aligned}$$

for any $t>0$, where, to simplify the notation, we denote by $o_t(t^k)$ ($k\ge 0$) any function of t (possibly depending also on α) such that $\lim_{t\to 0}t^{-k}o_t(t^k)=0$, and by $o_\alpha(\alpha^k)$ ($k\in\mathbb{R}$) any function depending only on α and such that $\lim_{\alpha\to+\infty}\alpha^{-k}o_\alpha(\alpha^k)=0$. Note that, according to the proof of Theorem 14.2.4, $\lambda_{\min}(L_5L_5^*)$ and $\lambda_{\min}(M_8^*M_8)$ are both positive.

Let us now consider the function g_3. Taking (14.2.26)-(14.2.30) and Hypothesis 14.0.1(ii) into account, and applying (14.2.19) with $a=\sqrt{\kappa}|D^h_{*,i}u|$ and $b=|D^k_{*,j}u|$ ($i\le k$, $j=1,\dots,k+1$), it is easy to check that all the terms in the definition of the function g_3 are negligible (as t tends to 0^+) with respect to the terms in (14.4.36) and (14.4.37). Now, combining these two estimates, we get

$$\begin{aligned}
g^{(\varepsilon)}(t)\le{}&\{-\alpha^3+o_\alpha(\alpha^3)\}|D_{*,1}u(t)|^2+\{-\iota+o_\alpha(1)\}|D_{*,2}u(t)|^2\\
&+\{-2\alpha+o_\alpha(\alpha)+o_t(1)\}\kappa|D^2_{*,1}u(t)|^2\\
&+\{-2\iota+o_\alpha(1)+o_t(1)\}\kappa|D^2_{*,2}u(t)|^2\\
&+t^2\{-2\lambda_{\min}(L_5^*L_5)\alpha^{-\frac{4}{5}}+o_\alpha(\alpha^{-\frac{4}{5}})+o_t(1)\}|D^2_{*,3}u(t)|^2\\
&+\{-2\kappa+o_\alpha(1)+o_t(1)\}|D^3_{*,1}u(t)|^2\\
&+t\{-2\alpha^{-\frac{7}{16}}\kappa+o_\alpha(\alpha^{-\frac{7}{16}})+o_t(1)\}|D^3_{*,2}u(t)|^2\\
&+t^3\{-2\alpha^{-\frac{7}{8}}\kappa+o_\alpha(\alpha^{-\frac{7}{8}})+o_t(1)\}|D^3_{*,3}u(t)|^2\\
&+t^5\{-2\alpha^{-\frac{13}{12}}\lambda_{\min}(M_8^*M_8)+o_\alpha(\alpha^{-\frac{13}{12}})+o_t(1)\}|D^3_{*,4}u(t)|^2\\
&-\{2\alpha^{-\frac{4}{3}}+o_\alpha(\alpha^{-\frac{4}{3}})+o_t(1)\}\kappa|D^4_{*,1}u(t)|^2
\end{aligned}$$

$$-t^2\{2\alpha^{-\frac{8}{9}} + o_\alpha(\alpha^{-\frac{8}{9}}) + o_t(1)\}\kappa|D^4_{*,2}u(t)|^2$$
$$-t^4\{2\alpha^{-1} + o_\alpha(\alpha^{-1}) + o_t(1)\}\kappa|D^4_{*,3}u(t)|^2$$
$$-t^6\{2\alpha^{-\frac{9}{8}} + o_\alpha(\alpha^{-\frac{9}{8}})\}\kappa|D^4_{*,4}u(t)|^2. \qquad (14.4.38)$$

It is now clear that we can fix T_0 sufficiently close to 0 and α large enough, so as to make all the terms in (14.4.38) negative in $(0,T_0]\times\mathbb{R}^N$. ■

We are now able to prove (14.4.5).

Proposition 14.4.7 *For any* $0\le\alpha\le\beta\le 3$ *there exist two positive constants* $C=C(\alpha,\beta)$ *and* $\omega=\omega(\alpha,\beta)$ *such that* (14.4.5) *holds true.*

Proof. As it has been already mentioned, the proof is based on an interpolation argument and it is similar to that of Theorem 6.1.8. For the reader's convenience, we go into details.

As a first step, we apply (A.4.3) with $X_1 = X_2 = C_b(\mathbb{R}^N)$, $Y_1 = Y_2 = C^{3,1}_b(\mathbb{R}^N)$ and $\theta=\alpha$. Taking (14.4.6) into account, we get

$$\|T(t)\|_{L(\mathcal{C}^{3\alpha,\alpha}_b(\mathbb{R}^N))} \le C_1 e^{\omega_1 t}, \qquad t>0, \qquad (14.4.39)$$

for some positive constants C_1 and ω_1, independent of t. Next, applying (A.4.3) with $X_1 = C_b(\mathbb{R}^N)$, $X_2 = Y_1 = Y_2 = C^{3,1}_b(\mathbb{R}^N)$, $\theta=\alpha$ and, still taking (14.4.6) into account, we deduce that

$$\|T(t)\|_{L(\mathcal{C}^{3\alpha,\alpha}_b(\mathbb{R}^N),C^{3,1}_b(\mathbb{R}^N))} \le C_2 e^{\omega_2 t} t^{-\frac{3(1-\alpha)}{2}}, \qquad t>0, \qquad (14.4.40)$$

for some positive constants C_2 and ω_2, independent of t.

Finally, interpolating (14.4.39) and (14.4.40), with $\theta=(\beta-\alpha)/(1-\alpha)$, and taking (14.4.31) into account, we deduce (14.4.5) in its generality. ■

In order to prove Theorem 14.4.1, we show that for any $f\in C_b(\mathbb{R}^N)$ and any $\lambda>0$ the function

$$u(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)dt, \qquad x\in\mathbb{R}^N, \qquad (14.4.41)$$

is a distributional solution to the elliptic equation

$$\lambda u - \mathcal{A}u = f. \qquad (14.4.42)$$

Moreover, we prove that if $f\in\mathcal{C}^{\theta,\theta/3}_b(\mathbb{R}^N)$ for some $\theta\in(0,1)$, then $u\in\mathcal{C}^{2+\theta,(2+\theta)/3}_b(\mathbb{R}^N)$ and it satisfies (14.4.3). Finally, we show that the function u in (14.4.41) is the unique solution of the equation (14.4.42) among all the distributional solutions which are twice continuously differentiable with respect to the first r variables, and are bounded with their classical derivatives.

The following proposition provides some regularity properties of the functions belonging to $D(\widehat{A})$.

Proposition 14.4.8 *Let $u \in D(\widehat{A})$ be such that $\widehat{A}u \in \mathcal{C}_b^{\theta,\theta/3}(\mathbb{R}^N)$ for some $\theta \in [0,1)$. Then, $u \in \mathcal{C}_b^{2+\theta,(2+\theta)/3}(\mathbb{R}^N)$ and there exists a positive constant C_1, independent of u, such that*

$$\|u\|_{\mathcal{C}_b^{2+\theta,(2+\theta)/3}(\mathbb{R}^N)} \le C_1\big(\|u\|_\infty + \|\widehat{A}u\|_{\mathcal{C}_b^{\theta,\theta/3}(\mathbb{R}^N)}\big). \tag{14.4.43}$$

In particular, $D(\widehat{A})$ is continuously embedded in $\mathcal{C}_b^{2,2/3}(\mathbb{R}^N)$. Finally, if $f \in \mathcal{C}_b^{\theta,\theta/3}(\mathbb{R}^N)$ and u solves the equation $\lambda u - \widehat{A}u = f$, then

$$\|u\|_{\mathcal{C}_b^{2+\theta,(2+\theta)/3}(\mathbb{R}^N)} \le C_2\|f\|_{\mathcal{C}_b^{\theta,\theta/3}(\mathbb{R}^N)}, \tag{14.4.44}$$

where C_2 is independent of f and u.

Proof. In view of Corollary 14.4.4, to prove that $D(\widehat{A}) \subset \mathcal{C}_b^{2,2/3}(\mathbb{R}^N)$ with a continuous embedding, one can fix a suitable large λ_0 (i.e., λ_0 larger than the constant ω appearing in Proposition 14.4.7) and show that the function $R(\lambda_0, \widehat{A})f$ belongs to $(C_b^{\alpha,\alpha/3}(\mathbb{R}^N), C_b^{2+\alpha,(2+\alpha)/3}(\mathbb{R}^N))_{1-\alpha/2}$ for any $f \in C_b(\mathbb{R}^N)$. This can be done using the same techniques as in the proof of Theorem 6.2.2, which rely on the estimates (14.4.7). Similarly, writing $u = R(\lambda, \widehat{A})(\lambda u - \widehat{A}u)$, one can then easily show (14.4.43) and (14.4.44) (with $\lambda > \omega$, where ω is still given by Proposition 14.4.7). Hence, we skip the details.

To prove (14.4.44) (with $\lambda \in (0, \omega]$), it suffices to observe that, using the resolvent identity (A.3.2), one can easily check that if u satisfies the equation $\lambda u - \widehat{A}u = f$, then $\lambda_0 u - \widehat{A}u = g := f + (\lambda_0 - \lambda)u$ and $u = R(\lambda_0, A)g$.
The arguments needed to prove the first part of the proposition (see also the proof of (6.2.8)) show that

$$\|u\|_{C_b^{\theta,\theta/3}(\mathbb{R}^N)} \le K_1\|g\|_{C_b(\mathbb{R}^N)} \le C\left(\|f\|_\infty + |\lambda - \lambda_0|\|u\|_\infty\right) \le K_2\|f\|_\infty,$$

for some positive constants K_1 and K_2, independent of f. It follows that

$$\|g\|_{C_b^{\theta,\theta/3}(\mathbb{R}^N)} \le (K_2 + 1)\|f\|_{C_b^{\theta,\theta/3}(\mathbb{R}^N)}. \tag{14.4.45}$$

Hence, (14.4.44) (with $\lambda = \lambda_0$) and (14.4.45) imply that

$$\|u\|_{C_b^{2+\theta,(2+\theta)/3}(\mathbb{R}^N)} \le C_2\|g\|_{C_b^{\theta,\theta/3}(\mathbb{R}^N)} \le C_2(K_2 + 1)\|f\|_{C_b^{\theta,\theta/3}(\mathbb{R}^N)}$$

and we are done. ∎

In order to prove that the problem (14.4.42) admits actually a unique distributional solution which is twice continuously differentiable with respect to the first r variables, we need the following lemma.

Lemma 14.4.9 *Let $\mathcal{B}$ the first-order differential operator formally defined by*

$$\mathcal{B}u(x) = \langle Bx, Du(x)\rangle, \qquad x \in \mathbb{R}^N.$$

Then, for any $u \in C_b(\mathbb{R}^N)$ such that $\mathcal{B}u \in C_b(\mathbb{R}^N)$, there exists a sequence of smooth functions u_n such that $\mathcal{B}u_n \in C_b(\mathbb{R}^N)$ for any $n \in \mathbb{N}$ and it converges to $\mathcal{B}u$ locally uniformly.

Proof. Let φ be a smooth function compactly supported in $B(1)$, such that $0 \le \varphi(x) \le 1$ for any $x \in \mathbb{R}^N$ and $\|\varphi\|_{L^1(\mathbb{R}^N)} = 1$. For any $n \in \mathbb{N}$ we define the function $u_n : \mathbb{R}^N \to \mathbb{R}$ by setting

$$u_n(x) = (u \star \varphi_n)(x) := \int_{\mathbb{R}^N} u(x-y)\varphi_n(y)dy, \qquad x \in \mathbb{R}^N, \tag{14.4.46}$$

where $\varphi_n(x) = n^N\varphi(nx)$ for any $x \in \mathbb{R}^N$,

As it is easily seen, u_n converges to u as n tends to $+\infty$, locally uniformly in $\mathbb{R}^N$. Let us show that $\mathcal{B}u_n$ converges to $\mathcal{B}u$ locally uniformly in $\mathbb{R}^N$, as well. To compute $\mathcal{B}u_n$ we first assume that $u \in C^1_b(\mathbb{R}^N)$ and observe that, for any $i, j = 1, \ldots, N$, we have

$$\begin{aligned}
x_jD_iu_n(x) &= x_j\int_{\mathbb{R}^N}(D_iu)(x-y)\varphi_n(y)dy\\
&= \int_{\mathbb{R}^N}(x_j-y_j)(D_iu)(x-y)\varphi_n(y)dy + \int_{\mathbb{R}^N}(D_iu)(x-y)y_j\varphi_n(y)dy\\
&= \int_{\mathbb{R}^N}(x_j-y_j)(D_iu)(x-y)\varphi_n(y)dy + \int_{\mathbb{R}^N}u(x-y)D_i(y_j\varphi_n(y))dy\\
&= \int_{\mathbb{R}^N}(x_j-y_j)(D_iu)(x-y)\varphi_n(y)dy + \delta_{ij}\int_{\mathbb{R}^N}u(x-y)\varphi_n(y)dy\\
&\quad + \int_{\mathbb{R}^N}u(x-y)y_jD_i\varphi_n(y)dy.
\end{aligned}$$

Therefore,

$$\mathcal{B}u_n = \mathcal{B}u \star \varphi_n + \mathrm{Tr}(B)u_n + u \star \mathcal{B}\varphi_n. \tag{14.4.47}$$

Now, fix $u \in C_b(\mathbb{R}^N)$ and let $\{u_m\} \subset C^1_b(\mathbb{R}^N)$ be a sequence of smooth functions converging to u locally uniformly. Moreover, let $u_n^m = u_m \star \varphi_n$. As we can easily see, u_n^m and $\mathcal{B}u_n^m$ converge, respectively, to u_n and $\mathcal{B}u_n$, locally uniformly in $\mathbb{R}^N$, as m tends to $+\infty$. Moreover, $\mathcal{B}u^m$ converges to $\mathcal{B}u$ in the sense of distributions. This implies that $\mathcal{B}u^m \star \varphi_n$ converges to $\mathcal{B}u \star \varphi_n$ pointwise in $\mathbb{R}^N$ as m tends to $+\infty$. Therefore, writing (14.4.47) with u_n replaced with u_n^m and letting m go to $+\infty$, we see that (14.4.47) holds true also in the case when $u \in C_b(\mathbb{R}^N)$.

Now, we are almost done. Indeed, since $\mathcal{B}u \in C_b(\mathbb{R}^N)$, then $\mathcal{B}u \star \varphi_n$ converges locally uniformly to $\mathcal{B}u$ as n tends to $+\infty$. Finally,

$$\begin{aligned}(u \star \mathcal{B}\varphi_n)(x) &= \sum_{i,j=1}^N b_{ij} \int_{\mathbb{R}^N} u(x-y) y_j D_i\varphi_n(y)dy \\ &= \sum_{i,j=1}^N b_{ij} \int_{\mathbb{R}^N} u(x-z/n) z_j D_i\varphi(z)dz\end{aligned}$$

and the last side of the previous chain of equalities tends to

$$u(x) \sum_{i,j=1}^N b_{ij} \int_{\mathbb{R}^N} z_i D_j\varphi(z)dz = -u(x)\mathrm{Tr}(B),$$

as n tends to $+\infty$, locally uniformly in $\mathbb{R}^N$. Therefore, $\mathcal{B}u_n$ converges locally uniformly to $\mathcal{B}u$, as n tends to $+\infty$. ■

We are now able to prove Theorem 14.4.1.

Proof of Theorem 14.4.1. (*Existence*). The existence part follows immediately from Propositions 14.3.5 and 14.4.8. Indeed, Proposition 14.3.5 shows that the function $u = R(\lambda, \widehat{A})f$ is a distributional solution to the equation (14.4.42), whereas Proposition 14.4.8 ensures that u enjoys all the regularity properties claimed in the statement of the theorem.

(*Uniqueness*). Here, we prove that $v \equiv 0$ is the unique distributional solution to the equation $\lambda v - \mathcal{A}v = 0$ which is bounded and continuous in $\mathbb{R}^N$ and admits classical derivatives $D_{ij}v \in C_b(\mathbb{R}^N)$ for any $i,j = 1,\ldots,r$. This is equivalent to prove that any solution v to the previous equation, with the claimed regularity, belongs to $D(\widehat{A})$. For this purpose, let $\{\hat{v}_m\}$ be the sequence defined by $\hat{v}_m = T(1/m)v$ for any $m \in \mathbb{N}$. By Theorem 14.2.7, any function $\hat{v}_m$ belongs to $C_b^2(\mathbb{R}^N)$ and $\hat{v}_m$ converges locally uniformly to v as m tends to $+\infty$. Moreover, $\|\hat{v}_m\|_\infty \le \|v\|_\infty$ for any $m \in \mathbb{N}$. We now show that

$$\mathcal{A}\hat{v}_m = T(1/m)\mathcal{A}v, \qquad m \in \mathbb{N}. \tag{14.4.48}$$

This will imply that $\mathcal{A}\hat{v}_m$ converges locally uniformly to $\mathcal{A}v$ and $\|\mathcal{A}\hat{v}_m\|_\infty \le \|\mathcal{A}v\|_\infty$ for any $m \in \mathbb{N}$. Hence, by Proposition 14.3.5, $v \in D(A)$ and, consequently, $v \equiv 0$.

So, let us prove (14.4.48). For this purpose, we approximate v by convolution, by the sequence of smooth functions $\{v_n\} \in C_b^2(\mathbb{R}^N)$ defined as in (14.4.46). As we can easily see, v_n converges locally uniformly to v as n tends to $+\infty$ and D_iv_n, $D_{ij}v_n$ converge locally uniformly, respectively, to D_iv and $D_{ij}v$ for any $1 \le i,j \le r$. Moreover, by Lemma 14.4.9, $\mathcal{B}v_n$ converges to $\mathcal{B}v$ locally uniformly in $\mathbb{R}^N$. Therefore, $\mathcal{A}v_n$ converges to $\mathcal{A}v$ locally uniformly in $\mathbb{R}^N$, as well. Now, the formula (14.4.48) follows observing that

$T(1/m)\mathcal{A}v_n = \mathcal{A}T(1/m)v_n$ for any $m, n \in \mathbb{N}$ (see Lemma 14.3.4) and letting n tend to $+\infty$. Indeed, according to Proposition 14.3.1, $T(1/m)v_n$ converges to $\hat{v}_m$ in $C^2(K)$ and $T(1/m)\mathcal{A}v_n$ converges to $T(1/m)\mathcal{A}v$ in $C(K)$ for any compact set $K \subset \mathbb{R}^N$. This finishes the proof. ∎

The results of Theorem 14.4.1 can be improved if f is smoother. In such a case it can be proved that the function $u = R(\lambda, \widehat{A})f$ is a classical solution to the equation (14.4.1), in the sense that all the derivatives involved in the definition of $\mathcal{A}u$ exists in the classical sense and they are bounded and continuous in $\mathbb{R}^N$. Since the proof of the next corollary is similar to, and even simpler than the proof of Corollary 14.4.11, we do not go into details.

Corollary 14.4.10 *Suppose that $f \in C_b^\theta(\mathbb{R}^N)$ for some $\theta > 1/3$. Then, the function $u = R(\lambda, A)f$ belongs to $C_b^{\theta+2/3}(\mathbb{R}^N)$ and there exists a positive constant C, independent of f, such that*

$$\|u\|_{C_b^{\theta+2/3}(\mathbb{R}^N)} \le C\|f\|_{C_b^\theta(\mathbb{R}^N)}.$$

Now, we consider the parabolic nonhomogeneous Cauchy problem (14.4.2). We are going to show that its unique distributional solution, with the regularity properties claimed in Theorem 14.4.1, is the function u given by the variation-of-constants formula

$$u(t,x) = (T(t)f)(x) + \int_0^t (T(t-s)g(s,\cdot))(x)ds, \qquad t \in [0,T], \quad x \in \mathbb{R}^N. \tag{14.4.49}$$

As a first step we observe that the convolution term in (14.4.49) is well defined. Of course, it suffices to check that for any $x \in \mathbb{R}^N$, the function $w : [0,T] \times [0,T] \to \mathbb{R}$ defined by $w(r,s) = (T(r)f(s))(x)$ for any $r, s \in [0,T]$ is continuous. For this purpose we observe that, by Theorem 14.2.7, the function $w(\cdot, s)$ is continuous for any $s \in [0,T]$, and by Proposition 14.3.1, $w(r,\cdot)$ is continuous in $[0,T]$, uniformly with respect to $r \in [0,T]$.

We are now in a position to prove Theorem 14.4.2.

Proof of Theorem 14.4.2. (*Existence*). Of course, according to Theorem 14.2.7, we can restrict ourselves to proving the assertion in the case when $f \equiv 0$. For this purpose, we begin by proving that the function

$$v(t,x) = \int_0^t (T(t-s)g(s,\cdot))(x)ds, \qquad t \in [0,T], \ x \in \mathbb{R}^N \tag{14.4.50}$$

is a distributional solution to (14.4.2) (with $f \equiv 0$). For this purpose, we introduce a sequence $\{g_n\} \in C_b^{1,2}([0,T] \times \mathbb{R}^N)$ converging locally uniformly in $[0,T] \times \mathbb{R}^N$ to g, and we denote by v_n the function defined by (14.4.50) with g being replaced with g_n. Using the estimate (14.2.34) with $h = k = 2$ we can show that v_n is a classical solution to the problem (14.4.2), where we take $f \equiv 0$ and we replace g with g_n (see also the proof of the forthcoming Corollary 14.4.11). Moreover, v_n converges to v uniformly in $[0,T] \times K$ for any compact set $K \subset \mathbb{R}^N$. Indeed,

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times K} |v_n(t,x) - v(t,x)| \\
&= \sup_{(t,x)\in[0,T]\times K} \left| \int_0^t (T(s)(g_n(t-s,\cdot) - g(t-s,\cdot)))(x) ds \right| \\
&\leq \sup_{x\in K} \int_0^T \big(\sup_{r\in[0,T]} |g_n(r,\cdot) - g(r,\cdot)| \big)(x) ds
\end{aligned}$$

(recall that $\{T(t)\}$ is an order preserving semigroup, see Theorem 14.2.7). Since $\sup_{t\in[0,T]} |g_n(r,\cdot) - g(r,\cdot)|$ converges to 0 locally uniformly in $\mathbb{R}^N$, as n tends to $+\infty$, Proposition 14.3.1 implies that the last side of the previous chain of inequalities vanishes as n tends to $+\infty$. Hence, v_n tends to v uniformly in $[0,T] \times K$.

Now, we observe that, for any smooth function $\varphi \in C_c^\infty((0,T) \times \mathbb{R}^N)$,

$$\begin{aligned}
\int_{(0,T)\times\mathbb{R}^N} g_n \varphi \, dt \, dx &= \int_{(0,T)\times\mathbb{R}^N} (D_t v_n - \mathcal{A} v_n) \varphi \, dt \, dx \\
&= \int_{(0,T)\times\mathbb{R}^N} v_n (-D_t \varphi - \mathcal{A}^* \varphi) \, dt \, dx,
\end{aligned}$$

where $\mathcal{A}^*$ is given by (14.3.9). Letting n go to $+\infty$ we deduce that v is a distributional solution of (14.4.2), with $f \equiv 0$.

The regularity properties of u and the estimate (14.4.4) can be proved arguing as in the proof of Proposition 6.2.6, taking Proposition 14.4.7 into account.

(*Uniqueness*). Let us now prove that the function u in (14.4.49) is the unique distributional solution to the problem (14.4.2) which is twice continuously differentiable in $[0,T] \times \mathbb{R}^N$ with respect to the space variables x_j ($j = 1, \ldots, r$). We adapt to this situation the proof of Theorem 14.4.1. For this purpose, let w be a solution to the Cauchy problem (14.4.2) with $f \equiv 0$ and $g \equiv 0$, such that $w, D_i w, D_{ij} w \in C_b([0,T] \times \mathbb{R}^N)$ for any $i,j = 1, \ldots, r$. We extend w by continuity to $\mathbb{R} \times \mathbb{R}^N$ by setting $\overline{w}(t,\cdot) = 0$ for any $t < 0$ and $\overline{w}(t,\cdot) = w(T,\cdot)$ for any $t > T$. The so extended function is as smooth as w is, i.e., $D_i \overline{w}$ and $D_{ij} \overline{w}$ are continuous in $\mathbb{R} \times \mathbb{R}^N$ for any $i,j \leq r$. Then, we regularize $\overline{w}$ by convolution by setting

$$w_n(t,x) = (\overline{w} \star (\varrho_n \varphi_n))(t,x) := \int_{\mathbb{R}\times\mathbb{R}^N} \overline{w}(t-s, x-y) \varrho_n(s) \varphi_n(y) \, ds \, dy, \tag{14.4.51}$$

with $\varrho_n(s) = n\varrho(ns)$, $\varphi_n(x) = n^N \varphi(nx)$, where $\varrho \in C_c^\infty(\mathbb{R})$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$ are such that $\chi_{(-1/2,1/2)} \leq \varrho \leq \chi_{(-1,1)}$, $\chi_{B(1/2)} \leq \varphi \leq \chi_{B(1)}$, and $\|\varrho\|_{L^1(\mathbb{R})} = \|\varphi\|_{L^1(\mathbb{R}^N)} = 1$.

Let us show that $D_tw_n-\mathcal{A}w_n$ converges to 0 locally uniformly in $(0,T)\times\mathbb{R}^N$. For this purpose, we observe that the smoothness of w implies that $\mathcal{A}w\in C([0,T]\times\mathbb{R}^N)$ and that $\mathcal{A}w_n$ converges to $\mathcal{A}w$ locally uniformly in $(0,T)\times\mathbb{R}^N$. Here, $D_tw-\mathcal{B}w$ is meant in the sense of distributions. Therefore, we can restrict ourselves to showing that $(D_t-\mathcal{B})w_n$ converges to $(D_t-\mathcal{B})w\in C([0,T]\times\mathbb{R}^N)$ locally uniformly in $(0,T)\times\mathbb{R}^N$. To show such a property, we first prove that

$$(D_t-\mathcal{B})w_n=\overline{(D_tw-\mathcal{B}w)}\star\varrho_n\varphi_n+\mathrm{Tr}(B)w_n+\overline{w}\star(\varrho_n\mathcal{B}(\varphi_n)),\quad(14.4.52)$$

in $(1/n,T-1/n)\times\mathbb{R}^N$, where $\overline{D_tw-\mathcal{B}w}$ denotes any continuous extension of $D_tw-\mathcal{B}w$ to the whole of $\mathbb{R}\times\mathbb{R}^N$.

An integration by parts shows that (14.4.52) holds true in the case when $\overline{w}\in C_b^{1,2}([0,T]\times\mathbb{R}^N)$. In the general case, we approximate $\overline{w}$ by a sequence of smooth functions $\{w_m\}\subset C_b^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ converging to $\overline{w}$ locally uniformly in $\mathbb{R}\times\mathbb{R}^N$, and we define the function w_n^m $(m,n\in\mathbb{N})$ accordingly to (14.4.51), with $\overline{w}$ being replaced with w_m. As it is immediately seen,

$$(D_t-\mathcal{B})w_n^m=(D_tw^m-\mathcal{B}w^m)\star\varrho_n\varphi_n+\mathrm{Tr}(B)w_n^m+w^m\star(\varrho_n\mathcal{B}\varphi_n),$$

in $[0,T]\times\mathbb{R}^N$. Since w^m converges to $\overline{w}$ locally uniformly, then the function $\mathrm{Tr}(B)w_n^m+w^m\star(\varrho_n\mathcal{B}\varphi_n)$ converges to $\mathrm{Tr}(B)w_n+\overline{w}\star(\varrho_n\mathcal{B}\varphi_n)$, locally uniformly in $[0,T]\times\mathbb{R}^N$. As far as the term $(D_tw^m-\mathcal{B}w^m)\star\varrho_n\varphi_n$ is concerned, we observe that

$$\begin{aligned}&((D_tw^m-\mathcal{B}w^m)\star\varrho_n\varphi_n)(t,x)\\&=\int_{[t-1/n,t+1/n]\times(x+B(1/n))}(D_tw^m-\mathcal{B}w^m)(s,y)\varrho_n(t-s)\varphi_n(x-y)\,ds\,dy,\end{aligned}\quad(14.4.53)$$

for any $t>0$ and any $x\in\mathbb{R}^N$. Suppose that $t\in(1/n,T-1/n)$. Then the function $\varrho_n(t-\cdot)\varphi_n(x-\cdot)$ is compactly supported in $(0,T)\times\mathbb{R}^N$. Therefore, since $D_tw^m-\mathcal{B}w^m$ converges to $D_tw-\mathcal{B}w\in C([0,T]\times\mathbb{R}^N)$ in the sense of distributions, letting m go to $+\infty$ in (14.4.53), we deduce that

$$\begin{aligned}&\lim_{m\to+\infty}((D_tw^m-\mathcal{B}w^m)\star\varrho_n\varphi_n)(t,x)\\&=\int_{[t-1/n,t+1/n]\times(x+B(1/n))}(D_tw-\mathcal{B}w)(s,y)\varrho_n(t-s)\varphi_n(x-y)\,ds\,dy\\&=\int_{\mathbb{R}\times\mathbb{R}^N}(\overline{D_tw-\mathcal{B}w})(t-s,x-y)\varrho_n(s)\varphi_n(y)\,ds\,dy,\end{aligned}$$

for any $t\in(1/n,T-1/n)$ and any $x\in\mathbb{R}^N$. The formula (14.4.52) follows.

Now, arguing as in the proof of Lemma 14.4.9, from (14.4.52) we can conclude that $D_tw_n-\mathcal{B}w_n$ converges locally uniformly in $(0,T)\times\mathbb{R}^N$ to $D_tw-\mathcal{B}w$.

Therefore, $D_t w_n - \mathcal{A} w_n$ converges to 0 locally uniformly in $(0,T)\times\mathbb{R}^N$, as n tends to $+\infty$.

To conclude the proof, we observe that, for any $n\in\mathbb{N}$ and any $t_0\in(0,T)$, the function $z_n = w_n(\cdot+t_0,\cdot)$ is a classical solution to the Cauchy problem

$$\begin{cases} D_t z_n(t,x) = \mathcal{A} z_n(t,x) + f_n(t+t_0,x), \; t\in[0,T-t_0], \; x\in\mathbb{R}^N, \\ z_n(0,x) = w_n(t_0,x), \qquad\qquad\qquad\qquad\qquad\qquad x\in\mathbb{R}^N, \end{cases} \tag{14.4.54}$$

where $f_n := D_t z_n - \mathcal{A} z_n$. Since $f_n \in C_b^{1,2}([0,T]\times\mathbb{R}^N)$, the function $h_n : [0,T-t_0]\times\mathbb{R}^N \to \mathbb{R}$ defined by

$$h_n(t,x) = (T(t)w_n(t_0,\cdot))(x) + \int_0^t (T(t-s)f_n(t+t_0-s,\cdot))(x)ds,$$

for any $t\in[0,T-t_0]$ and any $x\in\mathbb{R}^N$, is a classical solution to the Cauchy problem (14.4.54) (see also the proof of Corollary 14.4.11). The maximum principle (see Theorem 14.2.6) implies that $h_n = z_n$. Now, since z_n, $f_n(\cdot+t_0,\cdot)$ converge, locally uniformly in $[0,T-t_0]\times\mathbb{R}^N$ to $w(\cdot+t_0,\cdot)$ and 0, respectively, taking Proposition 14.3.1 into account, we get

$$w(t+t_0,x) = (T(t)w(t_0,\cdot))(x), \qquad t\in[0,T-t_0], \quad x\in\mathbb{R}^N,$$

for any $t_0\in(0,T)$. Letting t_0 go to 0, and recalling that $u(0,\cdot)=0$, we deduce that $w\equiv 0$. This finishes the proof. ∎

To conclude this section we show that the results in Theorem 14.4.2 can be improved if f and g are smoother. In such a case the function u in (14.4.49) is a classical solution to the Cauchy problem (14.4.2), in the sense that the time derivative of u as well as all the space derivatives involved in the definition of $\mathcal{A}u$ exist in the classical sense and they are bounded and continuous in $[0,T]\times\mathbb{R}^N$.

Corollary 14.4.11 *Suppose that the assumptions of Theorem* 14.4.2 *are satisfied. Further, suppose that $f\in C_b^{\theta+2/3}(\mathbb{R}^N)$, $g(t,\cdot)\in C_b^{\theta}(\mathbb{R}^N)$ for some $\theta>1/3$ and any $t\in[0,T]$, with*

$$\sup_{t\in[0,T]} \|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)} < +\infty.$$

Then, the function u in (14.4.49) *is once continuously differentiable in $[0,T]\times\mathbb{R}^N$ with respect to the time and space variables. Moreover, for any $t\in[0,T]$, $u(t,\cdot)$ belongs to $C_b^{\theta+2/3}(\mathbb{R}^N)$ and there exists a positive constant C, independent of f and g, such that*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{C_b^{\theta+2/3}(\mathbb{R}^N)} \le C\Big(\|f\|_{C_b^{\theta+2/3}(\mathbb{R}^N)} + \sup_{t\in[0,T]} \|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)}\Big). \tag{14.4.55}$$

Proof. As in the proof of Theorem 14.4.2, we can restrict ourselves to dealing with the function v defined in (14.4.50). Using an interpolation argument similar to that used in the proof of Proposition 14.4.7, from (14.2.1)-(14.2.3) and (14.2.34) (with $h=k$) we deduce that, for any $0\le\alpha\le\beta\le 3$ and any $\omega>0$, there exist two positive constants C and ω such that

$$\|T(t)\|_{L(C_b^\alpha(\mathbb{R}^N),C_b^\beta(\mathbb{R}^N))}\le Ce^{\omega t}t^{-\frac{3(\beta-\alpha)}{2}},\qquad t>0.\tag{14.4.56}$$

Now, arguing as in the proof of Proposition 6.2.6, we can prove that $v(t,\cdot)\in C_b^{\theta+2/3}(\mathbb{R}^N)$ for any $t>0$ and

$$\sup_{t\in[0,T]}\|v(t,\cdot)\|_{C_b^{\theta+2/3}(\mathbb{R}^N)}\le C_1\sup_{t\in[0,T]}\|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)},\tag{14.4.57}$$

for some positive constant C_1, independent of g. Therefore, (14.4.55) follows from (14.2.34) and (14.4.57).

To conclude the proof, let us show that v is continuously differentiable with respect to time in $(0,T]\times\mathbb{R}^N$ and $D_tv=\mathcal{A}v+g$. This can be done arguing as in the proof of Proposition 6.2.5, with minor changes. For this reason we limit ourselves to sketching the proof, pointing out the main differences.

Using (14.4.5) with $\alpha=\theta/3$ and $\beta=2/3+\theta/6$, we deduce that

$$\|D_{ij}T(t-s+r)g(s,\cdot)\|_\infty\le c(t-s)\sup_{t\in[0,T]}\|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)},\tag{14.4.58}$$

for any $i,j\le r$ and some positive function $c\in L^1(0,T)$. Now, (14.4.56) (with $(\alpha,\beta)=(\theta,1)$) and (14.4.58) imply that, for any compact set $K\subset\mathbb{R}^N$, there exists a function $\hat{c}_K\in L^1(0,T)$ such that

$$\|\mathcal{A}T(t-s+r)g(s,\cdot)\|_{C(K)}\le\hat{c}_K(t-s)\sup_{t\in[0,T]}\|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)}.\tag{14.4.59}$$

Hence, the same arguments as in the proof of Proposition 6.2.5 show that v is differentiable from the right, with respect to the time variable, in $(0,T)\times\mathbb{R}^N$ and $D_t^+v=\mathcal{A}v+g$ in $(0,T)\times\mathbb{R}^N$.

To conclude that v is continuously differentiable with respect to the time variable in $[0,T]\times\mathbb{R}^N$, we just need to show that $\mathcal{A}v$ is continuous in $[0,T]\times\mathbb{R}^N$. Here, the proof differs a bit from that of Proposition 6.2.5. Hence, we go into the details. Fix $t_0,t\in[0,T]$. From Proposition A.4.4 and the estimate (14.4.4) we get

$$\begin{aligned}
&\|v(t,\cdot,y)-v(t_0,\cdot,y)\|_{C^2(\overline{B}(R))}\\
&\le C_R\|v(t,\cdot,y)-v(t_0,\cdot,y)\|_{C(\overline{B}(R))}^{\frac{\theta}{2+\theta}}\|v(t,\cdot,y)-v(t_0,\cdot,y)\|_{C_b^{2+\theta}(\overline{B}(R))}^{\frac{2}{(2+\theta)}}\\
&\le 2C_R'\|v(t,\cdot,y)-v(t_0,\cdot,y)\|_{C(\overline{B}(R))}^{\frac{\theta}{2+\theta}},
\end{aligned}$$

for any $R>0$ and any $B(R)\subset\mathbb{R}^r$. Since $v\in C_b([0,T]\times\mathbb{R}^N)$, we deduce that the functions $t\mapsto D_iv(t,x,y)$ and $t\mapsto D_{ij}v(t,x,y)$ $(i,j=1,\dots,r)$ are continuous in $[0,T]$, uniformly with respect to (x,y) in bounded subsets of $\mathbb{R}^N$. Therefore, $D_i v, D_{ij} v \in C_b([0,T]\times\mathbb{R}^N)$. Similarly, taking the estimate (14.4.57) into account, one can show that $D_i v \in C_b([0,T]\times\mathbb{R}^N)$ for any $j > r$. It follows that $\mathcal{A}v \in C([0,T]\times\mathbb{R}^N)$ and we are done. ■

Part IV

Appendices

Appendix A

Basic notions of functional analysis in Banach spaces

In this appendix we collect a few basic results on linear operators, and on elementary spectral theory that we use in this book. For more details and for the proofs of the results that we present, we refer the reader mainly to [81, 147].

A.1 Bounded, compact and closed linear operators

Let X and Y be two Banach spaces. We denote by $L(X, Y)$ the vector space of linear and bounded operators $T : X \to Y$. We endow it with the norm

$$\|T\|_{L(X,Y)} = \sup_{\substack{x \in X \\ \|x\|=1}} \|Tx\|_Y = \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X}. \tag{A.1.1}$$

If $Y = X$, we write $L(X)$ instead of $L(X, X)$.

The norm in (A.1.1) makes $L(X, Y)$ a Banach space.

An operator $T \in L(X, Y)$ is called *compact* if it maps the closed ball $B(1)$ of X into a relatively compact set in Y.

Proposition A.1.1 ([147], Chapter 10, Section 2) *Let X, Y, Z be three Banach spaces. Then, the following properties are met:*

(i) the set of all the compact operators from X to Y is a closed subspace of $L(X, Y)$;

(ii) for any compact operator $L : X \to Y$ and any bounded operator $M \in B(Y, Z)$ (resp. $M \in B(Z, X)$), the operator $M \circ L$ (resp. $L \circ M$) is a compact operator.

A very useful criterion for the convergence of a given sequence of bounded linear operators is given in the following proposition.

Proposition A.1.2 *Let $\{M_n\} \in L(X)$ be a sequence of bounded operators with* $\sup_{n\in\mathbb{N}} \|M_n\|_{L(X)} < +\infty$ *and such that $M_n x$ converges in X, as n tends* *to $+\infty$, for any x in a dense subspace of X. Then, M_n converges pointwise in X to a bounded linear operator M which satisfies $\|M\|_{L(X)} \le C$.*

We now introduce another class of linear operators which we use in this book. If $D(A)$ is a vector subspace of X and $A : D(A) \subset X \to Y$ is a linear operator, we say that A is *closed* if its graph

$$\mathcal{G}_A = \{(x, y) \in X \times Y : x \in D(A),\ y = Ax\}$$

is a closed subset of $X \times Y$. As it is easily seen, A is closed if and only if, for any sequence $\{x_n\} \subset D(A)$ such that x_n and Ax_n converge, respectively, to some elements $x \in X$ and $y \in Y$, as n tends to $+\infty$, then $x \in D(A)$ and $y = Ax$.

In general, a closed operator is not continuous in $(X, \|\cdot\|)$. It turns out to be bounded if we endow $D(A)$ with the *graph norm*

$$\|x\|_{D(A)} = \|x\|_X + \|Ax\|_Y, \qquad x \in D(A). \tag{A.1.2}$$

Note that $D(A)$ is a Banach space when it is endowed with the graph norm.

An operator $A : D(A) \subset X \to Y$ is said to be *closable* if there exists a (closed) operator $\overline{A}$ whose graph coincides with the closure of $\mathcal{G}_A$. The operator $\overline{A}$ is called the *closure* of A. In such a case any $x \in D(\overline{A})$ is the limit of a sequence $\{x_n\} \subset D(A)$ which is a Cauchy sequence with respect to the graph norm (A.1.2). Moreover, $\overline{A}x := \lim_{n\to+\infty} Ax_n$.

Equivalently, A is closable if, for any sequence $\{x_n\} \in D(A)$ converging to 0 in X and such that Ax_n converges to some $y \in Y$ as n tends to $+\infty$, then $y = 0$.

Now, let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_X$.

For any linear operator $A : D(A) \subset X \to X$ with dense domain, the *adjoint* A^* of A is the operator $A^* : D(A^*) \subset X \to X$ defined as follows:

$$D(A^*) = \{x \in X : \exists y \in X \text{ s.t. } \langle Az, x\rangle_X = \langle z, y\rangle_X,\ \ \forall z \in D(A)\}, \qquad A^*x = y.$$

The operator $A : D(A) \subset X \to X$ is said to be *self-adjoint* if $D(A) = D(A^*)$ and $A = A^*$.

Finally, $A : D(A) \subset X \to X$ is nonpositive if $\langle Ax, x\rangle_X \le 0$ for any $x \in D(A)$.

A.2 Vector valued Riemann integral

In this section we define the Riemann integral for vector-valued functions. For more details and for the proof of the results that we present here, we refer the reader to [46, Chapter 2], [48, Chapter 3] and [72, Chapter 3].

Definition A.2.1 *A bounded function* $f : [a, b] \mapsto X$ $(-\infty < a < b < +\infty)$ *is said to be integrable on* $[a, b]$ *if there exists* $x \in X$ *with the following property: for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for any partition* $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_n = b\}$ *of* $[a, b]$*, with* $\max_{i=1,\ldots,n}(t_i - t_{i-1}) < \delta$*, and for any choice of the points* $\xi_i \in [t_{i-1}, t_i]$*, we have*

$$\left\| x - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right\| < \varepsilon.$$

In this case we define

$$\int_a^b f(t)dt = x.$$

Arguing as in the real-valued case, one can easily check that the set of functions that are integrable in $[a, b]$ is a vector space and that the integral over $[a, b]$ is a linear operator in this vector space. In particular, if $f : [a, b] \to X$ is continuous, then it is integrable. Moreover, if f is integrable in $[a, b]$, then the map $t \mapsto \|f(t)\|_X$ is integrable in $[a, b]$ as well. Finally, if f is integrable in $[a, b]$, then it is integrable in any $[c, d] \subset [a, b]$ and

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^d f(t)dt,$$

for any $c \in (a, b)$.

As in the real-valued case, the definition of Riemann integral can be easily extended to the case of unbounded intervals or unbounded functions.

Definition A.2.2 *Let* $I \subset \mathbb{R}$ *be an interval with endpoints* a *and* b $(-\infty \le a < b \le +\infty)$ *with* a *and* b *not necessarily in* I*. Moreover, let* $f : I \to X$ *be Riemann integrable in* $[c, d]$ *for any* $a < c < d < b$*. We say that* f *admits an improper integral in* I *if, for any* $t_0 \in I$*, the limits*

$$\lim_{c \to a^+} \int_c^{t_0} f(t)dt, \qquad \lim_{d \to b^-} \int_{t_0}^d f(t)dt$$

exist in X*. In this case we set*

$$\int_I f(x)dx = \lim_{c \to a^+} \int_c^{t_0} f(t)dt + \lim_{d \to b^-} \int_{t_0}^d f(t)dt.$$

Note that the previous definition is independent of t_0.

We now recall the definition of the integral of vector-valued functions of a complex variable, along a smooth curve γ.

Definition A.2.3 *Let Ω be an open subset of $\mathbb{C}$, $f : \Omega \to X$ be a continuous function and $\gamma : [a, b] \to \Omega$ be a piecewise C^1-curve. The integral of f along γ is defined as follows:*

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

As in the case of vector-valued functions defined in a real interval, we can define the *improper complex integrals* in an obvious way.

Definition A.2.4 *Let $\Omega \subset \mathbb{C}$ be a (possibly) unbounded open set. Moreover, let $I = (a, b)$ be a (possibly unbounded) interval and $\gamma : I \to \mathbb{C}$ be a (piecewise) C^1 curve in Ω. We say that f admits an improper integral along γ if for any $t_0 \in (a, b)$ the limits*

$$\lim_{s\to a^+} \int_s^{t_0} f(\gamma(\tau))\gamma'(\tau)d\tau \qquad \text{and} \qquad \lim_{s\to b^-} \int_{t_0}^{b} f(\gamma(\tau))\gamma'(\tau)d\tau$$

exist in X. In such a case, we set

$$\int_\gamma f(z)dz = \lim_{s\to a^+} \int_s^{t_0} f(\gamma(\tau))\gamma'(\tau)d\tau + \lim_{s\to b^-} \int_{t_0}^{s} f(\gamma(\tau))\gamma'(\tau)d\tau.$$

Note that the definition of the improper integral is independent of the choice of t_0. Moreover, when I is bounded and the integral of f along γ exists, then f admits an improper integral along γ and the two integrals coincide.

A.3 Spectrum and resolvent

In this section, we recall the notions of resolvent and spectrum of a linear operator.

Definition A.3.1 *Let $A : D(A) \subset X \to X$ be a linear operator. The* resolvent set *$\rho(A)$ and the* spectrum *$\sigma(A)$ of A are defined by*

$$\rho(A) = \{\lambda \in \mathbb{C} : \exists\, (\lambda I - A)^{-1} \in L(X)\}, \qquad \sigma(A) = \mathbb{C} \setminus \rho(A). \qquad \text{(A.3.1)}$$

The complex numbers $\lambda \in \sigma(A)$ such that $\lambda I - A$ is not injective are called the eigenvalues of A, and the elements $x \in D(A)$ such that $x \neq 0$ and $Ax = \lambda x$ are called the eigenvectors (or eigenfunctions, when X is a function space) of A relative to the eigenvalue λ. The set $\sigma_p(A)$ whose elements are the eigenvalues of A is called the point spectrum of A.

If $\lambda \in \rho(A)$, the operator $R(\lambda, A) := (\lambda I - A)^{-1}$ is called the resolvent operator.

Let us recall some simple properties of resolvent and spectrum. First of all, since the inverse of a bounded operator is a closed operator, it is clear that, if $\rho(A) \neq \varnothing$, then A is closed.

The family of operators $\{R(\lambda, A) : \lambda \in \rho(A)\}$ satisfies the so-called *resolvent identity*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \qquad \lambda, \mu \in \rho(A). \tag{A.3.2}$$

Such a formula characterizes the resolvent operators, as specified in the following proposition.

Proposition A.3.2 ([147], Section 8.4) *Let $\Omega \subset \mathbb{C}$ be an open set, and let $\{F(\lambda) : \lambda \in \Omega\} \subset L(X)$ be a family of linear operators verifying the resolvent identity*

$$F(\lambda) - F(\mu) = (\mu - \lambda)F(\lambda)F(\mu), \quad \lambda, \mu \in \Omega.$$

If the operator $F(\lambda_0)$ is invertible, for some $\lambda_0 \in \Omega$, then there exists a closed linear operator $A : D(A) \subset X \to X$ such that $\rho(A)$ contains Ω, and $R(\lambda, A) = F(\lambda)$ for any $\lambda \in \Omega$.

As the following proposition shows, the spectrum of an operator A is always closed in $\mathbb{C}$.

Proposition A.3.3 *Let λ_0 be in $\rho(A)$. Then, $|\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}_{L(X)}$ implies that λ belongs to $\rho(A)$ and the equality*

$$R(\lambda, A) = R(\lambda_0, A)(I + (\lambda - \lambda_0)R(\lambda_0, A))^{-1} \tag{A.3.3}$$

holds. As a consequence, $\rho(A)$ is open and $\sigma(A)$ is closed.

Further properties of the resolvent operator are listed in the next proposition.

Proposition A.3.4 *The function $R(\cdot, A)$ is holomorphic in $\rho(A)$. Moreover, the domain of analyticity of the function $\lambda \mapsto R(\lambda, A)$ is $\rho(A)$ and the estimate*

$$\|R(\lambda, A)\|_{L(X)} \geq \frac{1}{\mathrm{dist}(\lambda, \sigma(A))}, \qquad \lambda \in \rho(A) \tag{A.3.4}$$

holds.

As a consequence of Proposition A.3.4 we obtain the following Laurent expansion of $R(\cdot, A)$ in a neighborhood of an isolated point of $\sigma(A)$.

Proposition A.3.5 ([81], Chapter 3, Section 6.5) *Suppose that $\lambda \in \mathbb{C}$ is an isolated point in $\sigma(A)$. Then, there exists $r > 0$ such that*

$$R(\mu, A) = \sum_{n=0}^{+\infty} (\mu - \lambda)^n S^{n+1} + P + \sum_{n=1}^{+\infty} \frac{D^n}{(\mu - \lambda)^{n+1}}, \tag{A.3.5}$$

for any $0 < |\mu - \lambda| < r$. *Here,* P *is the projection defined by*

$$P = \frac{1}{2\pi i}\int_\gamma R(\xi, A)d\xi, \tag{A.3.6}$$

where γ *is the boundary of the ball* $\lambda + B(r)$ *(oriented counterclockwise), such that* $\overline{\lambda + B(r)} \cap \sigma(A) = \{\lambda\}$. *The operators* D *and* S *are defined, respectively, by*

$$D = (\lambda I - A)P, \qquad S = \lim_{\xi\to\lambda} R(\xi, A)(I - P).$$

In particular, λ *is the unique element of the spectrum of the restriction of* A *to* $P(X)$*, whereas the spectrum of the restriction of* A *to* $(I - P)(X)$ *is* $\sigma(A) \setminus \{\lambda\}$. *Finally,* $X = P(X) \oplus (I - P)(X)$.

The next theorem gives an estimate from above of the norm of the resolvent set of a densely defined operator $A : D(A) \subset X \to X$, in terms of its numerical range

$$r(A) = \{\langle x', Ax\rangle : x \in D(A),\ \|x\| = 1,\ x' \in X',\ \|x'\| = 1,\ \langle x', x\rangle = 1\}.$$

Theorem A.3.6 ([123], Chapter 1, Theorem 3.9) *Let* $A : D(A) \subset X \to X$ *be a densely defined closed operator in* X. *If* $\lambda \in \mathbb{C} \setminus \overline{r(A)}$, *then the operator* $\lambda I - A$ *is one to one and has closed range. Moreover, if* Σ *is a component of* $\mathbb{C} \setminus \overline{r(A)}$ *which intersects* $\rho(A)$*, then the spectrum of* A *is contained in* $\mathbb{C} \setminus \Sigma$ *and*

$$\|R(\lambda, A)\|_{L(X)} \le \frac{1}{\mathrm{dist}(\lambda, \overline{r(A)})}, \qquad \lambda \in \mathbb{C} \setminus \Sigma.$$

In the case when $R(\lambda, A)$ is a compact operator for some (and, hence, for any) $\lambda \in \rho(A)$, then $\sigma(A)$ has a very easy structure, as the following proposition shows.

Proposition A.3.7 *Let* $A : X \to X$ *be a closed operator with* $\sigma(A) \neq \varnothing$ *and compact resolvent operator. Then, the following properties are met:*

(*i*) $\sigma(A)$ *consists of an at most countable set of points. Any* $\lambda \in \sigma(A)$ *is an eigenvalue of* A *and the corresponding eigenspace has finite dimension;*

(*ii*) *for any* $\lambda \in \sigma(A)$*, the vector space of the generalized eigenvectors of* A *(i.e., the vector space of all* $x \in X$ *such that* $(\lambda I - A)^k x = 0$ *for some* $k \in \mathbb{N}$*) has finite dimension;*

(*iii*) *any* $\lambda \in \sigma(A)$*, is a pole of the function* $R(\cdot, A)$ *whose order coincides with the dimension of the vector space of all the generalized eigenvectors associated with the eigenvalue* λ. *Finally, the order of the pole coincides also with the index* $i_\lambda(A)$ *of the eigenvalue* λ *(i.e., with the smallest integer* k *such that* $\mathrm{Ker}((\lambda I - A)^k) = \mathrm{Ker}((\lambda I - A)^{k+1})$.

Proof. The proof can be obtained combining the results in [81, Chapter 3, Sections 6.4, 6.5 and 6.8]. For the reader's convenience we give some details of how this can be done.

Since, for any $\xi \in \rho(A)$, $R(\xi, A)$ is a compact operator then, according to Proposition A.1.1, the projection P in (A.3.6) is a compact operator as well. Hence, $P(X)$ is a finite dimensional subspace of X (see [81, Chapter 3, Section 4]).

Now, the arguments in [81, Chapter 3, Section 6.5] show that any $\lambda \in \sigma(A)$ is a pole of the resolvent operator, with finite multiplicity that coincides with the dimension m of $P(X)$. According to the decomposition theorem (see [81, Chapter 3, Theorem 6.17]), λ is the unique element in the spectrum of the restriction of A to $P(X)$, and it belongs to the resolvent set of the restriction of A to $(I-P)(X)$. It follows that $P(X)$ is the space of generalized eigenvectors of A corresponding to λ. Now, the equality $i_\lambda(A) = m$ follows observing that $i_\lambda(A) = i_\lambda(\tilde{A})$, since, according to a well-known result of linear algebra, $i_\lambda(\tilde{A}) = m$. ■

A.4 Some results from interpolation theory

In this section we recall some results from the interpolation theory. For the proof of the results that we present here, if not otherwise specified, we refer the reader to [104] and [141].

Let us introduce the interpolation spaces $(X, Y)_{\theta,\infty}$.

Definition A.4.1 *Let X, Y be two Banach spaces with Y continuously embedded in X. Moreover, let $K : (0, +\infty) \times X \to \mathbb{R}$ be the function defined by*

$$K(\xi, x) = \inf_{x=a+b} \left(\|a\|_X + \xi\|b\|_Y\right), \qquad \xi > 0, \quad x \in X.$$

For any $\theta \in [0, 1]$, the interpolation space $(X, Y)_{\theta,\infty}$ is defined by

$$(X, Y)_{\theta,\infty} = \{x \in X : \sup_{0<\xi<1} \xi^{-\theta} K(\xi, x) < +\infty\},$$

and it is normed by

$$\|x\|_{\theta,\infty} := \sup_{0<\xi<1} \xi^{-\theta} K(\xi, x), \qquad x \in (X, Y)_{\theta,\infty}. \tag{A.4.1}$$

For any $\theta \in [0, 1]$, $(X, Y)_{\theta,\infty}$ is a Banach space when it is endowed with the norm in (A.4.1) and

$$Y \subset (X, Y)_{\theta,\infty} \subset X \tag{A.4.2}$$

with continuous embeddings.

The topological inclusions in (A.4.2) and the following proposition show the reason why $(X,Y)_{\theta,\infty}$ are called interpolation spaces.

Proposition A.4.2 *Let X_1, X_2, Y_1, Y_2 be four Banach spaces with Y_j continuously embedded in X_j for $j=1,2$. Moreover, let $T\in L(X_1,X_2)\cap L(Y_1,Y_2)$. Then, for any $\theta\in(0,1)$, $T\in L((X_1,Y_1)_{\theta,\infty},(X_2,Y_2)_{\theta,\infty})$. Moreover,*

$$\|T\|_{L((X_1,Y_1)_{\theta,\infty},(X_2,Y_2)_{\theta,\infty})}\le \|T\|^{1-\theta}_{L(X_1,X_2)}\|T\|^{\theta}_{L(Y_1,Y_2)}. \tag{A.4.3}$$

We now introduce the definition of spaces of class J_θ and K_θ $(\theta\in[0,1])$ which play a central role in the interpolation theory.

Definition A.4.3 *Let X,Y,Z be a triplet of Banach spaces such that $Y\subset Z\subset X$ with continuous embeddings. Moreover, let $\theta\in[0,1]$. We say that Z belongs to the class J_θ between X and Y (in short $Z\in J_\theta(X,Y)$) if there exists a positive constant $C=C(\theta)$ such that*

$$\|x\|_Z\le C\|x\|_X^{1-\theta}\|x\|_Y^{\theta},\qquad x\in Z.$$

Similarly, we say that Z belongs to the class K_θ between X and Y (in short $Z\in K_\theta(X,Y)$) if $Z\subset (X,Y)_{\theta,\infty}$ with a continuous embedding.

Abstract and concrete examples of spaces of class J_θ are given in the following two propositions.

Proposition A.4.4 *Let $0<\theta<\alpha$ and let $\Omega\subset\mathbb{R}^N$ $(N\ge 1)$ be an open set with boundary uniformly of class C^α (possibly $\Omega=\mathbb{R}^N$). Then, $C^\theta_b(\overline{\Omega})$ belongs to the class $J_{\theta/\alpha}$ between $C_b(\overline{\Omega})$ and $C^\alpha_b(\overline{\Omega})$.*

Proposition A.4.5 *For any $\theta\in[0,1]$ and any pair of Banach spaces X and Y, with Y continuously embedded in X, $(X,Y)_{\theta,\infty}\in J_\theta(X,Y)\cap K_\theta(X,Y)$.*

The next proposition show some interesting properties of the spaces of class J_α.

Proposition A.4.6 *Let $Y\subset Z\subset X$ be a triplet of Banach spaces such that $Z\in J_\alpha(X,Y)$ for some $\alpha\in(0,1)$. Further, let $I\subset\mathbb{R}$ be an interval. Then, the following properties are met:*

(i) *if $u\in B(I;Y)\cap C^\theta_b(I;X)$ for some $\theta\in[0,1)$, then $u\in C^{\theta(1-\alpha)}_b(I;Z)$. Moreover, there exists a positive constant C such that*

$$\|u\|_{C^{\theta(1-\alpha)}_b(I;Z)}\le C\|u\|^{1-\alpha}_{C^\theta_b(I;X)}\|u\|^{\alpha}_{B(I;Y)};$$

(*ii*) *if* $u \in B(I;Y) \cap \mathrm{Lip}(I;X)$*, then* $u \in C_b^{1-\alpha}(I;Z)$*. Moreover, there exists a positive constant* C *such that*

$$\|u\|_{C_b^{1-\alpha}(I;Z)} \le C\|u\|_{\mathrm{Lip}(I;X)}^{1-\alpha}\|u\|_{B(I;Y)}^{\alpha}.$$

We now state one of the most useful tools from interpolation theory: the so-called *reiteration theorem.*

Theorem A.4.7 *Let* $0 \le \theta_1 < \theta_2 \le 1$*. Suppose that* $Z_j \in K_{\theta_j}(X,Y)$ $(j = 1,2)$*. Then, for any* $\theta \in (0,1)$

$$(Z_1, Z_2)_{\theta,\infty} \subset (X,Y)_{(1-\theta)\theta_1+\theta\theta_2,\infty},$$

with a continuous embedding. Similarly, if $Z_j \in J_{\theta_j}(X,Y)$ $(j = 1,2)$*, then, for any* $\theta \in (0,1)$

$$(X,Y)_{(1-\theta)\theta_1+\theta\theta_2,\infty} \subset (Z_1, Z_2)_{\theta,\infty},$$

with a continuous embedding. In particular, if $Z_j \in J_{\theta_j}(X,Y) \cap K_{\theta_j}(X,Y)$ $(j = 1,2)$*, then*

$$(X,Y)_{(1-\theta)\theta_1+\theta\theta_2,\infty} = (Z_1, Z_2)_{\theta,\infty},$$

with equivalence of the corresponding norms.

Using the interpolation theorem some spaces of Hölder continuous functions can be characterized as interpolation spaces.

Theorem A.4.8 *For any* $0 \le \alpha < \beta$ *and any* $\theta \in (0,1)$ *it holds that*

$$(C_b^{\alpha}(\mathbb{R}^N), C_b^{\beta}(\mathbb{R}^N))_{\theta,\infty} = \mathcal{C}_b^{\alpha+\theta(\beta-\alpha)}(\mathbb{R}^N), \tag{A.4.4}$$

with equivalence of the corresponding norms. Moreover,

$$(BUC(\mathbb{R}^N), C_b^{\beta}(\mathbb{R}^N))_{\theta,\infty} = \mathcal{C}_b^{\theta\beta}(\mathbb{R}^N). \tag{A.4.5}$$

Proof. For the proof of (A.4.4) see [104, Theorem 1.2.17 & Corollary 1.2.18]. The last part of the theorem follows from the previous one through the reiteration theorem, recalling that $BUC(\mathbb{R}^N)$ belongs to $J_0(C_b(\mathbb{R}^N), C_b^{\beta}(\mathbb{R}^N)) \cap K_0(C_b(\mathbb{R}^N), C_b^{\beta}(\mathbb{R}^N))$. ■

To conclude this section we state the Riesz-Thorin interpolation theorem. For the proof we refer the reader to [133, Theorem IX.17].

Theorem A.4.9 (Riesz-Thorin interpolation theorem) *Let* (Ω_1, μ_1) *and* (Ω_2, μ_2) *be two* σ*-finite spaces. Moreover, let* T *be a linear operator such that*

$$T \in L(L^{p_0}(\Omega_1, \mu_1), L^{q_0}(\Omega_2, \mu_2)) \cap L(L^{p_1}(\Omega_1, \mu_1), L^{q_1}(\Omega_2, \mu_2)),$$

for some $p_0, p_1, q_0, q_1 \in [1, +\infty]$. *Then,*

$$T \in L(L^{p_\theta}(\Omega_1, \mu_1), L^{q_\theta}(\Omega_2, \mu_2)), \qquad \theta \in (0,1),$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Moreover,

$$\|T\|_{L(L^{p_\theta}(\Omega_1,\mu_1),L^{q_\theta}(\Omega_2,\mu_2))} \leq \|T\|^{1-\theta}_{L(L^{p_0}(\Omega_1,\mu_1),L^{q_0}(\Omega_2,\mu_2))} \|T\|^{\theta}_{L(L^{p_1}(\Omega_1,\mu_1),L^{q_1}(\Omega_2,\mu_2))}.$$

Appendix B

An overview on strongly continuous and analytic semigroups

In this appendix we recall the definitions and the main properties of strongly continuous and analytic semigroups that we use in this book. If not otherwise specified, we refer the reader to [51, 72, 104, 123] for the proofs of the results of this appendix.

To begin with we give the definition of semigroup of bounded linear operators. Throughout this appendix, if not otherwise specified, X will denote a Banach space.

Definition B.0.1 *A one-parameter family $\{T(t)\}$ of linear bounded operators in X is said to be a semigroup of bounded linear operators if*

$$T(t+s) = T(t) \circ T(s), \qquad s, t \geq 0. \tag{B.0.1}$$

We simply write $T(t)T(s)$ for $T(t) \circ T(s)$ when there is no damage of confusion.

B.1 Strongly continuous semigroups

We begin this section with the definition of the strongly continuous semigroups.

Definition B.1.1 *A semigroup $\{T(t)\}$ of bounded linear operators, defined in X, is strongly continuous if, for any $x \in X$, $T(t)x$ tends to x as t tends to* 0 *from the right.*

Using the semigroup rule (B.0.1), one can see that, if $\{T(t)\}$ is a strongly continuous semigroup, then the function $t \mapsto T(t)x$ is continuous in $[0, +\infty)$ for any $x \in X$. Moreover, it is also possible to show that there exist two constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\|_{L(X)} \leq Me^{\omega t}, \qquad t > 0. \tag{B.1.1}$$

When $\omega = 0$ and $M_\omega = 1$, we say that $\{T(t)\}$ is a *strongly continuous semigroup of contractions in X.*

For any strongly continuous semigroup we define its *exponential growth bound* as the infimum of the set of all the $\omega \in \mathbb{R}$ such that (B.1.1) holds for some $M \geq 1$. It is possible to show that the exponential growth bound of $\{T(t)\}$ coincides with the spectral bound $s(A)$ of its infinitesimal generator, where

$$s(A) = \sup\{\mathrm{Re}\,\lambda : \ \lambda \in \sigma(A)\}.$$

See, e.g., [8, Theorem 5.3.1]).

Remark B.1.2 If $\{T(t)\}$ is a strongly continuous semigroup, then the mapping $(t, x) \mapsto T(t)x$ is locally uniformly continuous in $[0, +\infty) \times X$. See, e.g., [51, Chapter 1, Lemma 5.2]. As a consequence, for any compact set $K \subset X$ and any $T > 0$, the mapping $t \mapsto T(t)x$ is continuous in $[0, T]$, uniformly with respect to $x \in K$.

We can now give the definition of the infinitesimal generator of a strongly continuous semigroup.

Definition B.1.3 *Let $\{T(t)\}$ be a strongly continuous semigroup in X. The infinitesimal generator of $\{T(t)\}$ is the operator $A : D(A) \subset X \to X$ defined by*

$$\begin{cases} D(A) = \left\{x \in X : \exists g \in X \text{ s.t. } g = \displaystyle\lim_{t\to 0^+} \frac{T(t)x - x}{t}\right\}, \\ Ax = \displaystyle\lim_{t\to 0^+} \frac{T(t)x - x}{t}. \end{cases} \tag{B.1.2}$$

It is easy to see that the function $t \mapsto T(t)x$ is continuously Fréchet differentiable in $[0, +\infty)$ with values in X if and only if $x \in D(A)$. In this case $D_t T(t)x = AT(t)x = T(t)Ax$ for any $t \geq 0$.

As the following proposition shows, the infinitesimal generator characterizes uniquely the strongly continuous semigroups.

Proposition B.1.4 *Let $\{S(t)\}$ and $\{T(t)\}$ be two strongly continuous semigroups in X and denote by A and B their infinitesimal generators. If $A = B$, then $S(t) \equiv T(t)$ for any $t > 0$.*

The Hille-Yosida theorem is a keystone in the theory of strongly continuous semigroups since it allows us to give a complete characterization of the infinitesimal generators of strongly continuous semigroups.

Theorem B.1.5 (Hille-Yosida) *A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}$ satisfying* (B.1.1) *for some $M \geq 1$ and some $\omega \in \mathbb{R}$, if and only if*

(*i*) *A is closed and its domain is dense in X;*

(*ii*) *the resolvent set* $\rho(A)$ *contains the half line* $(\omega, +\infty)$ *and, for any* $\lambda > \omega$ *and any* $n \in \mathbb{N}$,

$$\|R(\lambda, A)\|_{L(X)} \le \frac{M}{(\lambda - \omega)^n}.$$

The proof of the Hille-Yosida theorem shows a connection between the resolvent operator and the semigroup. In fact, the resolvent operator is the Laplace transform of the semigroup, namely

$$R(\lambda, A)x = \int_0^{+\infty} e^{-\lambda t} T(t)x\, dt, \qquad \mathrm{Re}\,\lambda > \omega, \quad x \in X. \tag{B.1.3}$$

Hence, many properties of $R(\lambda, A)$ can be deduced from the corresponding properties of the semigroup.

Another important result in the theory of strongly continuous semigroups is the Lumer-Phillips theorem. It provides an alternative characterization of the infinitesimal generators of strongly continuous semigroups of contractions.

Definition B.1.6 *Let* $A : D(A) \subset X \to X$ *be a linear operator.* A *is dissipative if*

$$\|\lambda x - Ax\|_X \ge \lambda \|x\|_X, \qquad \lambda > 0, \quad x \in D(A).$$

Theorem B.1.7 (Lumer-Phillips) *Let* $A : D(A) \subset X \to X$ *be a linear operator with dense domain. Then, the following properties are met:*

(*i*) *if* A *is dissipative and there exists* $\lambda_0 > 0$ *such that the range of the operator* $\lambda_0 I - A$ *is* X, *then* A *is the infinitesimal generator of a strongly continuous semigroup of contractions;*

(*ii*) *if* A *is the infinitesimal generator of a strongly continuous semigroup of contractions in* X, *then the range of the operator* $\lambda I - A$ *is* X *for any* $\lambda > 0$. *Moreover,* A *is dissipative.*

A straightforward consequence of the Lumer-Phillips theorem is the following proposition.

Proposition B.1.8 *Let* $A : D(A) \subset X \to X$ *be a dissipative operator. Then, the following properties are met:*

(*i*) *if* A *is closable, then the closure* $\overline{A}$ *is also dissipative;*

(*ii*) *if* $D(A)$ *is dense in* X *and* $(\lambda I - A)(D(A))$ *is dense in* X *for some* (*and hence all* $\lambda > 0$), *then the closure* $\overline{A}$ *is the infinitesimal generator of a strongly continuous semigroup of contractions.*

To conclude this section, we recall the definition of a core of a linear operator A and we provide a useful sufficient condition guaranteeing that a set E is a core of A, when A is the infinitesimal generator of a strongly continuous semigroup. Finally, we state the famous Trotter-Kato theorem.

Definition B.1.9 *Let A be a linear operator in a Banach space X with domain $D(A)$. A subspace E of $D(A)$ is a core of A if E is dense in $D(A)$ in the graph norm (see* (A.1.2)*).*

Proposition B.1.10 *Let $A : D(A) \subset X \to X$ be the generator of a strongly continuous semigroup of contractions $\{S(t)\}$ in X. Moreover, let E be a subspace of $D(A)$ dense in X such that $S(t)(E) \subset E$ for any $t > 0$. Then, E is a core of A.*

Theorem B.1.11 (Trotter-Kato) *Let $\{T(t)\}$, $\{T_n(t)\}$ $(n \in \mathbb{N})$ be strongly continuous semigroups of contractions in X, with infinitesimal generators $A : D(A) \subset X \to X$ and $A_n : D(A_n) \subset X \to X$, respectively. Further, let D be a core of the operator A such that $D \subset D(A_n)$ for any $n \in \mathbb{N}$. If, for any $x \in D$, $A_n x$ tends to Ax in X, as n tends to $+\infty$, then $R(\lambda, A_n)x$ tends to $R(\lambda, A)x$, as n tends to $+\infty$, for any $\lambda \in \mathbb{C}$ with positive real part.*

B.1.1 On the closure of the sum of generators of strongly continuous semigroups

Here we briefly recall some results on the sum of closed operators that we use in Chapter 9. Throughout the subsection we make the following hypotheses.

Hypotheses B.1.12 (i) $A : D(A) \subset X \to X$ is a closed operator such that $-A$ is the infinitesimal generator of a strongly continuous semigroup of contractions in X;

(ii) $0 \in \rho(A)$.

The analyticity of the function $\lambda R(\lambda, A)$ in $\rho(A)$ implies that $\rho(A)$ contains a sector $\Sigma_\varphi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \varphi\}$ for some $\varphi \in [0, \pi)$ and that there exists a positive constant C_φ such that $\|\lambda R(\lambda, -A)\|_{L(X)} \le C_\varphi$ for any $\lambda \in \Sigma_\varphi$.

Therefore, one can introduce the *spectral angle* φ_A of A, defined by

$$\varphi_A = \inf\{\varphi > 0 : \Sigma_{\pi-\varphi} \subset \rho(-A) \text{ and } C_\varphi < +\infty\}.$$

Moreover, for $s \in \mathbb{R}\setminus\{0\}$, one can define the operator A^{is} through the formula

$$A^{is}x = -\frac{\sin(i\pi s)}{\pi}\bigg\{i\frac{x}{s} + \frac{1}{1+is}A^{-1}x + \int_0^1 \lambda^{is+1}R(-\lambda, A)A^{-1}x\,d\lambda + \int_1^{+\infty} \lambda^{is-1}R(-\lambda, A)Ax\,d\lambda\bigg\}, \qquad x \in D(A)$$

(see [86, 87]). A^{is} is a closed and densely defined operator for any $s \neq 0$ but, in general, it is not bounded in $\mathbb{R}$.

If the operators A^{is} are bounded for any $s \neq 0$ and the map $s \mapsto A^{is}$ ($A^0 := I$), is locally bounded in $\mathbb{R}$, we say that A admits bounded imaginary powers or simply that $A \in \mathcal{BIP}(X)$.

If $A \in \mathcal{BIP}(X)$, then the limit

$$\theta_A := \lim_{|s|\to+\infty} |s|^{-1} \log\left(\|A^{is}\|_{L(X)}\right) \tag{B.1.4}$$

is well defined and it is called the *power angle* of A.

We summarize in the following proposition a few properties of the class $\mathcal{BIP}(X)$. For a proof, we refer the reader to [34, Theorem 5.8], [35], [131] and [132, Theorem 3].

Proposition B.1.13 *The following properties are met:*

(*i*) *if $A \in \mathcal{BIP}(X)$, then $\omega I + A \in \mathcal{BIP}(X)$ for any $\omega > 0$ and $\theta_{\omega I + A} = \theta_A$;*

(*ii*) *if $A \in \mathcal{BIP}(X)$, then $\theta_A \geq \varphi_A$;*

(*iii*) *if $A : D(A) \subset L^p(\mathbb{R}^N, \mu) \to L^p(\mathbb{R}^N, \mu)$ (μ being a σ-finite measure, and $p \in (1, +\infty)$) is the generator of a strongly continuous semigroup of contractions $\{T(t)\}$ such that $T(t)f \geq 0$ for any $f \geq 0$, then, for any $\omega > 0$, $\omega I - A \in \mathcal{BIP}(L^p(\mathbb{R}^N, \mu))$ and $\theta_{\omega I - A} \leq \pi/2$;*

(*iv*) *if A is the realization in $L^p(\mathbb{R}^N, dx)$ ($p \in (1, +\infty)$) of the uniformly elliptic operator $\mathcal{A}u = \sum_{i,j=1}^N q_{ij} D_{ij} u$ with constant coefficients, then $\theta_{\omega I - A} = 0$ for any $\omega > 0$;*

(*v*) *if X is a Hilbert space and $A : D(A) \subset X \to X$ is a self-adjoint operator such that $-A$ generates a strongly continuous semigroup of contractions in X, then $A \in \mathcal{BIP}(X)$ and $\theta_A = 0$.*

The property (iii) in Proposition B.1.13 is known as the *transference principle*. The property (iv) holds for a more general class of elliptic operators in divergence form than those we considered here. We refer the reader to [132, Theorem C] for more details.

We now introduce a class of Banach spaces, the so-called ξ-convex spaces.

Definition B.1.14 *A Banach space X is said to be ξ-convex if there exists a function $\xi : X \times X \to \mathbb{R}$ such that*

(*i*) *ξ is convex with respect to both the variables;*

(*ii*) *$\xi(0, 0) > 0$;*

(*iii*) *$\xi(x + y) \leq |x + y|$ for any $x, y \in \partial B(1)$.*

Examples of ξ-convex Banach spaces are the L^p-spaces ($p \in [1, +\infty)$) relevant to σ-finite measures in $\mathbb{R}^N$. In particular $L^p(\mathbb{R}^N)$ are all ξ-convex Banach spaces.

For further results on the operators which admit bounded imaginary powers and on ξ-convex Banach spaces, we refer the reader to [141, Chapter 1, Section 15], [47, Appendix], the survey paper [23] and the monograph [130].

To conclude this subsection we state two theorems which guarantee the closedness of the sum of n sectorial operators which admits bounded imaginary powers. The first one deals with the sum of n operators A_i commuting in the resolvent sense (i.e., $R(\lambda, A_i)R(\mu, A_j) = R(\mu, A_j)R(\lambda, A_i)$ for any $\lambda \in \rho(A_i)$, any $\mu \in \rho(A_j)$ and any $i, j = 1, \ldots, N$). It extends a well known result by Dore and Venni (see [47]).

Theorem B.1.15 ([132], Corollary 4) *Let X be a ξ-convex space and let $A_i : D(A_i) \subset X \to X$ ($i = 1, \ldots, N$) belong to $\mathcal{BIP}(X)$ and commute in the resolvent sense. Further, suppose that the power angles θ_i satisfy $\theta_i + \theta_j < \pi$ for any $i, j = 1, \ldots, n$, $i \neq j$, and there is only one index j such that $\theta_j = \max_{i=1,\ldots,n} \theta_i$. Then, the operator $A = \sum_{i=1}^n A_i$ with domain $D(A) = \bigcap_{i=1}^n D(A_i)$ is closed, it belongs to $\mathcal{BIP}(X)$ and there exists a positive constant C such that*

$$\sum_{i=1}^{n} \|A_i x\|_X \le C\|Ax\|_X, \qquad x \in D(A). \tag{B.1.5}$$

The second theorem deals with the case of two operators which do not commute in resolvent sense. In such a case the noncommutativity of the resolvent operators of A and B is balanced by a suitable assumption on their commutators.

Theorem B.1.16 ([120], Corollary 2) *Let X be a ξ-convex Banach space and let $A, B \in \mathcal{BIP}(X)$. Further suppose that the power angles θ_A and θ_B satisfy $\theta_A + \theta_B < \pi$ and that there exist $C > 0$, α, β satisfying $0 \le \alpha < \beta \le 1$ and φ_A, φ_B with $\varphi_A > \theta_A$, $\varphi_B > \theta_B$ and $\varphi_A + \varphi_B < \pi$ such that*

$$\|AR(\lambda, -A)[A^{-1}, R(\mu, -B)]\|_{L(X)} \le \frac{C}{(1 + |\lambda|^{1-\alpha})|\mu|^{1+\beta}}, \tag{B.1.6}$$

for any $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ such that $|\arg \lambda| < \pi - \varphi_A$ and $|\arg \mu| < \pi - \varphi_B$. Then, the operator $A + B$ with domain $D(A + B) = D(A) \cap D(B)$ is closed in X.

B.2 Analytic semigroups

In this section we introduce the analytic semigroups and we describe their main properties.

Definition B.2.1 *A linear operator $A : D(A) \subset X \to X$ is said to be sectorial if there exist three constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $M > 0$ such that*

$$\begin{cases} (i) \;\; \rho(A) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, \;\; |\arg(\lambda - \omega)| < \theta\}, \\ (ii) \; \|R(\lambda, A)\|_{L(X)} \le \dfrac{M}{|\lambda - \omega|}, \qquad \lambda \in S_{\theta,\omega}. \end{cases} \tag{B.2.1}$$

For any $t > 0$, the conditions (B.2.1) allow us to define a bounded linear operator $T(t)$ on X as follows. For any $r > 0$ and any $\eta \in (\pi/2, \theta]$, let $\gamma_{r,\eta}$ be the curve

$$\{\lambda \in \mathbb{C} : |\arg\lambda| = \eta, \; |\lambda| \ge r\} \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \le \eta, \; |\lambda| = r\},$$

oriented counterclockwise. We set

$$T(t) = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}+\omega} e^{t\lambda} R(\lambda, A)\, d\lambda, \qquad t > 0, \qquad T(0) = I. \tag{B.2.2}$$

It is easy to check that the integral in (B.2.2) is well defined and it is independent of $r > 0$ and $\eta \in (\pi/2, \theta)$.

In the following theorem we summarize the main properties of $T(t)$ for $t > 0$.

Theorem B.2.2 *Let A be a sectorial operator and let $T(t)$ be given by* (B.2.2), *for any $t > 0$. Then, the following properties are met:*

(*i*) *the family of bounded operators $\{T(t)\}$ is a semigroup;*

(*ii*) *$T(t)x \in D(A^k)$ for any $t > 0$, any $x \in X$ and any $k \in \mathbb{N}$. Moreover, if $x \in D(A^k)$, then*

$$A^k T(t)x = T(t)A^k x, \qquad t \ge 0;$$

(*iii*) *there exists a constant $M > 0$ such that*

$$\|T(t)\|_{L(X)} \le M e^{\omega t}, \qquad t > 0, \tag{B.2.3}$$

where ω is the number in (B.2.1). *Moreover, for any $\varepsilon > 0$ and any $k \in \mathbb{N}$, there exists a positive constant $C_{k,\varepsilon}$ such that*

$$\|t^k A^k T(t)\|_{L(X)} \le C_{k,\varepsilon} e^{(\omega+\varepsilon)t}, \qquad t > 0; \tag{B.2.4}$$

(*iv*) *the function $t \mapsto T(t)$ is analytic in $(0 + \infty)$ with values in $L(X)$ and*

$$\frac{d^k}{dt^k} T(t) = A^k T(t), \qquad t > 0, \;\; k \in \mathbb{N}. \tag{B.2.5}$$

We can now give the following definition.

Definition B.2.3 *Let A be a sectorial operator. The semigroup $\{T(t)\}$ defined by* (B.2.2) *is called the analytic semigroup generated by A (in X).*

The property (iv) in Theorem B.2.2 implies that $\{T(t)\}$ is norm continuous in $(0, +\infty)$ according to the following definition.

Definition B.2.4 *A semigroup of bounded linear operators $\{T(t)\}$ in X is said to be norm continuous in the interval $I \subset [0, +\infty)$ if the map $t \mapsto T(t)$ is continuous in I with respect to the topology of $L(X)$.*

The norm continuity of the semigroup allows us to write

$$R(\lambda, A) = \int_0^{+\infty} e^{-\lambda t} T(t) dt, \tag{B.2.6}$$

for any $\lambda \in \mathbb{C}$, with $\mathrm{Re}\lambda > \omega$, where the integral converges in the topology of $L(X)$.

In the following proposition we exploit the behaviour of the function $t \mapsto T(t)x$ at $t = 0$.

Proposition B.2.5 *The following properties are met:*

(*i*) *if $x \in \overline{D(A)}$, then $\lim_{t \to 0^+} T(t)x = x$. Conversely, if $y = \lim_{t \to 0^+} T(t)x$ exists, then $x \in \overline{D(A)}$ and $y = x$;*

(*ii*) *if $x \in D(A)$ and $Ax \in \overline{D(A)}$, then $\lim_{t \to 0^+} (T(t)x - x)/t = Ax$. Conversely, if $z := \lim_{t \to 0^+} (T(t)x - x)/t$ exists, then $x \in D(A)$ and $Ax = z \in \overline{D(A)}$;*

(*iii*) *if $x \in D(A)$ and $Ax \in \overline{D(A)}$, then $\lim_{t \to 0^+} AT(t)x = Ax$.*

Remark B.2.6 Let $x \in X$ and denote by $u : [0, +\infty) \to X$ the function defined by $u(t) = T(t)x$ for any $t > 0$ and $u(0) = x$. Combining the results in Theorem B.2.2 and Proposition B.2.5, we easily deduce that $u \in C([0, +\infty); X)$ if and only if $x \in \overline{D(A)}$. Similarly, $u \in C([0, +\infty); D(A))$ (i.e., both u and Au belong to $C([0, +\infty); X)$) if and only if $x \in D(A)$ and $Ax \in \overline{D(A)}$. Since, according to Theorem B.2.2, u is differentiable in $(0, +\infty)$ and $u'(t) = Au(t)$ for any $t > 0$, then $u \in C^1([0, +\infty); X)$ if and only if $x \in D(A)$ and $Ax \in \overline{D(A)}$. Moreover, $u \in C^1([0, +\infty); D(A))$ if and only if $x \in D(A^2)$ and $A^2x \in \overline{D(A)}$.

As a straightforward consequence of Proposition B.2.5(i), we deduce that an analytic semigroup $\{T(t)\}$ fails to be strongly continuous in X unless $D(A)$ is dense in X.

Useful criteria to establish if an operator $A : D(A) \subset X \to X$ is sectorial are given by the following three theorems.

Theorem B.2.7 *Let $A : D(A) \subset X \to X$ be a linear operator such that $\rho(A)$ contains the halfplane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geq K\}$ for some $K \geq 0$, and*

$$\|\lambda R(\lambda, A)\|_{L(X)} \leq M, \qquad \mathrm{Re}\,\lambda \geq K, \tag{B.2.7}$$

for some $M \geq 1$. Then, A is sectorial.

Theorem B.2.8 *Let $\{T(t)\}$ be a strongly continuous semigroup in X such that the function $t \mapsto T(t)x$ is differentiable in $(0, +\infty)$ with values in X for any $x \in X$ and there exists a positive constant M such that $t\|T'(t)\|_{L(X)} \leq M$ for any $t \in (0, 1)$. Then, $\{T(t)\}$ is analytic.*

Theorem B.2.9 *Let X be a Hilbert space and $A : D(A) \subset X \to X$ be a self-adjoint dissipative operator. Then, A is sectorial in X and it satisfies the conditions* (B.2.1) *with $\omega = 0$ and arbitrary $\theta < \pi$.*

Next, we recall the following classical perturbation result for analytic semigroups.

Theorem B.2.10 *If $A : D(A) \subset X \to X$ is sectorial and $B : D(B) \subset X \to X$ is a linear operator such that*

(i) *$D(A) \subset D(B)$;*

(ii) *there exist $\theta \in (0, 1)$ and $C > 0$ such that*

$$\|Bx\| \leq C\|x\|_{D(A)}^{\theta}\|x\|_X^{1-\theta}, \qquad x \in D(A).$$

Then, $A + B : D(A + B) := D(A) \to X$ is sectorial. Moreover, if A satisfies (B.2.1) *with $\theta > 3\pi/4$, then the operator $-A^2$ is sectorial as well.*

We now introduce the interpolation spaces $D_A(\alpha, \infty)$ ($\alpha \in (0, 1)$), which play a crucial role in the theory of analytic semigroups.

Definition B.2.11 *Let $\alpha \in (0, 1)$. We set $D_A(\alpha, \infty) = (X, D(A))_{\alpha,\infty}$ (see Section* A.4*) and we call it the interpolation space of order α.*

According to Proposition A.4.2, and Theorem B.2.2, one can easily see that, if $x \in D_A(\alpha, \infty)$, then the function $t \mapsto g_x(t) := t^{-\alpha}(T(t)x - x)$ is bounded in a right neighborhood of $t = 0$. Actually, as the following proposition shows, the function g_x is bounded near $t = 0$ if and only if $x \in D_A(\alpha, \infty)$.

Proposition B.2.12 *The interpolation space $D_A(\alpha, \infty)$ is the set of all the $x \in X$ such that*

$$[[x]]_{D_A(\alpha,\infty)} := \sup_{0<t\leq 1} t^{-\alpha}\|T(t)x - x\|_X < +\infty.$$

Moreover, the norm

$$x \mapsto \|x\|_X + [[x]]_{D_A(\alpha,\infty)}$$

is equivalent to the norm $\|\cdot\|_{(X,D(A))_{\alpha,\infty}}$ *defined in* (A.4.1).

Equivalently, $D_A(\alpha,\infty)$ *can be defined as the set of all the* $x \in X$ *such that*

$$[[[x]]]_{D_A(\alpha,\infty)} := \sup_{0<\xi\leq 1} \|\xi^{1-\theta} AT(\xi)x\|_X < +\infty.$$

Moreover, the norm

$$x \mapsto \|x\|_X + [[[x]]]_{D_A(\alpha,\infty)}$$

is equivalent to the norm $\|\cdot\|_{(X,D(A))_{\alpha,\infty}}$.

As a consequence of Theorem B.2.2 and Proposition B.2.12, it follows that the function $u = T(\cdot)x$ belongs to $C^\alpha([0,T];X)$ for some (and hence any) $T > 0$ if and only if $x \in D_A(\alpha,\infty)$. Similarly, $u \in C^\alpha([0,T];D(A))$ (or, equivalently, $u \in C^{1+\alpha}([0,T];X)$) if and only if $x \in D_A(1+\alpha,\infty)$, where

$$D_A(1+\alpha,\infty) := \{x \in D(A) : \ Ax \in D_A(\alpha,\infty)\}.$$

To conclude this section, let us consider the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in (0,T], \\ u(0) = x, \end{cases} \tag{B.2.8}$$

where $f : [0,T] \to X$ is a given continuous function and $x \in X$.

Definition B.2.13 *We say that* $u : [0,T] \to X$ *is a strict solution to* (B.2.8) *if* $u \in C^1([0,T];X) \cap C([0,T];D(A))$ *and it satisfies* (B.2.8).
Similarly, a function $u : [0,T] \to X$ *is a classical solution to* (B.2.8) *if* $u \in C^1((0,T];X) \cap C((0,T];D(A)) \cap C([0,T];X)$ *and it satisfies* (B.2.8).

We have seen that, in the case when $f = 0$ and $x \in \overline{D(A)}$, the function $u = T(\cdot)x$ is a classical solution to (B.2.8) and it is strict provided that $x \in D(A)$ and $Ax \in \overline{D(A)}$. Actually, one can show that the previous one is the unique classical (resp. strict) solution to (B.2.8) with $f = 0$. Indeed, if $f \in C((0,T];X)$ and the function $t \mapsto \|f(t)\|_X$ is integrable in $(0,T)$, then any classical (resp. strict) solution u to (B.2.8) is given by the variation-of-constants formula

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \qquad t \in [0,T]. \tag{B.2.9}$$

Whenever the integral in (B.2.9) does make sense, the function u defined by (B.2.9) is said to be a *mild solution* of (B.2.8).

Appendix C

PDE's and analytic semigroups

In this appendix we recall some classical results on PDE's of elliptic and parabolic problems that we use in this book. Namely, we state some well-known (interior) Schauder estimates, some maximum principle and we see the relations between the theory of PDE's and the theory of analytic semigroups.

Throughout this appendix, by $\mathcal{A}$ we denote the elliptic operator defined on smooth functions by

$$\mathcal{A}u(x) = \sum_{i,j=1}^{N} q_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{N} b_i(x)D_i u(x) + c(x)u(x), \qquad \text{(C.0.1)}$$

with real coefficients q_{ij}, b_i $(i,j = 1,\ldots,N)$ and c defined in an open set Ω. We do not assume any smoothness assumption on the domain Ω unless it is explicitly mentioned.

The following condition is always assumed in this appendix.

Hypothesis C.0.1 $q_{ij} \equiv q_{ji}$ for any $i,j = 1,\ldots,N$ and

$$\sum_{i,j=1}^{N} q_{ij}(x)\xi_i\xi_j \geq \mu_0|\xi|^2, \qquad \xi \in \Omega, \ x \in \overline{\Omega},$$

for some positive constant μ_0.

C.1 *A priori* estimates

In this section we recall some classical L^p- and Schauder interior estimates for solutions to elliptic and parabolic problems in bounded domains Ω or in the whole of $\mathbb{R}^N$.

We begin with the elliptic case and recall well-known interior L^p-estimates and L^p-estimates up to a part of the boundary of Ω, where homogeneous Neumann boundary conditions are prescribed.

Theorem C.1.1 ([66], Theorems 9.11 & 9.19) *Let Ω be an open set and p any real number in the interval $(1,+\infty)$. Then, the following properties are met:*

(*i*) *if the coefficients of the operator* $\mathcal{A}$ *are bounded and continuous in* Ω, *then, for any open set* $\Omega' \subset\subset \Omega$, *there exists a positive constant* C, *depending on* p, Ω, Ω', μ_0 *and the moduli of continuity of the coefficients* q_{ij} *in* Ω' $(i,j = 1,\ldots,N)$, *such that*

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\|u\|_{L^p(\Omega)} + \|\mathcal{A}u\|_{L^p(\Omega)}),$$

for any $u \in L^p(\Omega) \cap W^{2,p}_{\rm loc}(\Omega)$ *such that* $\mathcal{A}u \in L^p(\Omega)$;

(*ii*) *if the coefficients* q_{ij}, b_i $(i,j = 1,\ldots,N)$ *and* c *are locally Lipschitz continuous in* Ω, $u \in W^{2,p}_{\rm loc}(\Omega)$ *and* $\mathcal{A}u \in W^{1,p}_{\rm loc}(\Omega)$, *then* $u \in W^{3,p}_{\rm loc}(\Omega)$ *and, for any* $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, *there exists a positive constant* $\widetilde{C}$, *depending on* p, Ω', Ω'' *and the coefficients of the operator* $\mathcal{A}$, *such that*

$$\|u\|_{W^{3,p}(\Omega')} \leq \widetilde{C}(\|u\|_{L^p(\Omega'')} + \|\mathcal{A}u\|_{W^{1,p}(\Omega'')}).$$

We also recall the following L^p-estimates up to the boundary of Ω.

Theorem C.1.2 *Let* $p \in (1,+\infty)$ *and let* Ω *be an open set of class* C^2. *Moreover, let* Ω_0 *and* Ω_1 *be two bounded subsets of* Ω, *with* Ω_1 *of class* C^2, *such that their boundaries intersect the boundary of* Ω *and* $\mathrm{dist}(\Omega_0, \Omega\backslash\Omega_1) > 0$. *Further, assume that the coefficients of the operator* $\mathcal{A}$ *are continuous in* $\overline{\Omega}$. *Then, there exists a positive constant* C, *depending on* $p, N, \mu, \Omega_0, \Omega_1$, *the sup-norm of the coefficients of the operator* $\mathcal{A}$ *and the modulus of continuity of the leading coefficients in* Ω_1, *such that*

$$\|u\|_{W^{2,p}(\Omega_0)} \leq C\left(\|\mathcal{A}u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega_1)}\right),$$

for any function $u \in W^{2,p}(\Omega)$ *such that* $\dfrac{\partial u}{\partial \nu} = 0$ *on* $\partial\Omega \cap \partial\Omega_1$.

To conclude the analysis of the elliptic case, we consider the following result concerning the regularity of distributional solutions to elliptic equations. Such results are essentially due to S. Agmon (see [3] and [4], in the case when $p = 2$), which deals with the case when $\mathbb{R}^N$ is replaced with a bounded set Ω. Since as far as we know they seem not to be written, at least in the most used textbooks, in the form that we need, for the reader's convenience we provide a proof of them. Such a proof is a variant of a proof due to G. Metafune, D. Pallara and A. Rhandi.

Theorem C.1.3 *Assume that Hypothesis C.0.1 is satisfied. Moreover, let* $p \in (1,+\infty)$, *denote by* p' *the conjugate exponent to* p *(i.e.,* $1/p' = 1 - 1/p$*) and set*

$$\mathcal{A} = \sum_{i,j=1}^N q_{ij}D_{ij} + \sum_{i=1}^N b_iD_i.$$

Then, the following properties are met:

(*i*) *if* $q_{ij} \in C_b^1(\mathbb{R}^N)$, $b_i \in C_b(\mathbb{R}^N)$ $(i,j=1,\dots,N)$ *and* $u \in L^p(\mathbb{R}^N)$ *satisfies the estimate*

$$\left|\int_{\mathbb{R}^N} u\mathcal{A}\varphi dx\right| \le C\|\varphi\|_{W^{1,p'}(\mathbb{R}^N)}, \tag{C.1.1}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, *then* u *belongs to* $W^{1,p}(\mathbb{R}^N)$;

(*ii*) *if* $q_{ij} \in C^1(\mathbb{R}^N)$, $b_i \in C(\mathbb{R}^N)$ $(i,j=1,\dots,N)$ (*not necessarily bounded*) *and* $u \in L^p_{\rm loc}(\mathbb{R}^N)$ *satisfies* (C.1.1) *for any* $\varphi \in C_c^\infty(\mathbb{R}^N)$, *then* $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$;

(*iii*) *if* $q_{ij} \in C_b^2(\mathbb{R}^N)$, $b_i \in C_b^1(\mathbb{R}^N)$ $(i,j=1,\dots,N)$, ${\rm div}\, b \in L^p(\mathbb{R}^N)$ *and* $f, u \in L^p(\mathbb{R}^N)$ *are such that*

$$\int_{\mathbb{R}^N} u\mathcal{A}\varphi dx = \int_{\mathbb{R}^N} f\varphi dx, \tag{C.1.2}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, *then* $u \in W^{2,p}(\mathbb{R}^N)$. *In the case when* $\mathcal{A} = \sum_{i,j=1}^N D_i(q_{ij}D_{ij}) + \sum_{i=1}^N b_iD_i$ *the same result holds under the weaker assumptions* $q_{ij}, b_i \in C_b^1(\mathbb{R}^N)$ $(i,j=1,\dots,N)$;

(*iv*) *if* $q_{ij} \in C^2(\mathbb{R}^N)$, $b_i \in C^1(\mathbb{R}^N)$ $(i,j=1,\dots,N)$ (*not necessarily bounded*) *and* $f, u \in L^p_{\rm loc}(\mathbb{R}^N)$ *satisfy* (C.1.2) *for any* $\varphi \in C_c^\infty(\mathbb{R}^N)$, *then* $u \in W^{2,p}_{\rm loc}(\mathbb{R}^N)$. *In the case when* $\mathcal{A} = \sum_{i,j=1}^N D_i(q_{ij}D_{ij}) + \sum_{i=1}^N b_iD_i$ *the same result holds under the weaker assumptions* $q_{ij}, b_i \in C^1(\mathbb{R}^N)$ $(i,j=1,\dots,N)$.

Proof. We begin by proving the property (i). For this purpose, let $A_1 : W^{2,p'}(\mathbb{R}^N) \to L^{p'}(\mathbb{R}^N)$ be the operator defined by

$$A_1v = \sum_{i,j=1}^N q_{ij}D_{ij}v, \tag{C.1.3}$$

for any $v \in W^{2,p'}(\mathbb{R}^N)$. From (C.1.1) we deduce that, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N} uA_1\varphi dx\right| \le C\|\varphi\|_{W^{1,p'}(\mathbb{R}^N)}. \tag{C.1.4}$$

For any $h \in \mathbb{R}^N$, set $\tau_h u = |h|^{-1}(u(\cdot+h)-u)$. An explicit computation shows that

$$\int_{\mathbb{R}^N} \tau_h u\, \mathcal{A}_1\varphi\, dx = \int_{\mathbb{R}^N} u\mathcal{A}_1(\tau_{-h}\varphi)\, dx + \int_{\mathbb{R}^N} u(\tau_{-h}q_{ij})D_{ij}\varphi(\cdot-h)\, dx,$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. Using the boundedness of the diffusion coefficients q_{ij} in $\mathbb{R}^N$ $(i,j=1,\dots,N)$ and (C.1.4), we obtain

$$\left|\int_{\mathbb{R}^N} \tau_h u(\lambda\varphi - A_1\varphi)dx\right| \le C_1\|\varphi\|_{W^{2,p'}(\mathbb{R}^N)}, \tag{C.1.5}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, any $\lambda > 0$ and some positive constant $C_1 = C_1(\lambda)$, independent of φ and h. Since $u \in L^p(\mathbb{R}^N)$, by density, (C.1.5) extends to any function in $W^{2,p'}(\mathbb{R}^N)$. We fix a large λ so that the operator $\lambda - A_1$ is invertible from $W^{2,p'}(\mathbb{R}^N)$ to $L^{p'}(\mathbb{R}^N)$ (see the forthcoming Theorem C.3.6(i)), and we set $\varphi = R(\lambda, A_1)(\tau_h u|\tau_h u|^{p-2})$. The function φ satisfies

$$\|\varphi\|_{W^{2,p'}(\mathbb{R}^N)} \le C_2\|\tau_h u\|_{L^p(\mathbb{R}^N)}^{p-1}, \tag{C.1.6}$$

for some positive constant C_2, independent of h. Then, plugging this particular φ into (C.1.5) and taking (C.1.6) into account, we obtain

$$\int_{\mathbb{R}^N} |\tau_h u|^p dx \le C_3, \tag{C.1.7}$$

with C_3 being independent of h. Now, from (C.1.7), we immediately deduce that $u \in W^{1,p}(\mathbb{R}^N)$.

Proof of (ii). Let us fix $r > 0$ and let $\varphi, \psi_r \in C_c^\infty(\mathbb{R}^N)$ be two smooth functions with $\chi_{B(r)} \le \psi_r \le \chi_{B(2r)}$. Since

$$\psi_r \mathcal{A}\varphi = \mathcal{A}(\psi_r\varphi) - \varphi\mathcal{A}\psi_r - 2\sum_{i,j=1}^N q_{ij}D_i\varphi D_j\psi_r, \tag{C.1.8}$$

we immediately deduce that

$$\int_{\mathbb{R}^N} u\psi_r\mathcal{A}\varphi\, dx = \int_{\mathbb{R}^N} u\mathcal{A}(\varphi\psi_r)dx - \int_{\mathbb{R}^N} u\varphi\mathcal{A}\psi_r dx \\ -2\int_{\mathbb{R}^N} u\sum_{i,j=1}^N q_{ij}D_i\varphi D_j\psi_r dx.$$

Hence from (C.1.1), it follows that

$$\left|\int u\psi_r\mathcal{A}\varphi dx\right| \le C\|\varphi\|_{W^{1,p'}(\mathbb{R}^N)}.$$

Now, we consider a function $\eta \in C_c^\infty(\mathbb{R}^N)$ such that $\chi_{B(2r)} \le \eta \le \chi_{B(4r)}$ and we introduce the operator $\widetilde{\mathcal{A}} = \sum_{i,j=1}^N \widetilde{q}_{ij}D_{ij} + \sum_{i=1}^N \widetilde{b}_i D_i$, where

$$\widetilde{q}_{ij} = \eta q_{ij} + (1-\eta)\delta_{ij}, \qquad \widetilde{b}_i = \eta b_i, \qquad i,j = 1,\dots,N. \tag{C.1.9}$$

Of course,

$$\int u\psi_r\mathcal{A}\varphi dx = \int u\psi_r\widetilde{\mathcal{A}}\varphi dx$$

and the operator $\widetilde{\mathcal{A}}$ is uniformly elliptic in $\mathbb{R}^N$ and has the diffusion and the drift coefficients belonging, respectively, to $C_b^1(\mathbb{R}^N)$ and to $C_b(\mathbb{R}^N)$. Hence, we can apply the property (i) to the function $u\psi_r$, obtaining that $u\psi_r \in W^{1,p}(\mathbb{R}^N)$ and, consequently, that $u \in W^{1,p}(B(r))$. The arbitrariness of $r > 0$ implies that $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$.

Proof of (iii). We first assume that $\mathcal{A}$ is not in divergence form. According to the property (i), we know that $u \in W^{1,p}(\mathbb{R}^N)$. Therefore, taking (C.1.2) into account, we get

$$\int_{\mathbb{R}^N} uA_1\varphi dx = \int_{\mathbb{R}^N} f_1\varphi dx, \tag{C.1.10}$$

for any $\varphi \in C^\infty_c(\mathbb{R}^N)$ where A_1 is given by (C.1.3), $f_1 = f + u\,{\rm div}\,b + \sum_{i=1}^N b_iD_iu$ belong to $L^p(\mathbb{R}^N)$. Hence,

$$\int_{\mathbb{R}^N} u(\lambda\varphi - A_1\varphi)dx = \int_{\mathbb{R}^N} g\varphi dx \tag{C.1.11}$$

for any $\varphi \in C^\infty_c(\mathbb{R}^N)$, where $g = \lambda u - f_1$ belongs to $L^p(\mathbb{R}^N)$. By density, we can extend (C.1.11) to any function $\varphi \in W^{2,p'}(\mathbb{R}^N)$. Now, we fix λ in the resolvent sets of both the operator A_1 and its adjoint A_1^* defined by

$$A_1^*u = \sum_{i,j=1}^N q_{ij}D_{ij}u + 2\sum_{i,j=1}^N D_jq_{ij}D_iu + \sum_{i,j=1}^N (D_{ij}q_{ij})u,$$

for any $u \in D(A_1^*) = W^{2,p}(\mathbb{R}^N)$ (see again the forthcoming Theorem C.3.6(i)). To prove that $u \in W^{2,p}(\mathbb{R}^N)$, we show that $u = R(\lambda, A_1^*)g$. For this purpose, it suffices to observe that (C.1.11) is satisfied if we insert $R(\lambda, A_1^*)g$ in place of u. Therefore, the function $z = R(\lambda, A_1^*)g - u$ satisfies

$$\int_{\mathbb{R}^N} z(\lambda\varphi - A_1\varphi)dx = 0, \tag{C.1.12}$$

for any $\varphi \in W^{2,p'}(\mathbb{R}^N)$. Since $\lambda I - A_1$ is surjective from $W^{2,p'}(\mathbb{R}^N)$ into $L^{p'}(\mathbb{R}^N)$, it is now immediate to check that $z = 0$.

In the case when $\mathcal{A}$ is in divergence form, repeating the same computations as above (with the obvious changes), one can easily check that the same result holds also in the case when q_{ij} $(i, j = 1, \ldots, N)$ belong to $C^1_b(\mathbb{R}^N)$. Indeed, in the above proof, the second-order derivatives of the diffusion coefficients q_{ij} $(i, j = 1, \ldots, N)$ appeared only in the definition of the adjoint operator A_1^*. In the case when $\mathcal{A}$ is in divergence form the operator A_1^* is given by

$$A_1^*u = \sum_{i,j=1}^N D_i(q_{ij}D_ju), \qquad u \in D(A_1^*) = W^{2,p}(\mathbb{R}^N)$$

and, therefore, there is no need of requiring the existence of the second order derivatives of the diffusion coefficients.

Proof of (iv). We just consider the case when $\mathcal{A}$ is not in divergence form, since, as it has been pointed out at the end of the proof of the property (iii), the other case is completely similar. According to the property (ii), $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$. Therefore, also in this situation, u satisfies (C.1.10) for any $\varphi \in C^\infty_c(\mathbb{R}^N)$, where now $f_1 \in L^p_{\rm loc}(\mathbb{R}^N)$. Fix $r > 0$ and let ψ_r be as in the proof of the property (ii). Taking (C.1.8) (written with A_1 instead of $\mathcal{A}$) into account, one can easily show that the function $v = u\psi_r$ satisfies

$$\int_{\mathbb{R}^N} (\lambda\varphi - A_1\varphi)v dx = \int_{\mathbb{R}^N} g\varphi dx, \tag{C.1.13}$$

for any $\varphi \in C^\infty_c(\mathbb{R}^N)$ and any $\lambda > 0$, where

$$g = \lambda v - \psi_r \left(f + u \operatorname{div} b + \sum_{i=1}^N b_i D_i u \right) + uA_1\psi_r - 2 \sum_{i,j=1}^N D_i(q_{ij} u D_j \psi_r)$$

belongs to $L^p(\mathbb{R}^N)$. As it is immediately seen the function v satisfies the formula (C.1.13) also if we replace A_1 with the operator $\widetilde{A}_1 = \sum_{i,j=1}^N \widetilde{q}_{ij} D_{ij}$ when the coefficients $\widetilde{q}_{ij}$ $(i,j = 1,\ldots,N)$ are given by (C.1.9). Now, since $\widetilde{q}_{ij} \in C^2_b(\mathbb{R}^N)$ $(i,j = 1,\ldots,N)$, according to the property (iii), $v \in W^{2,p}(\mathbb{R}^N)$ so that $u \in W^{2,p}(B(r))$ and the arbitrariness of $r > 0$ allows us to conclude. ■

We now consider the parabolic problems associated with the operator $\mathcal{A}$. For this purpose, we denote by $d : [0, +\infty) \times \overline{\Omega} \to \mathbb{R}$ the function defined by

$$d(t,x) = d(t,x) = \operatorname{dist}(x, \partial\Omega) \wedge \sqrt{t}, \qquad t > 0, \ \ x \in \overline{\Omega}.$$

Theorem C.1.4 ([60], Thms. 3.5, 3.10 and [101]) *Assume that the coefficients q_{ij}, b_i $(i,j = 1,\ldots,N)$ and c of the operator $\mathcal{A}$ belong to $C^\alpha_b(\overline{\Omega})$ for some $\alpha \in (0,1)$. Further assume that $u \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,T)\times\Omega)$ is a bounded (with respect to the sup-norm) solution of the equation*

$$D_t u(t,x) - \mathcal{A}u(t,x) = 0, \qquad t \in (0,T), \ \ x \in \Omega.$$

Then, the following properties are met:

(*i*) *there exists a positive constant C_1, depending only on the coefficients of $\mathcal{A}$, such that*

$$|d^0, u|_\alpha + \sum_{i=1}^N |d, D_i u|_\alpha + \sum_{i,j=1}^N |d^2, D_i D_j u|_\alpha + |d^2, D_t u|_\alpha \le C_1 \sup_{(0,T)\times\Omega} |u|, \tag{C.1.14}$$

where

$$|d^m, u|_\alpha = \sup_{(t,x)\in(0,T)\times\Omega} |(d(t,x))^m u(t,x)| + \sup_{\substack{(t,x),(s,y)\in(0,T)\times\Omega \\ (t,x)\neq(s,y)}} (d(t,x)\wedge d(s,y))^{m+\alpha} \frac{|u(t,x)-u(s,y)|}{(|x-y|^2+|t-s|)^{\alpha/2}}.$$

In particular, for any open set $\Omega' \subset\subset \Omega$ and any $s\in(0,T)$, there exists a positive constant C_2 depending on s, the coefficients of the operator $\mathcal{A}$, Ω, Ω' and T, such that

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([s,T]\times\Omega')} \le C_2 \sup_{(0,T)\times\Omega} |u|. \tag{C.1.15}$$

Moreover, if $\mathrm{dist}(\Omega,\Omega') > \sqrt{T}$, *then*

$$\sup_{(t,x)\in(0,T)\times\Omega'} \Big(t^{\frac{1}{2}}|Du(t,x)| + t|D^2u(t,x)|\Big) \le C_3 \sup_{(t,x)\in(0,T)\times\Omega} |u(t,x)|, \tag{C.1.16}$$

for some positive constant C_3, depending on the coefficients of the operator $\mathcal{A}$, Ω, Ω' and T;

(*ii*) *let the coefficients q_{ij}, b_i and c belong to $C^{k+\alpha}(\Omega)$ for any $i,j=1,\ldots,N$ and some $k\in\mathbb{N}$. Then, u is continuously differentiable in Ω_T up to the $(k+2)$-th-order, with respect to the space variables. Moreover, $D^k u \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,T)\times\Omega)$ and*

$$\|D^k u\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon',T]\times\Omega')} \le C_4 \sup_{(t,x)\in[\varepsilon,T]\times\Omega} |u(t,x)|, \tag{C.1.17}$$

for any $\varepsilon,\varepsilon'>0$ ($\varepsilon<\varepsilon'<T$), any open set $\Omega'\subset\subset\Omega$ and some positive constant C_4, depending on ε,ε', T, Ω,Ω', $\|q_{ij}\|_{C^{1+\alpha}(\Omega)}$ and $\|b_i\|_{C^{1+\alpha}(\Omega)}$ $(i,j=1,\ldots,N)$ and $\|c\|_{C^{1+\alpha}(\Omega)}$.

Finally, if $\mathrm{dist}(\Omega',\Omega)\ge\sqrt{T}$, *then*

$$\sup_{(t,x)\in(0,T)\times\Omega'} t^{\frac{3}{2}}|D^3u(t,x)| \le C \sup_{(t,x)\in(0,T)\times\Omega} |u(t,x)|. \tag{C.1.18}$$

The next theorem provides us with Schauder estimates up to (a part of) the boundary of Ω.

Theorem C.1.5 ([94], Theorem 4.10.1) *Let Ω be an open subset of $\mathbb{R}^N$ with boundary of class $C^{2+\alpha}$ for some $\alpha\in(0,1)$, and let Ω' and Ω'' be two bounded subsets of Ω such that $\Omega'\subset\Omega''\subset\Omega$ and* $\mathrm{dist}(\Omega',\Omega\setminus\Omega'')>0$. *Moreover, assume that the coefficients q_{ij}, b_i $(i,j=1,\ldots,N)$ and c of the operator $\mathcal{A}$ belong to $C^\alpha_{\rm loc}(\overline{\Omega})$. Finally assume that $u\in C^{1+\alpha/2,2+\alpha}([T_1,T_2]\times\overline{\Omega}'')$ solves the differential equation $D_t u-\mathcal{A}u=0$ in $(T_1,T_2)\times\Omega''$ for some $0\le T_1<T_2$. Then, the following properties are met:*

(*i*) *if* $u \equiv 0$ *(resp.* $\frac{\partial u}{\partial \nu} \equiv 0$)) *on* $(T_1, T_2) \times \partial\Omega''$, *then, for any* $T^* \in (T_1, T_2)$, *there exists a positive constant* $C = C_{T^*,T_1,T_2,\Omega',\Omega''}$ *such that*

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([T^*,T_2]\times\overline{\Omega}')} \le C\|u\|_{L^\infty((T_1,T_2)\times\Omega'')}; \tag{C.1.19}$$

(*ii*) *if* $T_1 = T^* = 0$, $u \equiv 0$ *(resp.* $\frac{\partial u}{\partial \nu} \equiv 0$)) *on* $(0, T_2) \times \partial\Omega''$, *then there exists a positive constant* $C = C_{T_1,T_2,\Omega',\Omega''}$ *such that*

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T_2]\times\overline{\Omega}')} \le C\left(\|u\|_{L^\infty((0,T_2)\times\Omega'')} + \|u(0,\cdot)\|_{C^{2+\alpha}(\Omega'')}\right). \tag{C.1.20}$$

C.2 Classical maximum principles

In this section we collect the classical maximum principles for continuous solutions to both the elliptic equations (C.2.1), (C.2.2) and for the parabolic Cauchy problems (C.3.2) and (C.3.3) that we use in this book. For the proofs we refer the reader to [22], [89, Chapter 8] and [129].

We make the following assumptions on Ω and on the coefficients of the operator $\mathcal{A}$.

Hypotheses C.2.1 (i) Ω is either an open bounded set with boundary of class C^2 or $\Omega = \mathbb{R}^N$;

(ii) q_{ij}, b_i $(i, j = 1, \ldots, N)$ and c belong to $C_b(\overline{\Omega})$;

(iii) Hypothesis C.0.1 is satisfied.

Moreover, we set

$$c_0 = \sup_{x\in\Omega} c(x)$$

and denote by $\nu = \nu(x)$ the outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$.

Theorem C.2.2 *Let* $\lambda > c_0$ *and suppose that* $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ *for any* $p \in (1, +\infty)$ *satisfies the differential inequality* $\lambda u - \mathcal{A}u \ge 0$. *Then, the following properties are met:*

(*i*) *if* $u \ge 0$ *on* $\partial\Omega$, *then* $u \ge 0$ *in* Ω;

(*ii*) *if* $\dfrac{\partial u}{\partial \nu} \ge 0$ *on* $\partial\Omega$, *then* $u \ge 0$ *on* Ω.

Moreover, if $\lambda > c_0$, $f \in C_b(\overline{\Omega})$ *and* $u \in \bigcap_{1\le p<+\infty} W^{2,p}_{\mathrm{loc}}(\Omega)$ *solves the problem*

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{C.2.1}$$

or

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) = f(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(x) = 0, & x \in \partial\Omega, \end{cases} \tag{C.2.2}$$

then

$$\|u\|_\infty \leq \frac{1}{\lambda - c_0}\|f\|_\infty. \tag{C.2.3}$$

Concerning the parabolic equation $D_t u - \mathcal{A}u = 0$ we have the following result.

Proposition C.2.3 *Fix $T > 0$. Then, the following properties are met:*

(*i*) *if $z \in C_b([0,T] \times \overline{\Omega}) \cap C^{1,2}((0,T] \times \Omega)$ satisfies*

$$\begin{cases} D_t z(t,x) - \mathcal{A}z(t,x) \leq 0, & t \in (0,T],\, x \in \Omega, \\ z(t,x) \leq 0, & t \in (0,T],\, x \in \partial\Omega, \\ z(0,x) \leq 0, & x \in \overline{\Omega}, \end{cases}$$

then $z \leq 0$ in $[0,T] \times \overline{\Omega}$.

(*ii*) *if $z \in C_b([0,T] \times \overline{\Omega}) \cap C^{0,1}((0,T] \times \overline{\Omega}) \times \cap C^{1,2}((0,T] \times \Omega)$ satisfies*

$$\begin{cases} D_t z(t,x) - \mathcal{A}z(t,x) \leq 0, & t \in (0,T], \quad x \in \Omega, \\ \dfrac{\partial z}{\partial \nu}(t,x) \leq 0, & t \in (0,T], \quad x \in \partial\Omega, \\ z(0,x) \leq 0, & x \in \Omega, \end{cases}$$

then $z \leq 0$ in $[0,T] \times \overline{\Omega}$.

(*iii*) *If $z \in C_b([0,T] \times \overline{\Omega}) \cap C^{0,1}((0,T] \times \overline{\Omega}) \times \cap C^{1,2}((0,T] \times \Omega)$ satisfies*

$$D_t z(t,x) - \mathcal{A}z(t,x) \leq 0, \qquad t \in (0,T], \; x \in \Omega,$$

and z attains its maximum value M in $(0,T] \times \Omega$ at some point (t_0, x_0), then $z = M$ in $[0,t_0] \times \Omega$.

Further, if Ω satisfies the interior sphere condition and z attains its maximum value at some point $(t,x) \in (0,T] \times \partial\Omega$, then $\partial z/\partial\nu > 0$ at (t,x).

C.3 Existence of classical solution to PDE's and analytic semigroups

Here, we make the following assumptions on Ω and on the coefficients of the operator $\mathcal{A}$.

Hypotheses C.3.1 (i) Ω is an open set with a boundary which is uniformly of class $C^{2+2\alpha}$ for some $\alpha \in (0,1)$ (or, possibly, $\Omega = \mathbb{R}^N$);

(ii) q_{ij}, b_i $(i,j = 1,\ldots,N)$ and c belong to $C_b^{2\alpha}(\overline{\Omega})$;

(iii) $q_{ij} \equiv q_{ji}$ for any $i,j = 1,\ldots,N$ and

$$\sum_{i,j=1}^{N} q_{ij}(x)\xi_i\xi_j \geq \mu_0|\xi|^2, \qquad \xi \in \mathbb{R}^N, \ x \in \overline{\Omega},$$

for some positive constant μ_0.

For notational convenience, we set

$$c_0 = \sup_{x\in\overline{\Omega}} c(x), \tag{C.3.1}$$

and we denote by $\nu = \nu(x)$ the exterior unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$.

$$\begin{cases} D_t u(t,x) - \mathcal{A}u(t,x) = 0, & t>0,\, x \in \Omega,\\ u(t,x) = 0, & t>0,\, x\in\partial\Omega,\\ u(0,x) = f(x), & x \in \Omega \end{cases} \tag{C.3.2}$$

and the Cauchy-Neumann problem

$$\begin{cases} D_t u(t,x) - \mathcal{A}u(t,x) = 0, & t>0,\, x \in \Omega,\\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, & t>0,\, x\in\partial\Omega,\\ u(0,x) = f(x), & x \in \Omega. \end{cases} \tag{C.3.3}$$

More precisely, the following results can be proved (see also [74], [78], [79], [94, Chapter 4, Theorem 5.2] and [140, Chapter 5, Sections 3 & 5]).

Proposition C.3.2 *Let Hypotheses* C.3.1 *be satisfied. For any* $f \in C_b(\overline{\Omega})$, *problem* (C.3.2) *admits a unique solution* $u \in C([0,+\infty)\times\overline{\Omega}\setminus(\{0\}\times\partial\Omega)) \cap C^{1,2}((0,+\infty)\times\Omega)$, *which is bounded in* $[0,T]\times\overline{\Omega}$ *for any* $T>0$. *Moreover, it satisfies*

$$\|u(t,\cdot)\|_\infty \leq e^{c_0 t}\|f\|_\infty, \qquad t>0; \tag{C.3.4}$$

and

$$t^{\frac{k}{2}}\|D^kT(t)f\|_\infty \leq C\|f\|_\infty, \qquad t\in(0,T),$$

for any $T>0$ *some constant* $C = C(T)$, *independent of* f, *and any* $k = 1,2,3$. *Here,* c_0 *is given by* C.3.1.

Moreover, there exists a unique function $G_\Omega \in C((0,+\infty)\times\Omega\times\Omega)$ *such that*

$$u(t,x)=\int_\Omega G_\Omega(t,x,y)f(y)dy,\quad t>0,\ x\in\Omega.$$

The function G_Ω *is called the fundamental solution of the problem* (C.3.2)*, it is positive in* $(0,+\infty)\times\Omega\times\Omega$ *and satisfies*

$$G_\Omega(t+s,x,y)=\int_\Omega G_\Omega(s,x,z)G_\Omega(t,z,y)dz, \tag{C.3.5}$$

for any $s,t>0$ *and any* $x,y\in\Omega$*. Moreover, for any* $t>0$ *the function* $G(t,\cdot,\cdot)$ *is bounded in* $\Omega\times\Omega$*.*

Finally, if $f\in C_0(\Omega)$*, then* $u\in C([0,+\infty)\times\overline{\Omega})\cap C^{1+\alpha,2+2\alpha}_{\rm loc}((0,+\infty)\times\overline{\Omega})$*; if* $f\in C^{2+\alpha}_c(\mathbb{R}^N)$*, then* $u\in C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})$ *for any* $T>0$*, and, if* $f\in C^{2+k+\alpha}_c(\Omega)$ *and* Ω *is of class* $C^{2+k+\alpha}$ *for some* $k\in\mathbb{N}$*, then all the space derivatives of* u *up to the* k*-th order belong to* $C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})$ *for any* $T>0$*.*

Similar results hold for the Cauchy-Neumann problem (C.3.3). Here, we limit ourselves to stating the results that we actually need in this book.

Proposition C.3.3 *For any* $f\in C_b(\overline{\Omega})$*, there exists a unique solution* $u\in C([0,+\infty)\times\overline{\Omega})\cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((0,+\infty)\times\overline{\Omega})$ *to the Cauchy-Neumann problem* (C.3.3)*, which is bounded in* $[0,T]\times\Omega$ *for any* $T>0$*. Moreover,*

$$\|u(t,\cdot)\|_\infty\le e^{c_0t}\|f\|_\infty,\qquad t>0; \tag{C.3.6}$$

and

$$t^{\frac{k}{2}}\|D^ku(t,\cdot)\|_\infty\le C\|f\|_\infty,\qquad t\in(0,T),$$

for any $T>0$*, any* $k=1,2,3$ *and some positive constant* $C=C(T)$*. Here,* c_0 *is given by* (C.3.1)*.*

Concerning the elliptic equation $\lambda u-\mathcal{A}u=f$, we have the following results.

Proposition C.3.4 *Under Hypotheses C.3.1, for any* $f\in C_b(\overline{\Omega})$ *and any* $\lambda>c_0$*, there exists a unique solution* $u\in\bigcap_{1\le p<+\infty}W^{2,p}(\Omega)$ *to the Dirichlet problem*

$$\begin{cases}\lambda u(x)-\mathcal{A}u(x)=f(x), & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega.\end{cases} \tag{C.3.7}$$

The function u *can be represented by*

$$u(x)=\int_\Omega K^\Omega_\lambda(x,y)f(y)dy,\qquad x\in\Omega, \tag{C.3.8}$$

where K_λ^Ω*, the so-called Green's function, is given by*

$$K_\lambda^\Omega(x,y) = \int_0^{+\infty} e^{-\lambda t} G_\Omega(t,x,y)dt, \qquad x,y \in \Omega. \tag{C.3.9}$$

Proposition C.3.5 *Under Hypotheses C.3.1, for any* $f \in C_b(\overline{\Omega})$ *and any* $\lambda > c_0$*, there exists a unique solution* $u \in \bigcap_{1\le p<+\infty} W^{2,p}(\Omega)$ *to the Neumann problem*

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) = f(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{C.3.10}$$

Most of the results of this section may be rephrased in terms of analytic semigroups. Indeed, the following theorem holds.

Theorem C.3.6 *Under Hypotheses C.3.1, the following properties are met:*

(*i*) *for any* $p \in (1,+\infty)$ *the operator* $A_p : D(A_p) \to L^p(\mathbb{R}^N)$*, defined by* $A_p u = \mathcal{A}u$ *for any* $u \in D(A_p) := W^{2,p}(\mathbb{R}^N)$*, is sectorial in* $L^p(\mathbb{R}^N)$ *and* $D(A_p)$ *is dense in* $L^p(\mathbb{R}^N)$*;*

(*ii*) *let* A_p *be the operator defined by*

$$\begin{cases} D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ A_p u = \mathcal{A}u, \quad u \in D(A_p). \end{cases}$$

Then, A_p *is sectorial in* $L^p(\Omega)$*, and* $D(A_p)$ *is dense in* $L^p(\Omega)$*;*

(*iii*) *for any* $p \in (1,+\infty)$ *let* A_p *be the operator defined by*

$$\begin{cases} D(A_p) = \left\{ u \in W^{2,p}(\Omega) : \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}, \\ A_p u = \mathcal{A}u, \quad u \in D(A_p). \end{cases}$$

Then, the operator A_p *is sectorial in* $L^p(\Omega)$ *and* $D(A_p)$ *is dense in* $L^p(\Omega)$*. Moreover,* $\rho(A) \supset (c_0,+\infty)$*;*

(*iv*) *let* $A : D(A) \to C_b(\overline{\Omega})$ *be defined by*

$$\begin{cases} D(A) = \left\{ u \in \bigcap_{1\le p<+\infty} W^{2,p}_{\mathrm{loc}}(\Omega) : \ u_{|\partial\Omega} = 0, \ \mathcal{A}u \in C_b(\overline{\Omega}) \right\}, \\ Au = \mathcal{A}u, \quad u \in D(A). \end{cases} \tag{C.3.11}$$

Then, A *is sectorial in* $C_b(\overline{\Omega})$ *and* $\overline{D(A)} = C_0(\overline{\Omega}) = \{u \in C_b(\overline{\Omega}) : \ u = 0 \text{ on } \partial\Omega\}$*. Moreover,* $\rho(A) \supset (c_0,+\infty)$ *and*

$$D_A(\alpha,\infty) = \left\{ u \in C_b^{2\alpha}(\overline{\Omega}) : u_{|_{\partial\Omega}} = 0 \right\},$$

$$D_A(1+\alpha,\infty) = \left\{u \in C_b^{2+2\alpha}(\overline{\Omega}) : u_{|\partial\Omega} = (\mathcal{A}u)_{|\partial\Omega} = 0\right\}.$$

Finally, $\{u \in C_b^1(\overline{\Omega}) : u_{|\partial\Omega} = 0\}$ *belongs to the class* $J_{1/2}$ *between* $C_b(\overline{\Omega})$ *and* $D(A)$.

All the previous results hold also in the case when Ω *is replaced with* $\mathbb{R}^N$*, provided we drop out any boundary condition.*

(*v*) *Let* $A : D(A) \to C_b(\overline{\Omega})$ *be defined by*

$$\begin{cases} D(A) = \left\{u \in \bigcap_{1\le p<+\infty} W^{2,p}_{\rm loc}(\Omega) : \dfrac{\partial u}{\partial \nu} = 0, \text{ on } \partial\Omega, \ u, \ \mathcal{A}u \in C_b(\overline{\Omega})\right\}, \\ Au = \mathcal{A}u, \quad u \in D(A). \end{cases} \tag{C.3.12}$$

Then, A *is sectorial in* $C_b(\overline{\Omega})$ *and its domain is dense in* $C(\overline{\Omega})$*. Moreover,*

$$D_A(\alpha,\infty) = \begin{cases} C_b^{2\alpha}(\overline{\Omega}), & \alpha \in (0,1/2), \\ \left\{u \in C_b^{2\alpha}(\overline{\Omega}) : \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}, & \alpha \in (1/2,1); \end{cases} \tag{C.3.13}$$

$$D_A(1+\alpha,\infty) = \begin{cases} \left\{u \in C_b^{2+2\alpha}(\overline{\Omega}) : \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}, & \alpha \in (0,1/2), \\ \left\{u \in C_b^{2+2\alpha}(\overline{\Omega}) : \dfrac{\partial u}{\partial \nu} = \dfrac{\partial(\mathcal{A}u)}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}, & \alpha \in (1/2,1). \end{cases} \tag{C.3.14}$$

Finally, $\{u \in C_b^1(\overline{\Omega}) : \partial u/\partial\nu = 0$ *on* $\partial\Omega\}$ *belongs to the class* $J_{1/2}$ *between* $C_b(\overline{\Omega})$ *and* $D(A)$.

For the proof we refer the reader to [1, 4, 5, 6, 24, 25, 137, 138].

Remark C.3.7 The properties (ii) and (iii) in Theorem C.3.6, combined with the results in Section B.2, show that the classical solution to problem (C.3.2) (resp. (C.3.3)) in Proposition C.3.2 (resp. in Proposition C.3.3) is given by $u = T(\cdot)f$, where $\{T(t)\}$ is the analytic semigroup generated in $C_b(\overline{\Omega})$ by the realization of the operator $\mathcal{A}$ with homogeneous Dirichlet boundary conditions (resp. with homogeneous Neumann boundary conditions).

Similar results hold for the solutions to the elliptic equations considered in Propositions C.3.4 and C.3.5. Namely, the solution u to (C.3.7) (resp. to

(C.3.10)) is given by $u = R(\lambda, A)f$, where $R(\lambda, A)$ is the resolvent operator associated with the operator A in Theorem C.3.6(iv) (resp. Theorem C.3.6(v)).

Appendix D

Some properties of the distance function

In this appendix we collect some regularity results of the distance function $d(x) = \mathrm{dist}(x, \partial\Omega)$, when $\partial\Omega$ is the boundary of a smooth open subset Ω of $\mathbb{R}^N$. We limit ourselves to state them in the case when Ω is bounded since they are classical results (see, e.g., [66, Section 14.6]). Moreover, we show that most of them may be extended also to the case where Ω is unbounded. The results we present here are taken from [59].

Definition D.0.1 *We say that an open subset Ω of $\mathbb{R}^N$ is uniformly of class $C^{2+\alpha}$, for some $\alpha \in [0,1)$, if there exist a countable open covering $\{U_n\}$ of $\partial\Omega$ and a family of diffeomorphisms $g_n : \overline{U}_n \to \overline{B}(1)$ of class $C^{2+\alpha}$, such that $g_n(U_n \cap \Omega) = \{x \in B(1) : x_N > 0\}$, $g_n(U_n \cap \partial\Omega) = \{x \in B(1) : x_N = 0\}$ (where x_N denotes the N-th component of the vector x) and, moreover,*

- (*i*) *there exists $k \in \mathbb{N}$ such that $\bigcap_{n \in J} U_n = \varnothing$ if $\mathrm{card}\,(J) > k$;*
- (*ii*) *there exists $\varepsilon > 0$ such that $\{x \in \Omega : d(x, \partial\Omega) < \varepsilon\} \subset \bigcup_{n \in \mathbb{N}} V_n$, where $V_n = g_n^{-1}(B(1/2))$;*
- (*iii*) *there exists $C > 0$ such that $\|g_n\|_{C^{2+\alpha}} + \|g_n^{-1}\|_{C^{2+\alpha}} \le C$, for any $n \in \mathbb{N}$.*

Remark D.0.2 It is possible to show that if Ω is uniformly of class $C^{2+\alpha}$, then the diffeomorphism g_n ($n \in \mathbb{N}$) can be chosen so that

$$(\mathrm{Jac}\, g_n)(x)\nu(x) = -\alpha_n e_N, \qquad x \in U_n \cap \partial\Omega, \quad n \in \mathbb{N},$$

for some positive function α_n, where ν denotes the unit outward normal and $e_N = (0, \ldots, 0, 1)$). Roughly speaking, the diffeomorphisms g_n leave the normal direction unchanged. We refer the reader to [146, Chapter 1, Section 2.4] for the proof of this property.

Remark D.0.3 As a consequence of the condition (iii) in Definition D.0.1, the principal curvatures $\lambda_1, \ldots, \lambda_{N-1}$ of $\partial\Omega$ are bounded both from above and below with respect to x, hence

$$M_0 := \inf_{x \in \partial\Omega} \left\{ \frac{\partial\nu}{\partial\tau}(x) \cdot \tau, \ |\tau| = 1, \ \tau \cdot \nu(x) = 0 \right\} > -\infty,$$

since, locally, $\partial\nu/\partial\tau \cdot \tau = \lambda_1 \tau_1^2 + \cdots + \lambda_{N-1}\tau_{N-1}^2$.

Definition D.0.4 *The open set Ω is said to satisfy the uniform interior sphere condition if, for any point $z_0 \in \partial\Omega$, there exists a ball $B(z_0, r_{z_0}) \subset \Omega$ such that $\overline{B}(z_0, r_{z_0}) \cap \partial\Omega = \{z_0\}$, and $\inf_{z_0 \in \Omega} r_{z_0} > 0$.*

As the following proposition shows, if $\partial\Omega$ is uniformly of class C^2, then Ω satisfies the interior sphere condition.

Proposition D.0.5 ([59], Prop. B.2) *If $\partial\Omega$ is uniformly of class C^2, then it satisfies a uniform interior sphere condition.*

Proof. Let $y_0 \in \partial\Omega$ and let $j \in \mathbb{N}$ be such that $y_0 \in V_j$. Further, denote by $\nu(y_0)$ and T_{y_0}, respectively, the unit inward normal vector and the tangent space to $\partial\Omega$ at y_0. By the implicit function theorem and by a rotation of coordinates, we may assume that the x_N axis lies in the direction $\nu(y_0)$ and that $\partial\Omega \cap V_j$ is given by the equation $x_N = \psi_j(x')$, where $x' = (x_1, \ldots, x_{N-1})$ and ψ_j is of class $C^{2+\alpha}$. Moreover $\Omega \cap V_j \subseteq \{x_N < \psi_j(x')\}$. Now, let B be a tangent ball to T_{y_0} at y_0 lying in the set $\{x_N < \psi_j(x')\}$. Taking a smaller radius r_0 if necessary, we find that $B \subseteq \Omega$ and $\overline{B} \cap \partial\Omega = \{y_0\}$. Moreover, comparing the curvatures of $T(y_0)$, B, and $\partial\Omega$ at y_0 we have $0 < 1/r_0 \le k_i(y_0)$, where k_i for $i = 1, \ldots, N-1$ denote the so-called principal curvatures of $\partial\Omega$. Since k_i are related to the second order derivatives of ψ_j (see [66, (14.93)]) which are uniformly bounded, we obtain that $k_i(y_0) \le C$, hence $r_0 \ge C^{-1}$. ■

Now, we are ready to prove the main result of this appendix.

Proposition D.0.6 ([59], Proposition B.3) *Let $\partial\Omega$ be uniformly of class C^2, Further, fix $\delta = 2/C$ and set $\Omega_\delta = \{y \in \overline{\Omega} \,|\, d(y) < \delta\}$. Then,*

(*i*) *for any $x \in \Omega_\delta$ there exists a unique $y = y(x) \in \partial\Omega$ such that $|x - y| = d(x)$;*

(*ii*) *$d \in C^2_b(\Omega_\delta)$;*

(*iii*) *$Dd(x) = \nu(y(x))$, for any $x \in \Omega_\delta$.*

Proof. We limit ourselves to proving the property (i). Indeed, the proof of the properties (ii) and (iii) relies on the property (i) and the implicit function theorem, and it is completely similar to that given in [66, Section 14.6] in the case when Ω is bounded.

Of course, the existence part in (i) is obvious. To prove the uniqueness of the point $y = y(x)$ at any $x \in \Omega_\delta$, let $y \in \partial\Omega$ be such that $d(x) = |x - y|$. According to Proposition D.0.5, there exists a ball $B = \xi + B(\rho)$ such that $B \subset \Omega$ and $\overline{B} \cap \partial\Omega = \{y\}$. Moreover from the definition of δ it follows that $x \in B$. It is easy to see that x and ξ lie on the normal direction $\nu(y)$ and that the balls $x + B(d(x))$, $\xi + B(\rho)$ are tangent at y. Then, $x + B(d(x))$ still verifies the interior sphere condition at y. It follows that for any $z \in \partial\Omega \setminus \{y\}$, one has $z \notin x + B(d(x))$, so that y is actually the unique point such that $|x - y| = d(x)$.

■

Appendix E

Function spaces: definitions and main properties

In this appendix we collect all the function spaces that we consider in this book.

E.1 Spaces of functions that are continuous or Hölder continuous in domains $\Omega \subset \mathbb{R}^N$

E.1.1 Isotropic spaces

Given an open set $\Omega \subset \mathbb{R}^N$ (even $\Omega = \mathbb{R}^N$), we denote by $C_b(\Omega)$ (resp. $C_b(\overline{\Omega})$) the space of bounded and continuous functions $f : \Omega \to \mathbb{R}$ (resp. $f : \overline{\Omega} \to \mathbb{R}$) and we endow it with the sup-norm, i.e.,

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)|, \qquad f \in C_b(\Omega) \quad (\text{resp. } f \in C_b(\overline{\Omega})).$$

Let Ω be bounded, $C_0(\Omega)$ denotes the subspace of $C_b(\overline{\Omega})$ of functions $f : \overline{\Omega} \to \mathbb{R}$ such that $f = 0$ on $\partial\Omega$. When Ω is unbounded, $C_0(\overline{\Omega})$ denotes the subset of $C_b(\overline{\Omega})$ of functions $f : \overline{\Omega} \to \mathbb{R}$ such that $f = 0$ on $\partial\Omega$ and

$$\lim_{\substack{|x|\to+\infty \\ x\in\Omega}} f(x) = 0.$$

Next, for any $k > 0$, we denote by $C^k_b(\overline{\Omega})$ the space of functions $f : \overline{\Omega} \to \mathbb{R}$ which are continuously differentiable in Ω up to the $[k]$-order and such that $D^\alpha f \in C_b(\overline{\Omega})$ for any $|\alpha| \le [k]$ ($[k]$ denoting the integer part of k) and $D^\alpha f$ is $(k-[k])$-Hölder continuous for any $|\alpha| = [k]$. We endow $C^k_b(\overline{\Omega})$ with the norm

$$\|f\|_{C^k_b(\overline{\Omega})} = \sum_{|\alpha|\le[k]} \|D^\alpha f\|_\infty + \sum_{|\alpha|=[k]} [D^\alpha f]_{C^{k-[k]}(\overline{\Omega})}, \qquad f \in C^k_b(\overline{\Omega}), \quad \text{(E.1.1)}$$

where

$$[f]_{C^{k-[k]}(\overline{\Omega})} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^{k-[k]}}.$$

Similarly, $C^k_{\rm loc}(\mathbb{R}^N)$, ($k \in \mathbb{R}_+$), is the set of functions $f : \mathbb{R}^N \to \mathbb{R}$ which are continuously differentiable up to the $[k]$-th-order in $\mathbb{R}^N$ and have k-th-order derivatives which are $(k-[k])$-Hölder continuous in K for any compact set $K \subset \mathbb{R}^N$.

We say that $f \in BUC^k(\mathbb{R}^N)$ $(k \in \mathbb{N})$ if $f \in C_b^k(\mathbb{R}^N)$ and $D^\alpha f$ is uniformly continuous in $\mathbb{R}^N$ for any $|\alpha| = k$. We endow it with the norm in (E.1.1).

Moreover, for any open set $\Omega \subset \mathbb{R}^N$ and any $k \geq 0$, we denote by $C_c^k(\Omega)$ (resp. $C_c^k(\overline{\Omega})$) the subset of $C_b^k(\overline{\Omega})$ of functions $f : \overline{\Omega} \to \mathbb{R}$ compactly supported in Ω (resp. in $\overline{\Omega}$). We endow both $C_c^k(\Omega)$ and $C_c^k(\overline{\Omega})$ with the norm of $C_b^k(\overline{\Omega})$. We say that $f \in C_c^\infty(\Omega)$ if $f \in C_c^k(\Omega)$ for any $k \in \mathbb{N}$.

Finally, for any smooth open set Ω, we denote by $C_\nu^1(\overline{\Omega})$ the set of functions $f \in C_b^1(\overline{\Omega})$ such that the normal derivative $\partial f/\partial \nu$ identically vanishes on $\partial\Omega$.

Sometimes we also consider spaces of vector-valued continuous functions. For any $k \in [0,+\infty)$, and any $m \in \mathbb{N}$, we denote by $C^k(\Omega,\mathbb{R}^m)$ (resp. $C_{\rm loc}^k(\Omega,\mathbb{R}^m)$) the set of functions $f = (f_1,\ldots,f_m) : \Omega \to \mathbb{R}^m$ such that $f_j \in C^k(\Omega)$ (resp. $f_j \in C_{\rm loc}^k(\Omega)$) for any $j = 1,\ldots,m$.

E.1.2 Anisotropic Hölder spaces in $\mathbb{R}^N$

We now define the anisotropic Hölder spaces that we use in this book. For this purpose, for any $r \in \{1,\ldots,N-1\}$, we split $\mathbb{R}^N := \mathbb{R}^r \times \mathbb{R}^{N-r}$.

To define the spaces $\mathcal{C}_{b,r}^{3\theta,\theta}(\mathbb{R}^N)$, let us introduce the Zygmund spaces $\mathcal{C}_b^\alpha(\mathbb{R}^m)$ $(m \in \mathbb{N})$. For any $\gamma \in (0,1]$, the Zygmund space $\mathcal{C}_b^\gamma(\mathbb{R}^m)$ is the set of functions $f \in C_b(\mathbb{R}^m)$ such that

$$[f]_{\mathcal{C}_b^\gamma(\mathbb{R}^m)} = \sup_{\substack{x,y\in\mathbb{R}^m\\ x\neq y}} \frac{|f(x) - 2f\left(\frac{x+y}{2}\right) + f(y)|}{|x-y|^\gamma} < +\infty.$$

$\mathcal{C}_b^\gamma(\mathbb{R}^m)$ is a Banach space when it is endowed with the norm

$$\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^m)} = \|f\|_\infty + [f]_{\mathcal{C}_b^\gamma(\mathbb{R}^m)}, \qquad f \in \mathcal{C}_b^\gamma(\mathbb{R}^m).$$

For $\gamma > 1$ such that $\gamma \notin \mathbb{N}$,

$$\mathcal{C}_b^\gamma(\mathbb{R}^m) = \left\{ f \in C_b^{[\gamma]}(\mathbb{R}^m) : D^\alpha f \in \mathcal{C}_b^{\gamma-[\gamma]}(\mathbb{R}^m) \text{ for any } |\alpha| = [\gamma] \right\},$$

and it is normed by

$$\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^m)} = \|f\|_{C_b^{[\gamma]}(\mathbb{R}^m)} + \sum_{|\alpha|=[\gamma]} [D^\alpha f]_{\mathcal{C}_b^{\gamma-[\gamma]}(\mathbb{R}^m)}.$$

Finally, if $\gamma \in \mathbb{N}$, $\gamma > 1$,

$$\mathcal{C}_b^\gamma(\mathbb{R}^m) = \left\{ f \in C_b^{\gamma-1}(\mathbb{R}^m) : D^\alpha f \in \mathcal{C}_b^1(\mathbb{R}^m), \quad \text{for any } |\alpha| = \gamma - 1 \right\}$$

and it is normed by

$$\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^m)} = \|f\|_{C_b^{\gamma-1}(\mathbb{R}^m)} + \sum_{|\alpha|=\gamma-1} [D^\alpha f]_{\mathcal{C}_b^1(\mathbb{R}^m)}.$$

We can now define the space $C^{3\theta,\theta}_{b,r}(\mathbb{R}^N)$ for any $\theta \in (0,1]$, by setting

$$\mathcal{C}^{3\theta,\theta}_{b,r}(\mathbb{R}^N) = \Big\{ f : \mathbb{R}^N \to \mathbb{R} : f(\cdot,y) \in \mathcal{C}^{3\theta}_b(\mathbb{R}^r) \quad \forall y \in \mathbb{R}^{N-r},$$

$$\sup_{y\in\mathbb{R}^{N-r}} \|f(\cdot,y)\|_{\mathcal{C}^{3\theta}_b(\mathbb{R}^r)} < +\infty,$$

$$f(x,\cdot) \in \mathcal{C}^{\theta}_b(\mathbb{R}^{N-r}) \;\; \forall x \in \mathbb{R}^r, \quad \sup_{x\in\mathbb{R}^r} \|f(x,\cdot)\|_{\mathcal{C}^{\theta}_b(\mathbb{R}^{N-r})} < +\infty \Big\}$$

and we norm it by

$$\|f\|_{\mathcal{C}^{3\theta,\theta}_{b,r}(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^{N-r}} \|f(\cdot,y)\|_{\mathcal{C}^{3\theta}_b(\mathbb{R}^r)} + \sup_{x\in\mathbb{R}^r} [f(x,\cdot)]_{\mathcal{C}^{\theta}_b(\mathbb{R}^{N-r})}.$$

Similarly, we define and norm the spaces $C^{3\theta,\theta}_{b,r}(\mathbb{R}^N)$, replacing everywhere the Zygmund spaces $\mathcal{C}^{\alpha}_b$ with the usual Hölder spaces C^{α}_b.

When no confusion may arise, we simply write $\mathcal{C}^{3\theta,\theta}_b(\mathbb{R}^N)$ and $C^{3\theta,\theta}_b(\mathbb{R}^N)$.

Remark E.1.1 It is well known (see, e.g., [104, Proposition 0.2.2]) that the spaces $\mathcal{C}^{\gamma}_b(\mathbb{R}^m)$ and $C^{\gamma}_b(\mathbb{R}^m)$ coincide when $\gamma \notin \mathbb{N}$ and the corresponding norms are equivalent, whereas, if $\gamma \in \mathbb{N}$, $C^{\gamma}_b(\mathbb{R}^m)$ is properly continuously embedded in $\mathcal{C}^{\gamma}_b(\mathbb{R}^m)$. It follows that $\mathcal{C}^{3\theta,\theta}_{b,r}(\mathbb{R}^N) = C^{3\theta,\theta}_{b,r}(\mathbb{R}^N)$ with equivalence of the corresponding norms, if $\theta \in \mathbb{R}_+ \setminus (\frac{1}{3}\mathbb{N})$, while $C^{3\theta,\theta}_{b,r}(\mathbb{R}^N)$ is properly and continuously embedded in $\mathcal{C}^{3\theta,\theta}_{b,r}(\mathbb{R}^N)$ otherwise.

Remark E.1.2 By interpolation, it is easy to check that, if $f \in \mathcal{C}^{3\theta,\theta}_{b,r}(\mathbb{R}^N)$ and $3\theta > 1$, then all the derivatives of f, with respect to the first r variables and up to the $[3\theta]$-order are (bounded and) continuous in $\mathbb{R}^N$. Indeed, for any $x \in \mathbb{R}^N$ split $x = (y,z)$ where $y = (x_1,\ldots,x_r)$ and $z = (x_{r+1},\ldots,x_N)$, and let $k \in \mathbb{N}$ be such that $k < 3\theta$. Since there exists a positive constant C such that

$$\|g\|_{C^k_{b,r}(\mathbb{R}^r)} \le C\|g\|^{1-\frac{k}{3\theta}}_{C_b(\mathbb{R}^r)} \|g\|^{\frac{k}{3\theta}}_{\mathcal{C}^{3\theta}_b(\mathbb{R}^r)}, \qquad g \in C^{3\theta}_{b,r}(\mathbb{R}^r)$$

(see Proposition A.4.4), we easily deduce that

$$\begin{aligned} &\|f(\cdot,y_2) - f(\cdot,y_1)\|_{C^k_b(\mathbb{R}^r)} \\ &\le C\|f(\cdot,y_2) - f(\cdot,y_1)\|^{1-\frac{k}{3\theta}}_{C_b(\mathbb{R}^r)} \|f(\cdot,y_2) - f(\cdot,y_1)\|^{\frac{k}{3\theta}}_{\mathcal{C}^{3\theta}_b(\mathbb{R}^r)} \\ &\le 2C\|f\|_{\mathcal{C}^{3\theta,\theta}_{b,r}(\mathbb{R}^N)} |y_2 - y_1|^{\theta-\frac{k}{3}}, \end{aligned} \tag{E.1.2}$$

for any $z_1, z_2 \in \mathbb{R}^{N-r}$. From (E.1.2) we deduce that the k-th derivatives of f with respect to the first r variables belong to $\mathcal{C}^{3\theta-k,\theta-k/3}_{b,r}(\mathbb{R}^N)$, so, in particular, they are continuous in $\mathbb{R}^N$.

E.2 Parabolic Hölder spaces

Given any interval $I \subset (0,+\infty)$, any set $D \subset \mathbb{R}^N$ with nonempty interior part, and two integers $h,k \in \mathbb{N}\cup\{0\}$, $h \le 1$, we denote by $C^{h,k}(I\times D)$, the set of functions $f : I\times D \to \mathbb{R}$ which are continuous in $I\times D$, continuously differentiable in the interior of $I\times D$ up to the h-th order with respect to the time variable and up to the k-th order with respect to the space variables and such with all the derivatives which can be extended by continuity to $I\times D$. When $I\times D$ is bounded, we norm $C^{h,k}(I\times D)$ by setting

$$\|f\|_{C_b^{h,k}(I\times D)} = \sum_{j=0}^{h} \|D_t^{(j)} f\|_\infty + \sum_{0<|\alpha|\le k} \|D^\alpha f\|_\infty. \tag{E.2.1}$$

Moreover, when $I\times D$ is unbounded we write $C_b^{h,k}(I\times D)$ to denote the set of functions belonging to $C^{h,k}(I\times D)$ which are bounded together with their derivatives. We endow $C_b^{h,k}(I\times D)$ with the norm in (E.2.1).

We denote by $C_c^\infty(I\times D)$ the set of functions $f : I\times D \to \mathbb{R}$ compactly supported in $I\times D$, which admit classical derivatives $D_t^j D^\alpha f$ of any order in $I\times D$.

Next, for any $\alpha\in(0,1)$ and any $k\in\mathbb{N}\cup\{0\}$, we denote by $C^{1+\alpha/2,2+\alpha}(I\times D)$ the subspace of $C^{1,2}(I\times\Omega)$ of functions $f : I\times D\to\mathbb{R}$ such that the functions $D_t f, D_x^\beta f$ ($|\beta|\le 2$) are bounded and satisfy

$$\|f\|_{C^{1+\alpha/2,2+\alpha}(I\times D)} := \sup_{x\in D}\|f(\cdot,x)\|_{C_b^{1+\alpha/2}(I)} + \sup_{t\in I}\|f(t,\cdot)\|_{C_b^{2+\alpha}(D)} < +\infty. \tag{E.2.2}$$

It is a Banach space when endowed with the norm in (E.2.2).

Similarly, we say that $f\in C^{1+\alpha/2,2+\alpha}_{\rm loc}(I\times D)$ if $f\in C^{1+\alpha/2,2+\alpha}(F)$ for any open set $F := (a,b)\times\Omega'$ with compact closure in $I\times D$.

Let Y be a subspace of $C_b(\overline{\Omega})$. By $B(I;Y)$ and $C_b^k(I;Y)$ ($k\ge 0$) we denote the set of functions $f : I\to Y$ which are, respectively, bounded in I and differentiable in I (with values in Y) up to the $[k]$-th order, with bounded derivatives, $f^{([k])}$ being Hölder continuous in I with exponent $k-[k]$. We norm these spaces by setting

$$\|f\|_{B(I;Y)} = \sup_{t\in I}\|f(t,\cdot)\|_Y,$$

and

$$\|f\|_{C^k(I;Y)} = \sum_{j=0}^{[k]}\|f^{(j)}\|_{B(I;Y)} + \sup_{\substack{s,t\in I\\ s\neq t}}\frac{\|f(t)-f(s)\|_Y}{|t-s|^{k-[k]}}.$$

Sometimes, we identify a bounded and continuous function $f : I\to\Omega\to\mathbb{R}$ with an element of the space $C(I;C_b(\Omega))$ in the natural way. Modulo this identification, we can say that $f \in C^{1+\alpha/2,2+k+\alpha}(I \times \Omega)$ ($k \in \mathbb{N} \cup \{0\}$) if and only if $f \in C^{1+\alpha/2}(I; C_b(\overline{\Omega})) \cap C(I; C_b^{2+k+\alpha}(\Omega))$.

E.3 L^p and Sobolev spaces

Given an open set $\Omega \subset \mathbb{R}^N$ (possibly $\Omega = \mathbb{R}^N$), we denote by $B_b(\Omega)$ the space of bounded Borel measurable functions defined in Ω.

We now define the usual Sobolev spaces. Let μ be a positive measure defined on the σ-algebra of all the Borel sets of Ω. We denote by $L^p(\Omega, \mu)$ ($p \in [1, +\infty)$) the space of Borel measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_{L^p(\Omega,\mu)} := \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} < +\infty, \tag{E.3.1}$$

if $p < +\infty$ and

$$\|f\|_\infty = \min\{\Lambda > 0 \text{ s.t. } |f(x)| \le \Lambda,\ \mu - a.e. \text{ in } \Omega\}. \tag{E.3.2}$$

$L^p(\Omega, \mu)$ ($p \in [1, +\infty]$) is a Banach space when endowed with the norm in (E.3.1) and (E.3.2).

By $L^p(\Omega)$ ($p \in [1, +\infty]$) we denote the usual L^p spaces related to the Lebesgue measure defined on the σ-algebra of the Lebesgue measurable sets.

By $L^p_{\rm loc}(\Omega)$ ($p \in [1, +\infty]$) we denote the space of functions f which belong to $L^p(F)$ for any open set $F \subset \mathbb{R}^N$ with compact closure.

The Sobolev space $W^{k,p}(\Omega, \mu)$ ($k \in \mathbb{N}$, $p \in [1, +\infty]$) is the subspace of $L^p(\Omega, \mu)$ of functions f which admits distributional derivatives up to the k-th-order in $L^p(\Omega, \mu)$. It is a Banach space when endowed with the norm

$$\|f\|_{W^{k,p}(\Omega,\mu)} = \sum_{|\alpha| \le k} \|D^\alpha f\|_{L^p(\Omega,\mu)}, \qquad f \in W^{k,p}(\Omega, \mu).$$

As above, we simply write $W^{k,p}(\Omega)$ when the underlying measure is the Lebesgue measure.

We say that $u \in W^{k,p}_{\rm loc}(\Omega)$ ($k \in \mathbb{N}$, $p \in [1, +\infty]$) if $u \in W^{k,p}(F)$ for any open set $F \subset \mathbb{R}^N$ with compact closure.

For more details on L^p and Sobolev spaces we refer the reader to [2].

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